Modified Cattaneo-Vernotte Equation for Heat Transfer in Solids

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Abstract

We propose a modified Cattaneo-Vernotte relation between heat flux and temperature gradient, which leads to a second-order equation describing the evolution of temperature in solids with finite rate of propagation. A comparison of the temperature field spreading in the framework of Fourier, Cattaneo-Vernotte (CV) and modified Cattaneo-Vernotte (MCV) equations is discussed. The comparative analysis of MCV and Fourier solutions is carried out on the example of simple problem of plate cooling.

1. Introduction

In classical consideration the process of heat transfer in solid is described by a phenomenological equation based on two assumptions [1]. The first is the continuity of heat propagation

$$c\rho \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q} = 0 , \tag{1}$$

where c is the specific heat capacity, ρ is the mass density, θ is the temperature, $\mathbf{q}$ is the vector of heat flux. The second assumption is Fourier's law, which establishes the relationship between heat flux and gradient of temperature

$$\mathbf{q} = -\kappa \nabla \theta , \tag{2}$$

where κ is the thermal conductivity. Substitution (2) into equation (1) gives the classical equation for the temperature

$$\frac{\partial \theta}{\partial t} - \beta_q \Delta \theta = 0 , \tag{3}$$

where $\beta_q = \kappa / c\rho$ is the thermal diffusivity, Δ is the Laplace operator.

The disadvantage of relation (3) is that it leads us to the equations of parabolic type (3), which describes the instantaneous propagation of heat [2,3]. However, this contradicts the physical nature of the heat transfer process.

To overcome the drawback in heat conduction, a modified Fourier's law was proposed, taking into account "inertia" of the heat transfer process [4-7]

$$\tau_q \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\kappa \nabla \theta , \tag{4}$$

where τ_q is a relaxation time depending on material properties. When $\tau_q = 0$ the expression (4) is transformed to the Fourier's law (2). The relation (4) in combination with continuity condition (1) leads us to the wave equation of hyperbolic type

$$\tau_q \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \beta_q \Delta \theta = 0 , \tag{5}$$

which is widely discussed as Cattaneo-Vernotte (CV) equation [8-19].

The parabolic Fourier equation (3) and hyperbolic CV equation (5) describe the same stationary states, which are determined by Laplace operator, but the dynamics of relaxation to these stationary states is different. However, eliminating the paradox of instantaneous heat propagation [2,8,9], the CV heat equation leads to other paradoxical results associated with interference of temperature waves, their reflection from the boundaries of the body and the formation of shock heat waves [10-19]. Therefore, discussions about the applicability of the Fourier and CV equations continue [20,21]. We also note that despite the fact that the phenomenological equations of diffusion and heat transfer are the same [22], the hyperbolic diffusion equation and diffusion waves are not discussed in a literature.

In this paper, we propose a modification of CV approach to the description of heat transfer, which leads to the alternative equation and describes a different dynamics of heat propagation.

2. Comparison of Fourier equation and Cattaneo-Vernotte equation

Let us compare Fourier and CV equations in detail. The equation (5) introduces a very important parameter τ_q that describes the time scale of heat relaxation and allows one to determine the rate of heat propagation as

$$s_q^2 = \frac{\beta_q}{\tau_q} . \tag{6}$$

Besides the spatial scale of heat diffusion is defined as

$$l_q = \sqrt{\beta_q \tau_q} = s_q \tau_q , \tag{7}$$

This allows one to rewrite Fourier and CV equations in the similar form

$$\frac{1}{\tau_q}\frac{\partial \theta}{\partial t} - s_q^2 \Delta\theta = 0 , \tag{8}$$

$$\frac{\partial^2 \theta}{\partial t^2} + \frac{1}{\tau_q}\frac{\partial \theta}{\partial t} - s_q^2 \Delta\theta = 0 . \tag{9}$$

The equations (8) and (9) admit the solutions in the form of plane waves

$$\theta = A\exp\left(i\omega t + i\left(\boldsymbol{k}\cdot\boldsymbol{r}\right)\right), \tag{10}$$

where ω is the frequency, $\boldsymbol{k}$ is the wave vector. The dispersion relation for parabolic Fourier equation (8) is

$$\omega = i\tau_q s_q^2 k^2 , \tag{11}$$

where k is the wave number ($k = |\boldsymbol{k}|$). In this relation, the frequency is an imaginary quantity. Thus, the solutions of the Fourier equation are spatial harmonics decaying with time. The damping factor is

$$i\omega = -\tau_q s_q^2 k^2 . \tag{12}$$

The dependence of the decrement (12) on the wave number is shown in fig. 1.

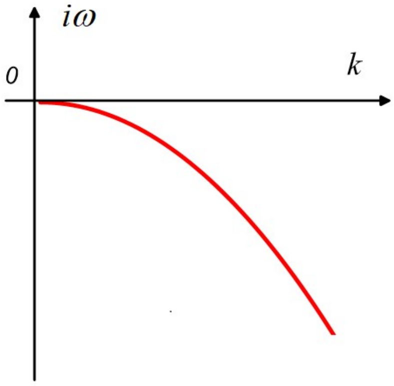

Fig. 1. The schematic plot of dispersion dependence for parabolic Fourier equation.

Also we can introduce the analog of group speed, which is the imaginary value

$$i v_F = i\frac{d\omega}{dk} = -2\tau_q s_q^2 k. \tag{13}$$

This value tends to infinity ($iv_F \to -\infty$) when $k \to \infty$, that is the reason of the infinitely fast damping of shortwave harmonics.

On the other hand, the dispersion relation for hyperbolic CV equation (9) is

$$\omega^2 + \frac{i\omega}{\tau_q} - s_q^2 k^2 = 0 . \tag{14}$$

From (14) we obtain

$$\omega = i\frac{1 \pm \sqrt{1 - 4l_q^2 k^2}}{2\tau_q}. \tag{15}$$

The behavior of spatial harmonics essentially depends on their wave number. In the region of wave numbers $k < k^*$ (where $k^* = \frac{1}{2l_q}$), solutions of CV equation also represent damped spatial harmonics. The damping factor is

$$i\omega = -\frac{1 \pm \sqrt{1 - 4l_q^2 k^2}}{2\tau_q} \tag{16}$$

Dispersion dependence (16) is shown in fig. 2.

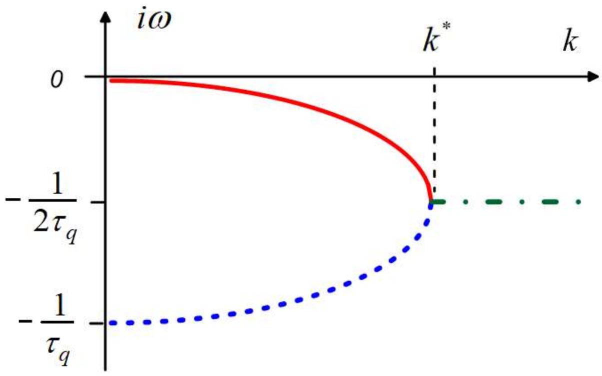

Fig. 2. The schematic plot of decrement for hyperbolic CV equation. The solid curve corresponds to the "-" sign in expression (16), the dotted curve corresponds to the "+" sign. The dash-dotted line in the region $k > k^*$ represents the decrement in expression (17).

In the region $k < k^*$, the decrement has two branches (shown by solid and dotted lines in Fig. 2) in accordance with the signs in expression (16). At small k on the upper branch of the dispersion curve the decrement is $i\omega \approx -\tau_q s_q^2 k^2$, that coincides with the decrement of the Fourier equation (12).

In the region $k > k^*$, the dispersion dependence (16) has both imaginary and real parts

$$\omega = i\frac{1}{2\tau_q} \pm \frac{\sqrt{4l_q^2 k^2 - 1}}{2\tau_q}. \tag{17}$$

The damping factor in this region of wave numbers is equal to

$$i\omega = -\frac{1}{2\tau_q} \tag{18}$$

It is shown by the dot-dashed line in Fig. 2. The real part of the dispersion relation (17) is shown in Fig. 3.

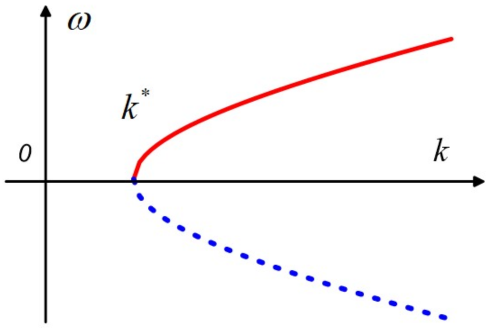

Fig. 3. The schematic plot of dispersion curves for Cattaneo-Vernotte equation.

The region of wave numbers $k > k^*$ corresponds to the spatial harmonics propagating in the form of traveling waves. In this region CV equation has the real group velocity

$$v_{CV} = \frac{d\omega}{dk} = \frac{2l_q}{\tau_q}\frac{k}{\sqrt{4l_q^2 k^2 - 1}}. \tag{19}$$

This value tends to be constant s_q ($v_{CV} \to s_q$) when $k \to \infty$.

3. Modified Cattaneo-Vernotte equation of heat transfer

Evidently, that the hyperbolic heat equation is a consequence of the concept of "inertia" for heat flow. However this concept raises doubts, since the macroscopic transfer of heat is associated not with their directed motion, but with chaotic vibrations of atoms in crystal lattice. Here we try to modify the Cattaneo-Vernotte condition and obtain alternative equation describing different dynamics of heat propagation.

We propose the modified Cattaneo-Vernotte (MCV) condition

$$-\tau_q \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\kappa \nabla \theta, \tag{20}$$

which differs from condition (4) by the sign in front of the time derivative. In this case, the term with the time derivative describes diffusive relaxation, and not the inertia of heat propagation.

Let us analyze the consequences of such modification. The modified system of equations describing the heat transfer is written as:

$$c\rho \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q} = 0, \tag{21}$$

$$-\tau_q \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} + \kappa \nabla \theta = 0 \ . \tag{22}$$

The equation (21) of this system is, as before, the continuity condition. Equation (22) describes the process of the heat flux relaxation. The system (21) - (22) is equivalent to the following MCV equation for the temperature field

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{1}{\tau_q}\frac{\partial \theta}{\partial t} + s_q^2 \Delta \theta = 0, \tag{23}$$

Note, that the stationary state of MCV equation (23) is the same as for Fourier and CV equations but the time evolution of temperature is different. Assuming harmonic solutions (10) we have the following dispersion relation for MCV equation:

$$\omega^2 + i\frac{1}{\tau_q}\omega + s_q^2 k^2 = 0 \ . \tag{24}$$

From (24) we have two roots

$$i\omega = \frac{1 \pm \sqrt{1 + 4\tau_q^2 s_q^2 k^2}}{2\tau_q}. \tag{25}$$

The schematic plots of dispersion curves (25) are represented in Fig. 4.

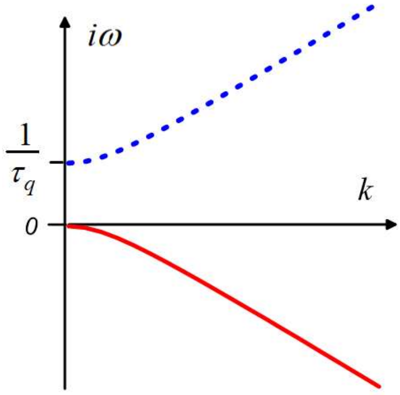

Fig. 4. The schematic plots of dispersion curves for MCV equation.

The upper branch of dispersion curve corresponds to

$$i\omega = \frac{1 + \sqrt{1 + 4\tau_q^2 s_q^2 k^2}}{2\tau_q} \tag{26}$$

and describes the solutions growing in time, that contradict the physical picture of the heat propagation process and are a consequence of the violation of the causality principle [23]. However, on the other hand, the solutions corresponding to the lower branch of the dispersion characteristic with

$$i\omega = \frac{1 - \sqrt{1 + 4\tau_q^2 s_q^2 k^2}}{2\tau_q} \tag{27}$$

describe damped in time spatial harmonics and can be used to describe the process of heat propagation.

4. Comparison of Fourier equation and modified Cattaneo-Vernotte equation

Let us compare Fourier and MCV equations. We write these equations in the similar form

$$\frac{1}{\tau_q}\frac{\partial\theta}{\partial t}-s_q^2\Delta\theta=0\,, \tag{28}$$

$$-\frac{\partial^2\theta}{\partial t^2}+\frac{1}{\tau_q}\frac{\partial\theta}{\partial t}-s_q^2\Delta\theta=0\,. \tag{29}$$

The dispersion relation for Fourier equation (28) is

$$i\omega=-\tau_q s_q^2 k^2\,. \tag{30}$$

The dispersion relation for MCV equation (29) is

$$i\omega=\frac{1-\sqrt{1+4\tau_q^2 s_q^2 k^2}}{2\tau_q}. \tag{31}$$

The schematic plots of (30) and (31) are represented in Fig. 1.

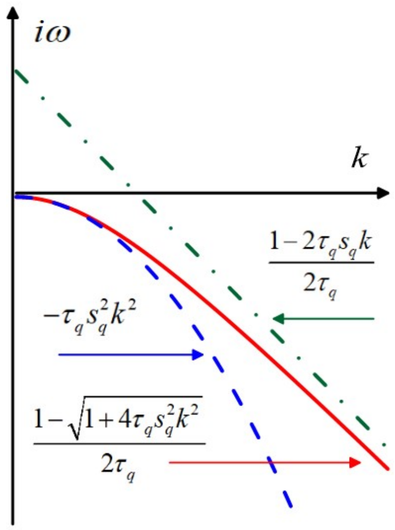

Fig. 5. The schematic plot of dispersion curves for Fourier (dashed blue line) and MCV (solid red line) equations. The asymptote (26) is shown by dot-dashed line.

In the region of small *k* the dependence (31) coincides with dependence (30), while at $k\to\infty$ it tends to the asymptote

$$i\omega=\frac{1-2\tau_q s_q k}{2\tau_q}. \tag{32}$$

This value tends to infinity when $k\to\infty$. The analog of group speed for MCV equation is

$$i\,v_{MCV}=i\frac{d\omega}{dk}=-\frac{2\tau_q s_q^2 k}{\sqrt{1+4\tau_q^2 s_q^2 k^2}}. \tag{33}$$

This quantity tends to be constant $-s_q$ at $k\to\infty$.

The plate cooling

As an example, let us consider one-dimensional problem of cooling a plate with thickness $2l$ uniformly heated to a temperature θ_0 and with zero temperature at the boundaries $x = \pm l$. In this case we have natural spatial scale l and we introduce new dimensionless variables $\tilde{t} = t/\tau_q$ and $\tilde{x} = x/l$. Then the Fourier equation is represented as

$$\frac{\partial\theta}{\partial\tilde{t}} - \lambda^2 \frac{\partial^2\theta}{\partial\tilde{x}^2} = 0\,, \tag{34}$$

while MCV equation is

$$\frac{\partial^2\theta}{\partial\tilde{t}^2} - \frac{\partial\theta}{\partial\tilde{t}} + \lambda^2 \frac{\partial^2\theta}{\partial\tilde{x}^2} = 0\,, \tag{35}$$

where $\lambda = l_q / l$ is the ratio of the diffusion length to half of the plate thickness. Corresponding dispersion relations are

$$i\omega = -\lambda^2 k^2\,. \tag{36}$$

and

$$i\omega = \frac{1 - \sqrt{1 + 4\lambda^2 k^2}}{2}. \tag{37}$$

The solution to this problem in the frame of Fourier equation (34) is expressed by the following Fourier series [1]:

$$\theta_F = \frac{4\theta_0}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \cos\left[\frac{(2m+1)\pi}{2}\tilde{x}\right] \exp\left[-\frac{\lambda^2 (2m+1)^2 \pi^2}{4}\tilde{t}\right], \tag{38}$$

with decrement of temperature damping

$$d_{F\,m} = \frac{\lambda^2 (2m+1)^2 \pi^2}{4}. \tag{39}$$

On the other hand, the solution to this problem in the case of MCV equation (35) is expressed by the following series:

$$\theta_M = \frac{4\theta_0}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} \cos\left[\frac{(2m+1)\pi}{2}\tilde{x}\right] \exp\left[\frac{1 - \sqrt{1 + \lambda^2 (2m+1)^2 \pi^2}}{2}\tilde{t}\right], \tag{40}$$

with damping parameter

$$d_{M\,m} = \frac{1 - \sqrt{1 + \lambda^2 (2m+1)^2 \pi^2}}{2}. \tag{41}$$

Thus, comparing damping parameters in (39) and (41) one can see that in case of MCV equation the higher harmonics decay more slowly than in case of Fourier equation in accordance with dispersion dependences (36) and (37).

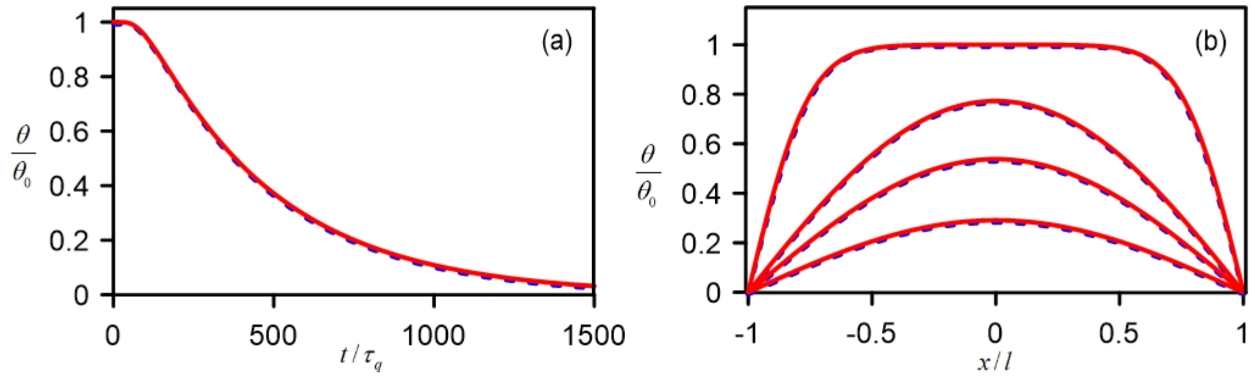

Fig. 6. The process of cooling the thick plate with $l > l_q$ ($\lambda^2 = 0.01$). (a) Time dependence of temperature at the point $\tilde{x} = 0$. (b) Temperature profiles at different time ($\tilde{t} = 20, 200, 350, 600$). The solutions of Fourier equation are indicated by dashed blue lines. Solutions of MCV equation are shown by solid red lines.

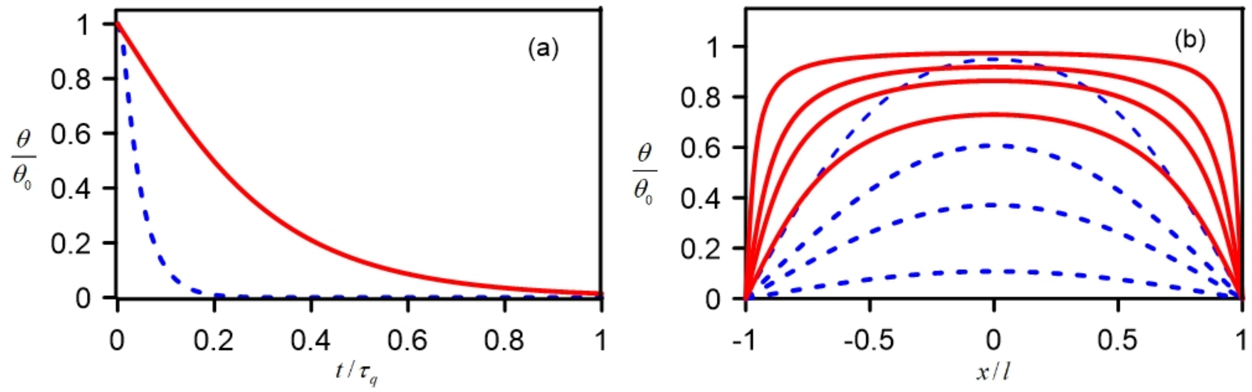

Fig. 7. The process of cooling the thin plate with $l < l_q$ ($\lambda^2 = 10$). (a) Time dependence of temperature at the point $\tilde{x} = 0$. (b) Temperature distributions at different time ($\tilde{t} = 0.01, 0.03, 0.05, 0.1$). The solutions of Fourier equation are indicated by dashed blue lines. Solutions of MCV equation are shown by solid red lines.

The results of numerical calculations for the plates with different thicknesses are represented in Fig. 6 and Fig. 7. It is seen that in the case of thick plates ($l > l_q$) the solution of MCV equation (red solid curves in Fig. 6a,b) coincides with the solution of Fourier equation (blue dashed curves in Fig. 6a,b). However, for thin plates ($l < l_q$) the solution to Fourier equation demonstrates a rapid decrease in temperature gradients and faster cooling of the plate (blue dashed curves in Fig. 7a,b) than in the case of the solution described by MCV equation (red solid curves in Fig. 7a,b).

To clarify the time evolution of Fourier and MCV solutions, we analyze the behavior of zero harmonics. Let us consider the cooling a plate (thickness $2l$) with $\theta_0 \cos(\pi x / 2l)$ initial temperature and with zero temperature at the boundaries $x = \pm l$. In this case

$$\theta_F = \theta_0 \cos\left(\frac{\pi}{2}\tilde{x}\right) \exp\left(-\frac{\lambda^2 \pi^2}{4}\tilde{t}\right), \tag{42}$$

and

$$\theta_M = \theta_0 \cos\left(\frac{\pi}{2}\tilde{x}\right) \exp\left(\frac{1-\sqrt{1+\lambda^2\pi^2}}{2}\tilde{t}\right). \tag{43}$$

The dependence of the ratio of damping parameters d_M / d_F as the function of λ is represented in Fig. 8.

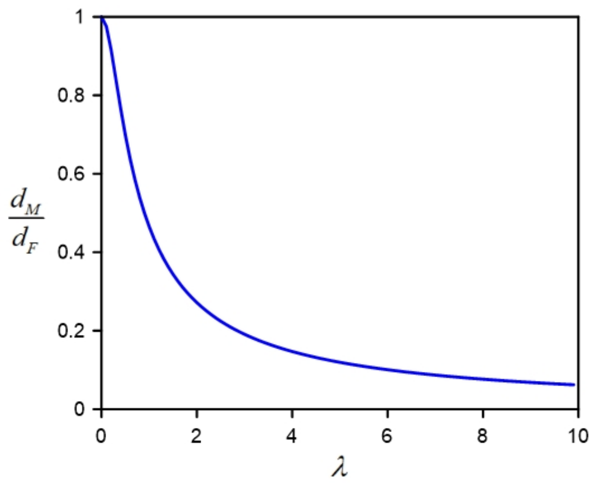

Fig. 8. The dependence of damping parameters ratio d_M / d_F on the parameter λ .

For thick plates when $\lambda^2\pi^2 \ll 1$ we have

$$d_M \approx -\frac{\lambda^2\pi^2}{4} = d_F \,, \tag{44}$$

and time behavior of Fourier and MCV solutions is practically the same. The temperature profiles at different time and the dependence of temperature at the central point of plate on time are shown in Fig. 9.

In opposite case of thin plate when $\lambda^2\pi^2 \gg 1$ we have

$$d_M \approx -\frac{\lambda\pi}{2} < d_F \,, \tag{45}$$

and MCV equation predicts slower cooling than Fourier equation. The corresponding profiles and time dependences are shown in Fig. 10.

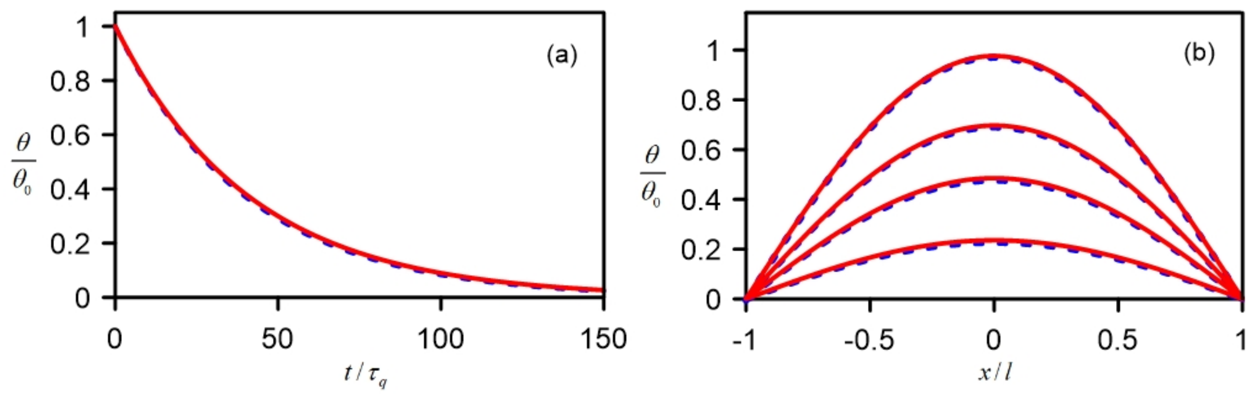

Fig. 9. The process of cooling the thick plate with $l > l_q$ ($\lambda^2 = 0.01$). (a) Time dependence of temperature at the point $\tilde{x} = 0$. (b) Temperature distributions at different time ($\tilde{t} = 1, 15, 30, 60$). The solutions of Fourier equation are indicated by dashed blue lines. Solutions of MCV equation are shown by solid red lines.

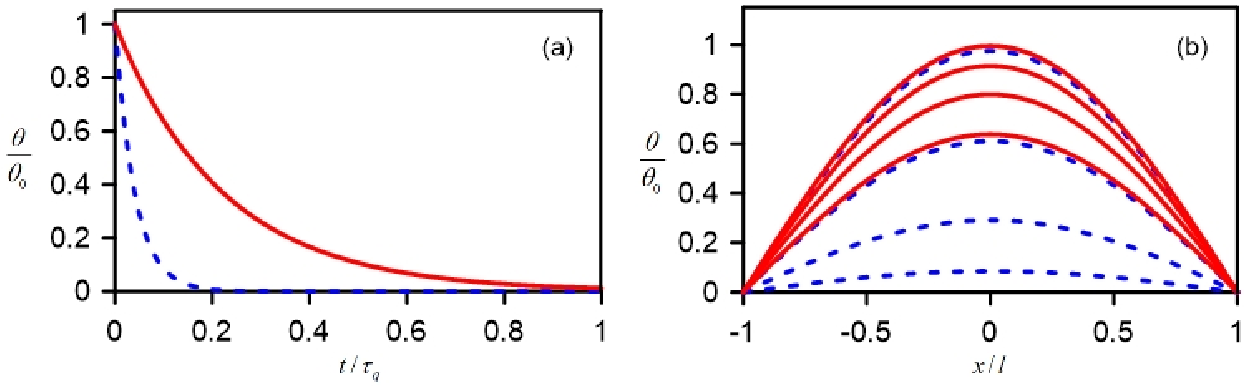

Fig. 10. The process of cooling the thick plate with $l > l_q$ ($\lambda^2 = 0.01$). (a) Time dependence of temperature at the point $\tilde{x} = 0$. (b) Temperature distributions at different time ($\tilde{t} = 0.001, 0.02, 0.05, 0.1$). The solutions of Fourier equation are indicated by dashed blue lines. Solutions of MCV equation are shown by solid red lines.

Thus it is seen that the differences between solutions of Fourier and MCV equations are noticeable only at small spatial scales, when the plate thickness is less than the diffusion length.

5. Conclusion

Thus we propose alternative relationship between heat flux and temperature gradient, which leads us to the MCV equation describing the evolution of temperature with finite rate. Solutions of MCV equation have the same spatial temperature distributions as in the case of Fourier and CV equations, but describe a different dynamics of heat transfer process. The peculiarities of MCV solutions and their comparison with Fourier solutions have been analyzed on the example of simple problem of plate cooling. It was shown that on large spatial scales, when the plate thickness is greater than the thermal diffusion length, the differences between the solutions of MCV and Fourier equations are insignificant. However, in the case when the plate thickness is less than the diffusion length, the MCV equation predicts a slower cooling in accordance with finite heat transfer rate.

Thus, it has been shown that MCV equation provides the finite rate of transfer processes, but it does not have the disadvantages of a CV equation, which predicts many paradoxical results associated with the possible propagation of heat in the form of real harmonic waves. The same approach can be applied to describe diffusion processes in solids.

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