Theoretical expansion of two operators in series

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October 2022

0- Abstract:

This paper is a clean and short approximation to the concept of the double serial operators in a theoretical way. It introduces with the help of combinatorics the full extension of theoretical formalism which is necessary to do double serial operators with the 6 basic mathematical tools.

Keywords: Mathematical Series, Calculus, Mathematical Operators, Summability.

- Dedicated to Paula, for giving me the necessary wings.

1- Introduction:

Here we can see the serial function of Addition and its opposite, Subtraction. Multiplication and its opposite, Division. Exponent and its opposite, Root. But this paper does not covers “Property” (the fourth positive operator in my words) also named “ultra exponential function” by M. H. Hooshmand when he analyzed the properties [2] and also named in its first definitions “Tetration” by Goodstein in his paper of 1947 and confirm by Wolfgang Beucher [4]. And it does not covers neither Scarcity (the fourth negative operator in my words) which could be also named “ultra root function”. These two operators are defined in them simple (not serial) form in my paper “Four-Operator Algebra” [1]. Anyway it is very interesting in my opinion to analyze the theoretical form of the serial expansions to understand in a generalist point of view a panoramic of the double series. And in this paper we are going to focus in the classical 6 first function operators.

I want to introduce the symbol tau for permutations of the 6 operators with 2 variables, I use the idea of the Kronecker delta [3], but in this time my symbol does not represent a matrix, instead it represents a permutation group.

$$\tau_{ij} := \{i+j\},\{i-j\},\{i\cdot j\},\{i\div j\},\{i\uparrow j\},\{\sqrt[j]{i}\}$$

2- The 36 possible theoretical expansions:

2.1- Summation of summation:

$$\sum_{m=c}^{d}\sum_{n=a}^{b}\tau_{ij} = f\left((c)_a+(c+1)_a+\ldots+(d-1)_a+(d)_a\right)+f\left((c)_{(a+1)}+(c+1)_{(a+1)}+\ldots+(d-1)_{(a+1)}+(d)_{(a+1)}\right)$$
$$+\ldots+f\left((c)_{(b-1)}+(c+1)_{(b-1)}+\ldots+(d-1)_{(b-1)}+(d)_{(b-1)}\right)+f\left((c)_{(b)}+(c+1)_{(b)}+\ldots+(d-1)_{(b)}+(d)_{(b)}\right)$$

2.2- Summation of subtractory:

$$\underset{m=c}{\overset{d}{\Sigma}}\ \underset{n=a}{\overset{b}{\mathrm{P}}}\ \tau_{ij}=f\left(-(c)_a-(c+1)_a-\ldots-(d-1)_a-(d)_a\right)+f\left(-(c)_{(a+1)}-(c+1)_{(a+1)}-\ldots-(d-1)_{(a+1)}-(d)_{(a+1)}\right)$$
$$+\ldots+f\left(-(c)_{(b-1)}-(c+1)_{(b-1)}-\ldots-(d-1)_{(b-1)}-(d)_{(b-1)}\right)+f\left(-(c)_{(b)}-(c+1)_{(b)}-\ldots-(d-1)_{(b)}-(d)_{(b)}\right)$$

2.3- Summation of productory:

$$\underset{m=c}{\overset{d}{\Sigma}}\ \underset{n=a}{\overset{b}{\Pi}}\ \tau_{ij}=f\left((c)_a\cdot(c+1)_a\cdot\ldots\cdot(d-1)_a\cdot(d)_a\right)+f\left((c)_{(a+1)}\cdot(c+1)_{(a+1)}\cdot\ldots\cdot(d-1)_{(a+1)}\cdot(d)_{(a+1)}\right)$$
$$+\ldots+f\left((c)_{(b-1)}\cdot(c+1)_{(b-1)}\cdot\ldots\cdot(d-1)_{(b-1)}\cdot(d)_{(b-1)}\right)+f\left((c)_{(b)}\cdot(c+1)_{(b)}\cdot\ldots\cdot(d-1)_{(b)}\cdot(d)_{(b)}\right)$$

2.4- Summation of divisory:

$$\underset{m=c}{\overset{d}{\Sigma}}\ \underset{n=a}{\overset{b}{\Delta}}\ \tau_{ij}=f\left((c)_a\div(c+1)_a\div\ldots\div(d-1)_a\div(d)_a\right)+f\left((c)_{(a+1)}\div(c+1)_{(a+1)}\div\ldots\div(d-1)_{(a+1)}\div(d)_{(a+1)}\right)$$
$$+\ldots+f\left((c)_{(b-1)}\div(c+1)_{(b-1)}\div\ldots\div(d-1)_{(b-1)}\div(d)_{(b-1)}\right)+f\left((c)_{(b)}\div(c+1)_{(b)}\div\ldots\div(d-1)_{(b)}\div(d)_{(b)}\right)$$

2.5- Summation of exponentory:

$$\underset{m=c}{\overset{d}{\Sigma}}\ \underset{n=a}{\overset{b}{\Theta}}\ \tau_{ij}=f\left((c)_a\uparrow(c+1)_a\uparrow\ldots\uparrow(d-1)_a\uparrow(d)_a\right)+f\left((c)_{(a+1)}\uparrow(c+1)_{(a+1)}\uparrow\ldots\uparrow(d-1)_{(a+1)}\uparrow(d)_{(a+1)}\right)$$
$$+\ldots+f\left((c)_{(b-1)}\uparrow(c+1)_{(b-1)}\uparrow\ldots\uparrow(d-1)_{(b-1)}\uparrow(d)_{(b-1)}\right)+f\left((c)_{(b)}\uparrow(c+1)_{(b)}\uparrow\ldots\uparrow(d-1)_{(b)}\uparrow(d)_{(b)}\right)$$

2.6- Summation of rootory:

$$\underset{m=c}{\overset{d}{\Sigma}}\ \underset{n=a}{\overset{b}{\mathrm{Z}}}\ \tau_{ij}=f\left(\sqrt[d_a]{ANS}\,\sqrt[d-1_a]{ANS}\cdots\sqrt{ANS}\,\sqrt[c+1_a]{c_a}\right)+f\left(\sqrt[d_{(a+1)}]{ANS}\,\sqrt[d-1_{(a+1)}]{ANS}\cdots\sqrt{ANS}\,\sqrt[c+1_{(a+1)}]{c_{(a+1)}}\right)$$
$$+\ldots+f\left(\sqrt[d_{(b-1)}]{ANS}\,\sqrt[d-1_{(b-1)}]{ANS}\cdots\sqrt{ANS}\,\sqrt[c+1_{(b-1)}]{c_{(b-1)}}\right)+f\left(\sqrt[d_b]{ANS}\,\sqrt[d-1_b]{ANS}\cdots\sqrt{ANS}\,\sqrt[c+1_b]{c_b}\right)$$

2.7- Subtractory of summation:

$$\underset{m=c}{\overset{d}{\mathrm{P}}}\ \underset{n=a}{\overset{b}{\Sigma}}\ \tau_{ij}=-f\left((c)_a+(c+1)_a+\ldots+(d-1)_a+(d)_a\right)-f\left((c)_{(a+1)}+(c+1)_{(a+1)}+\ldots+(d-1)_{(a+1)}+(d)_{(a+1)}\right)$$
$$-\ldots-f\left((c)_{(b-1)}+(c+1)_{(b-1)}+\ldots+(d-1)_{(b-1)}+(d)_{(b-1)}\right)-f\left((c)_{(b)}+(c+1)_{(b)}+\ldots+(d-1)_{(b)}+(d)_{(b)}\right)$$

2.8- Subtractory of subtractory:

$$\underset{m=c}{\overset{d}{\mathrm{P}}}\ \underset{n=a}{\overset{b}{\mathrm{P}}}\ \tau_{ij}=-f\left(-(c)_a-(c+1)_a-\ldots-(d-1)_a-(d)_a\right)-f\left(-(c)_{(a+1)}-(c+1)_{(a+1)}-\ldots-(d-1)_{(a+1)}-(d)_{(a+1)}\right)$$
$$-\ldots-f\left(-(c)_{(b-1)}-(c+1)_{(b-1)}-\ldots-(d-1)_{(b-1)}-(d)_{(b-1)}\right)-f\left(-(c)_{(b)}-(c+1)_{(b)}-\ldots-(d-1)_{(b)}-(d)_{(b)}\right)$$

2.9- Subtractory of productory:

$$\underset{m=c}{\overset{d}{\mathrm{P}}}\ \underset{n=a}{\overset{b}{\Pi}}\ \tau_{ij}=-f\left((c)_a\cdot(c+1)_a\cdot\ldots\cdot(d-1)_a\cdot(d)_a\right)-f\left((c)_{(a+1)}\cdot(c+1)_{(a+1)}\cdot\ldots\cdot(d-1)_{(a+1)}\cdot(d)_{(a+1)}\right)$$

$$-\ldots-f((c)_{(b-1)}\cdot(c+1)_{(b-1)}\cdot\ldots\cdot(d-1)_{(b-1)}\cdot(d)_{(b-1)})-f((c)_{(b)}\cdot(c+1)_{(b)}\cdot\ldots\cdot(d-1)_{(b)}\cdot(d)_{(b)})$$

2.10- Subtractory of divisory:

$$\underset{m=c}{\overset{d}{\mathrm{P}}}\ \underset{n=a}{\overset{b}{\Delta}}\ \tau_{ij}=-f((c)_a\div(c+1)_a\div\ldots\div(d-1)_a\div(d)_a)-f((c)_{(a+1)}\div(c+1)_{(a+1)}\div\ldots\div(d-1)_{(a+1)}\div(d)_{(a+1)})$$
$$-\ldots-f((c)_{(b-1)}\div(c+1)_{(b-1)}\div\ldots\div(d-1)_{(b-1)}\div(d)_{(b-1)})-f((c)_{(b)}\div(c+1)_{(b)}\div\ldots\div(d-1)_{(b)}\div(d)_{(b)})$$

2.11- Subtractory of exponentory:

$$\underset{m=c}{\overset{d}{\mathrm{P}}}\ \underset{n=a}{\overset{b}{\Theta}}\ \tau_{ij}=-f((c)_a\uparrow(c+1)_a\uparrow\ldots\uparrow(d-1)_a\uparrow(d)_a)-f((c)_{(a+1)}\uparrow(c+1)_{(a+1)}\uparrow\ldots\uparrow(d-1)_{(a+1)}\uparrow(d)_{(a+1)})$$
$$-\ldots-f((c)_{(b-1)}\uparrow(c+1)_{(b-1)}\uparrow\ldots\uparrow(d-1)_{(b-1)}\uparrow(d)_{(b-1)})-f((c)_{(b)}\uparrow(c+1)_{(b)}\uparrow\ldots\uparrow(d-1)_{(b)}\uparrow(d)_{(b)})$$

2.12- Subtractory of rootory:

$$\underset{m=c}{\overset{d}{\mathrm{P}}}\ \underset{n=a}{\overset{b}{\mathrm{Z}}}\ \tau_{ij}=-f(\sqrt[d_a]{ANS}\,{}^{d-1}\sqrt[a]{ANS}\,\ddot{\cdot}\sqrt{ANS}\,{}^{c+1}\sqrt[a]{c_a})-f(\sqrt[d_{(a+1)}]{ANS}\,{}^{d-1}\sqrt[(a+1)]{ANS}\,\ddot{\cdot}\sqrt{ANS}\,{}^{c+1}\sqrt[(a+1)]{c_{(a+1)}})$$
$$-\ldots-f(\sqrt[d_{(b-1)}]{ANS}\,{}^{d-1}\sqrt[(b-1)]{ANS}\,\ddot{\cdot}\sqrt{ANS}\,{}^{c+1}\sqrt[(b-1)]{c_{(b-1)}})-f(\sqrt[d_b]{ANS}\,{}^{d-1}\sqrt[b]{ANS}\,\ddot{\cdot}\sqrt{ANS}\,{}^{c+1}\sqrt[b]{c_b})$$

2.13- Productory of summation:

$$\underset{m=c}{\overset{d}{\Pi}}\ \underset{n=a}{\overset{b}{\Sigma}}\ \tau_{ij}=f((c)_a+(c+1)_a+\ldots+(d-1)_a+(d)_a)\cdot f((c)_{(a+1)}+(c+1)_{(a+1)}+\ldots+(d-1)_{(a+1)}+(d)_{(a+1)})$$
$$\ldots\cdot\ldots\cdot f((c)_{(b-1)}+(c+1)_{(b-1)}+\ldots+(d-1)_{(b-1)}+(d)_{(b-1)})\cdot f((c)_{(b)}+(c+1)_{(b)}+\ldots+(d-1)_{(b)}+(d)_{(b)})$$

2.14- Productory of subtractory:

$$\underset{m=c}{\overset{d}{\Pi}}\ \underset{n=a}{\overset{b}{\mathrm{P}}}\ \tau_{ij}=f(-(c)_a-(c+1)_a-\ldots-(d-1)_a-(d)_a)\cdot f(-(c)_{(a+1)}-(c+1)_{(a+1)}-\ldots-(d-1)_{(a+1)}-(d)_{(a+1)})$$
$$\ldots\cdot\ldots\cdot f(-(c)_{(b-1)}-(c+1)_{(b-1)}-\ldots-(d-1)_{(b-1)}-(d)_{(b-1)})\cdot f(-(c)_{(b)}-(c+1)_{(b)}-\ldots-(d-1)_{(b)}-(d)_{(b)})$$

2.15- Productory of productory:

$$\underset{m=c}{\overset{d}{\Pi}}\ \underset{n=a}{\overset{b}{\Pi}}\ \tau_{ij}=f((c)_a\cdot(c+1)_a\cdot\ldots\cdot(d-1)_a\cdot(d)_a)\cdot f((c)_{(a+1)}\cdot(c+1)_{(a+1)}\cdot\ldots\cdot(d-1)_{(a+1)}\cdot(d)_{(a+1)})$$
$$\ldots\cdot\ldots\cdot f((c)_{(b-1)}\cdot(c+1)_{(b-1)}\cdot\ldots\cdot(d-1)_{(b-1)}\cdot(d)_{(b-1)})\cdot((c)_{(b)}\cdot(c+1)_{(b)}\cdot\ldots\cdot(d-1)_{(b)}\cdot(d)_{(b)})$$

2.16- Productory of divisory:

$$\underset{m=c}{\overset{d}{\Pi}}\ \underset{n=a}{\overset{b}{\Delta}}\ \tau_{ij}=f((c)_a\div(c+1)_a\div\ldots\div(d-1)_a\div(d)_a)\cdot f((c)_{(a+1)}\div(c+1)_{(a+1)}\div\ldots\div(d-1)_{(a+1)}\div(d)_{(a+1)})$$
$$\ldots\cdot\ldots\cdot f((c)_{(b-1)}\div(c+1)_{(b-1)}\div\ldots\div(d-1)_{(b-1)}\div(d)_{(b-1)})\cdot f((c)_{(b)}\div(c+1)_{(b)}\div\ldots\div(d-1)_{(b)}\div(d)_{(b)})$$

2.17- Productory of exponentory:

$$\overset{d}{\underset{m=c}{\Pi}}\ \overset{b}{\underset{n=a}{\Theta}}\ \tau_{ij}=f\left((c)_a\uparrow(c+1)_a\uparrow\ldots\uparrow(d-1)_a\uparrow(d)_a\right)\cdot f\left((c)_{(a+1)}\uparrow(c+1)_{(a+1)}\uparrow\ldots\uparrow(d-1)_{(a+1)}\uparrow(d)_{(a+1)}\right)$$
$$\ldots\cdot\ldots\cdot f\left((c)_{(b-1)}\uparrow(c+1)_{(b-1)}\uparrow\ldots\uparrow(d-1)_{(b-1)}\uparrow(d)_{(b-1)}\right)\cdot f\left((c)_{(b)}\uparrow(c+1)_{(b)}\uparrow\ldots\uparrow(d-1)_{(b)}\uparrow(d)_{(b)}\right)$$

2.18- Productory of rootory:

$$\overset{d}{\underset{m=c}{\Pi}}\ \overset{b}{\underset{n=a}{Z}}\ \tau_{ij}=f\left(\sqrt[d_a]{ANS}\,{}^{d-1_a}\!\!\sqrt{ANS}\cdots\sqrt{ANS}\,{}^{c+1_a}\!\!\sqrt{c_a}\right)\cdot f\left(\sqrt[d_{(a+1)}]{ANS}\,{}^{d-1_{(a+1)}}\!\!\sqrt{ANS}\cdots\sqrt{ANS}\,{}^{c+1_{(a+1)}}\!\!\sqrt{c_{(a+1)}}\right)$$
$$\ldots\cdot\ldots\cdot f\left(\sqrt[d_{(b-1)}]{ANS}\,{}^{d-1_{(b-1)}}\!\!\sqrt{ANS}\cdots\sqrt{ANS}\,{}^{c+1_{(b-1)}}\!\!\sqrt{c_{(b-1)}}\right)\cdot f\left(\sqrt[d_b]{ANS}\,{}^{d-1_b}\!\!\sqrt{ANS}\cdots\sqrt{ANS}\,{}^{c+1_b}\!\!\sqrt{c_b}\right)$$

2.19- Divisory of summation:

$$\overset{d}{\underset{m=c}{\Delta}}\ \overset{b}{\underset{n=a}{\Sigma}}\ \tau_{ij}=f\left((c)_a+(c+1)_a+\ldots+(d-1)_a+(d)_a\right)\div f\left((c)_{(a+1)}+(c+1)_{(a+1)}+\ldots+(d-1)_{(a+1)}+(d)_{(a+1)}\right)$$
$$\ldots\div\ldots\div f\left((c)_{(b-1)}+(c+1)_{(b-1)}+\ldots+(d-1)_{(b-1)}+(d)_{(b-1)}\right)\div f\left((c)_{(b)}+(c+1)_{(b)}+\ldots+(d-1)_{(b)}+(d)_{(b)}\right)$$

2.20- Divisory of subtractory:

$$\overset{d}{\underset{m=c}{\Delta}}\ \overset{b}{\underset{n=a}{P}}\ \tau_{ij}=f\left(-(c)_a-(c+1)_a-\ldots-(d-1)_a-(d)_a\right)\div f\left(-(c)_{(a+1)}-(c+1)_{(a+1)}-\ldots-(d-1)_{(a+1)}-(d)_{(a+1)}\right)$$
$$\ldots\div\ldots\div f\left(-(c)_{(b-1)}-(c+1)_{(b-1)}-\ldots-(d-1)_{(b-1)}-(d)_{(b-1)}\right)\div f\left(-(c)_{(b)}-(c+1)_{(b)}-\ldots-(d-1)_{(b)}-(d)_{(b)}\right)$$

2.21- Divisory of productory:

$$\overset{d}{\underset{m=c}{\Delta}}\ \overset{b}{\underset{n=a}{\Pi}}\ \tau_{ij}=f\left((c)_a\cdot(c+1)_a\cdot\ldots\cdot(d-1)_a\cdot(d)_a\right)\div f\left((c)_{(a+1)}\cdot(c+1)_{(a+1)}\cdot\ldots\cdot(d-1)_{(a+1)}\cdot(d)_{(a+1)}\right)$$
$$\ldots\div\ldots\div f\left((c)_{(b-1)}\cdot(c+1)_{(b-1)}\cdot\ldots\cdot(d-1)_{(b-1)}\cdot(d)_{(b-1)}\right)\div\left((c)_{(b)}\cdot(c+1)_{(b)}\cdot\ldots\cdot(d-1)_{(b)}\cdot(d)_{(b)}\right)$$

2.22- Divisory of divisory:

$$\overset{d}{\underset{m=c}{\Delta}}\ \overset{b}{\underset{n=a}{\Delta}}\ \tau_{ij}=f\left((c)_a\div(c+1)_a\div\ldots\div(d-1)_a\div(d)_a\right)\div f\left((c)_{(a+1)}\div(c+1)_{(a+1)}\div\ldots\div(d-1)_{(a+1)}\div(d)_{(a+1)}\right)$$
$$\ldots\div\ldots\div f\left((c)_{(b-1)}\div(c+1)_{(b-1)}\div\ldots\div(d-1)_{(b-1)}\div(d)_{(b-1)}\right)\div f\left((c)_{(b)}\div(c+1)_{(b)}\div\ldots\div(d-1)_{(b)}\div(d)_{(b)}\right)$$

2.23- Divisory of exponentory:

$$\overset{d}{\underset{m=c}{\Delta}}\ \overset{b}{\underset{n=a}{\Theta}}\ \tau_{ij}=f\left((c)_a\uparrow(c+1)_a\uparrow\ldots\uparrow(d-1)_a\uparrow(d)_a\right)\div f\left((c)_{(a+1)}\uparrow(c+1)_{(a+1)}\uparrow\ldots\uparrow(d-1)_{(a+1)}\uparrow(d)_{(a+1)}\right)$$
$$\ldots\div\ldots\div f\left((c)_{(b-1)}\uparrow(c+1)_{(b-1)}\uparrow\ldots\uparrow(d-1)_{(b-1)}\uparrow(d)_{(b-1)}\right)\div f\left((c)_{(b)}\uparrow(c+1)_{(b)}\uparrow\ldots\uparrow(d-1)_{(b)}\uparrow(d)_{(b)}\right)$$

2.24- Divisory of rootory:

$$\underset{m=c}{\overset{d}{\Delta}}\ \underset{n=a}{\overset{b}{Z}}\ \tau_{ij}=f\left(\sqrt[d_a]{ANS}\sqrt[d-1_a]{ANS}\because\sqrt{ANS}\sqrt[c+1_a]{c_a}\right)\div f\left(\sqrt[d_{(a+1)}]{ANS}\sqrt[d-1_{(a+1)}]{ANS}\because\sqrt{ANS}\sqrt[c+1_{(a+1)}]{c_{(a+1)}}\right)$$
$$\ldots\div\ldots\div f\left(\sqrt[d_{(b-1)}]{ANS}\sqrt[d-1_{(b-1)}]{ANS}\because\sqrt{ANS}\sqrt[c+1_{(b-1)}]{c_{(b-1)}}\right)\div f\left(\sqrt[d_b]{ANS}\sqrt[d-1_b]{ANS}\because\sqrt{ANS}\sqrt[c+1_b]{c_b}\right)$$

2.25- Exponentory of summation:

$$\underset{m=c}{\overset{d}{\Theta}}\ \underset{n=a}{\overset{b}{\Sigma}}\ \tau_{ij}=f\left((c)_a+(c+1)_a+\ldots+(d-1)_a+(d)_a\right)\uparrow f\left((c)_{(a+1)}+(c+1)_{(a+1)}+\ldots+(d-1)_{(a+1)}+(d)_{(a+1)}\right)$$
$$\uparrow\ldots\uparrow f\left((c)_{(b-1)}+(c+1)_{(b-1)}+\ldots+(d-1)_{(b-1)}+(d)_{(b-1)}\right)\uparrow f\left((c)_{(b)}+(c+1)_{(b)}+\ldots+(d-1)_{(b)}+(d)_{(b)}\right)$$

2.26- Exponentory of subtractory:

$$\underset{m=c}{\overset{d}{\Theta}}\ \underset{n=a}{\overset{b}{P}}\ \tau_{ij}=f\left(-(c)_a-(c+1)_a-\ldots-(d-1)_a-(d)_a\right)\uparrow f\left(-(c)_{(a+1)}-(c+1)_{(a+1)}-\ldots-(d-1)_{(a+1)}-(d)_{(a+1)}\right)$$
$$\uparrow\ldots\uparrow f\left(-(c)_{(b-1)}-(c+1)_{(b-1)}-\ldots-(d-1)_{(b-1)}-(d)_{(b-1)}\right)\uparrow f\left(-(c)_{(b)}-(c+1)_{(b)}-\ldots-(d-1)_{(b)}-(d)_{(b)}\right)$$

2.27- Exponentory of productory:

$$\underset{m=c}{\overset{d}{\Theta}}\ \underset{n=a}{\overset{b}{\Pi}}\ \tau_{ij}=f\left((c)_a\cdot(c+1)_a\cdot\ldots\cdot(d-1)_a\cdot(d)_a\right)\uparrow f\left((c)_{(a+1)}\cdot(c+1)_{(a+1)}\cdot\ldots\cdot(d-1)_{(a+1)}\cdot(d)_{(a+1)}\right)$$
$$\uparrow\ldots\uparrow f\left((c)_{(b-1)}\cdot(c+1)_{(b-1)}\cdot\ldots\cdot(d-1)_{(b-1)}\cdot(d)_{(b-1)}\right)\uparrow\left((c)_{(b)}\cdot(c+1)_{(b)}\cdot\ldots\cdot(d-1)_{(b)}\cdot(d)_{(b)}\right)$$

2.28- Exponentory of divisory:

$$\underset{m=c}{\overset{d}{\Theta}}\ \underset{n=a}{\overset{b}{\Delta}}\ \tau_{ij}=f\left((c)_a\div(c+1)_a\div\ldots\div(d-1)_a\div(d)_a\right)\uparrow f\left((c)_{(a+1)}\div(c+1)_{(a+1)}\div\ldots\div(d-1)_{(a+1)}\div(d)_{(a+1)}\right)$$
$$\uparrow\ldots\uparrow f\left((c)_{(b-1)}\div(c+1)_{(b-1)}\div\ldots\div(d-1)_{(b-1)}\div(d)_{(b-1)}\right)\uparrow f\left((c)_{(b)}\div(c+1)_{(b)}\div\ldots\div(d-1)_{(b)}\div(d)_{(b)}\right)$$

2.29- Exponentory of exponentory:

$$\underset{m=c}{\overset{d}{\Theta}}\ \underset{n=a}{\overset{b}{\Theta}}\ \tau_{ij}=f\left((c)_a\uparrow(c+1)_a\uparrow\ldots\uparrow(d-1)_a\uparrow(d)_a\right)\uparrow f\left((c)_{(a+1)}\uparrow(c+1)_{(a+1)}\uparrow\ldots\uparrow(d-1)_{(a+1)}\uparrow(d)_{(a+1)}\right)$$
$$\uparrow\ldots\uparrow f\left((c)_{(b-1)}\uparrow(c+1)_{(b-1)}\uparrow\ldots\uparrow(d-1)_{(b-1)}\uparrow(d)_{(b-1)}\right)\uparrow f\left((c)_{(b)}\uparrow(c+1)_{(b)}\uparrow\ldots\uparrow(d-1)_{(b)}\uparrow(d)_{(b)}\right)$$

2.30- Exponentory of rootory:

$$\underset{m=c}{\overset{d}{\Theta}}\ \underset{n=a}{\overset{b}{Z}}\ \tau_{ij}=f\left(\sqrt[d_a]{ANS}\sqrt[d-1_a]{ANS}\because\sqrt{ANS}\sqrt[c+1_a]{c_a}\right)\uparrow f\left(\sqrt[d_{(a+1)}]{ANS}\sqrt[d-1_{(a+1)}]{ANS}\because\sqrt{ANS}\sqrt[c+1_{(a+1)}]{c_{(a+1)}}\right)$$
$$\uparrow\ldots\uparrow f\left(\sqrt[d_{(b-1)}]{ANS}\sqrt[d-1_{(b-1)}]{ANS}\because\sqrt{ANS}\sqrt[c+1_{(b-1)}]{c_{(b-1)}}\right)\uparrow f\left(\sqrt[d_b]{ANS}\sqrt[d-1_b]{ANS}\because\sqrt{ANS}\sqrt[c+1_b]{c_b}\right)$$

2.31- Rootory of summation:

$$\underset{m=c}{\overset{d}{Z}}\ \underset{n=a}{\overset{b}{\Sigma}}\ \tau_{ij}=f((c)_b+(c+1)_b+\ldots+(d-1)_b+(d)_b)\downarrow f((c)_{(b-1)}+(c+1)_{(b-1)}+\ldots+(d-1)_{(b-1)}+(d)_{(b-1)})$$
$$\downarrow\ldots\downarrow f((c)_{(a+1)}+(c+1)_{(a+1)}+\ldots+(d-1)_{(a+1)}+(d)_{(a+1)})\downarrow f((c)_{(a)}+(c+1)_{(a)}+\ldots+(d-1)_{(a)}+(d)_{(a)})$$

2.32- Rootory of subtractory:

$$\underset{m=c}{\overset{d}{Z}}\ \underset{n=a}{\overset{b}{P}}\ \tau_{ij}=f(-(c)_b-(c+1)_b-\ldots-(d-1)_b-(d)_b)\downarrow f(-(c)_{(b-1)}-(c+1)_{(b-1)}-\ldots-(d-1)_{(b-1)}-(d)_{(b-1)})$$
$$\downarrow\ldots\downarrow f(-(c)_{(a+1)}-(c+1)_{(a+1)}-\ldots-(d-1)_{(a+1)}-(d)_{(a+1)})\downarrow f(-(c)_{(a)}-(c+1)_{(a)}-\ldots-(d-1)_{(a)}-(d)_{(a)})$$

2.33- Rootory of productory:

$$\underset{m=c}{\overset{d}{Z}}\ \underset{n=a}{\overset{b}{\Pi}}\ \tau_{ij}=f((c)_b\cdot(c+1)_b\cdot\ldots\cdot(d-1)_b\cdot(d)_b)\downarrow f((c)_{(b-1)}\cdot(c+1)_{(b-1)}\cdot\ldots\cdot(d-1)_{(b-1)}\cdot(d)_{(b-1)})$$
$$\downarrow\ldots\downarrow f((c)_{(a+1)}\cdot(c+1)_{(a+1)}\cdot\ldots\cdot(d-1)_{(a+1)}\cdot(d)_{(a+1)})\downarrow f((c)_{(a)}\cdot(c+1)_{(a)}\cdot\ldots\cdot(d-1)_{(a)}\cdot(d)_{(a)})$$

2.34- Rootory of divisory:

$$\underset{m=c}{\overset{d}{Z}}\ \underset{n=a}{\overset{b}{\Delta}}\ \tau_{ij}=f((c)_b\div(c+1)_b\div\ldots\div(d-1)_b\div(d)_b)\downarrow f((c)_{(b-1)}\div(c+1)_{(b-1)}\div\ldots\div(d-1)_{(b-1)}\div(d)_{(b-1)})$$
$$\downarrow\ldots\downarrow f((c)_{(a+1)}\div(c+1)_{(a+1)}\div\ldots\div(d-1)_{(a+1)}\div(d)_{(a+1)})\downarrow f((c)_{(a)}\div(c+1)_{(a)}\div\ldots\div(d-1)_{(a)}\div(d)_{(a)})$$

2.35- Rootory of exponentory:

$$\underset{m=c}{\overset{d}{Z}}\ \underset{n=a}{\overset{b}{\Theta}}\ \tau_{ij}=f((c)_b\uparrow(c+1)_b\uparrow\ldots\uparrow(d-1)_b\uparrow(d)_b)\downarrow f((c)_{(b-1)}\uparrow(c+1)_{(b-1)}\uparrow\ldots\uparrow(d-1)_{(b-1)}\uparrow(d)_{(b-1)})$$
$$\downarrow\ldots\downarrow f((c)_{(a+1)}\uparrow(c+1)_{(a+1)}\uparrow\ldots\uparrow(d-1)_{(a+1)}\uparrow(d)_{(a+1)})\downarrow f((c)_{(a)}\uparrow(c+1)_{(a)}\uparrow\ldots\uparrow(d-1)_{(a)}\uparrow(d)_{(a)})$$

2.36- Rootory of rootory:

$$\underset{m=c}{\overset{d}{Z}}\ \underset{n=a}{\overset{b}{Z}}\ \tau_{ij}=f((c)_b\downarrow(c+1)_b\downarrow\ldots\downarrow(d-1)_b\downarrow(d)_b)\downarrow f((c)_{(b-1)}\downarrow(c+1)_{(b-1)}\downarrow\ldots\downarrow(d-1)_{(b-1)}\downarrow(d)_{(b-1)})$$
$$\downarrow\ldots\downarrow f((c)_{(a+1)}\downarrow(c+1)_{(a+1)}\downarrow\ldots\downarrow(d-1)_{(a+1)}\downarrow(d)_{(a+1)})\downarrow f((c)_{(a)}\downarrow(c+1)_{(a)}\downarrow\ldots\downarrow(d-1)_{(a)}\downarrow(d)_{(a)})$$

References:

[1] Millas Vera, Juan Elías - Four-Operator Algebra. (2021)

[2] Hooshmand, M. H. - ULTRA POWERS OF HIGHER ORDERS AND ULTRA EXPONENTIAL FUNCTIONAL SEQUENCES

[3] https://encyclopediaofmath.org/wiki/Kronecker_symbol

[4] Beucher, Wolfgang – Analyse for Waveshaping mittels Tetration. (2017)