REDUCTION FORMULAS OF THE COSINE OF INTEGER FRACTIONS OF π

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ABSTRACT. The power of some cosines of integer fractions π/n of the half circle allow a reduction to lower powers of the same angle. These are tabulated in the format

$$\sum_{i=0}^{\lfloor n/2 \rfloor} a_i^{(n)} \cos^i \frac{\pi}{n} = 0; \quad (n = 2, 3, 4, \ldots).$$

Related expansions of Chebyshev Polynomials $T_n(x)$ and factorizations of $T_n(x) + 1$ are also given.

1. Description

1.1. Chebyshev Coefficients. Each first formula in the list given in Section 2 is an expansion [7, 1.331.3] of

(1)
$$\cos(n\varphi) = 2^{n-1}\cos^n\varphi + n\sum_{k=1}^{\lfloor n/2 \rfloor}(-1)^k\frac{1}{k}\binom{n-k-1}{k-1}2^{n-2k-1}\cos^{n-2k}\varphi; \quad (n > 0),$$

providing an extension of the Tables [7, 1.335] which cover the cases up to $n = 7$. With

$$\Re e^{in\varphi} = \cos(n\varphi) = T_n(x), \tag{2}$$

this is also an extension of the Table [1, 22.3] of the coefficients of the Chebyshev Polynomials [1, 22.3.6][2, (7)], and of sequence A053120 of the OEIS [10]. The shortcut $x \equiv \cos\varphi$ is used throughout.

1.2. Cosine Reductions. Each second formula for a n is a Gröbner-base reduction [8] of the equation $\cos(n\varphi) = -1$ for $\varphi = \pi/n$, i.e., obtained through factorization of the polynomial $T_n(x)+1$ trivially derived from each first formula. Example: For $n = 9$, Eq. (11) the factorization is $T_9(x) + 1 = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x + 1 = (x + 1)(2x - 1)^2(8x^3 - 6x - 1)^2$.

The existence of a non-trivial factorization is demonstrated via [1, 22.7.24] and [1, 22.12.5]:

(3)
$$T_n(x)+1 = \begin{cases} T_{2m}(x) + 1 = 2T_m^2(x) & ; \quad (n \equiv 2m, \text{ even}); \\ T_{2m+1}(x) + 1 & \\ = (x + 1)[\sum_{i=0}^{2m-1} 2(-1)^i(i + 1)T_{2m-i}(x) + (2m + 1)] & \\ = (x + 1)[1 - 2(1 - x)\sum_{k=0}^{m-1} U_{2k+1}(x)] & , \quad (n \equiv 2m + 1, \text{ odd}). \end{cases}$$

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If n is not a prime number, the second line tabulated comprises the formula which appears for a multiple earlier in the table implicitly. Example: Eq. (14) for $n = 14$ is an implicit equation for $\cos(\pi/14)$ which is reduced to Eq. (10) by use of the duplication formula [7, 1.323] $\cos^2\frac{\pi}{14} = \frac{1}{2}[1+\cos\frac{\pi}{7}]$ and its powers $\cos^4\frac{\pi}{14} = \frac{1}{4}[1+\cos\frac{\pi}{7}]^2$ and $\cos^6\frac{\pi}{14} = \frac{1}{8}[1+\cos\frac{\pi}{7}]^3$. Construction of the results would also work for composite n in the other direction.

Higher cosine powers than the leading power of the second equations follow by taking the remainder of a polynomial division modulo the one actually shown. Example: Eq. (13) has a leading power 4 in $\cos\frac{\pi}{12}$, so polynomials of degree 4 and higher can be reduced to polynomials of degree 3 or less:

(4)
$$\cos^5\frac{\pi}{12} = \frac{1}{16}\cos\frac{\pi}{12}\left[16\cos^4\frac{\pi}{12} - 16\cos^2\frac{\pi}{12} + 1\right] + \cos^3\frac{\pi}{12} - \frac{1}{16}\cos\frac{\pi}{12} = \cos^3\frac{\pi}{12} - \frac{1}{16}\cos\frac{\pi}{12};$$

(5)
$$\cos^6\frac{\pi}{12} = \left[\frac{1}{16} + \frac{1}{16}\cos^2\frac{\pi}{12}\right]\left[16\cos^4\frac{\pi}{12} - 16\cos^2\frac{\pi}{12} + 1\right] + \frac{15}{16}\cos^2\frac{\pi}{12} - \frac{1}{16} = \frac{15}{16}\cos^2\frac{\pi}{12} - \frac{1}{16}.$$

1.3. **Representations by Irrational Numbers.** The table is also a quick guide to construct cosines in terms of square and cubic roots of rational numbers: if the second equation is quadratic, bi-quadratic etc. in $\cos(\pi/n)$, it can be solved for $\cos(\pi/n)$ with the standard formulas for this type of polynomial roots [9]. Example: Eq. (12) is bi-quadratic in $\cos(\pi/10)$, with a first root

$$\cos^4\frac{\pi}{10} - \frac{5}{4}\cos^2\frac{\pi}{10} + \frac{5}{16} = 0 \Rightarrow \cos^2\frac{\pi}{10} = \frac{5}{8} + \sqrt{(\frac{5}{8})^2 - \frac{5}{16}} = \frac{5}{8} + \frac{\sqrt{5}}{8}, \quad (6)$$

and therefore a second root

$$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}. \quad (7)$$

Bootstrapping from the expressions for $\cos\frac{\pi}{2}$, $\cos\frac{\pi}{3}$, $\cos\frac{\pi}{5}$ and $\cos\frac{\pi}{15}$ given below, and with [1, 4.3.21]

$$\cos\frac{z}{2} = \pm\sqrt{\frac{1+\cos z}{2}}, \quad (8)$$

one finds algebraic representations for all cosines of the form $\cos\frac{\pi}{2^k}$, $\cos\frac{\pi}{3\times 2^k}$, $\cos\frac{\pi}{5\times 2^k}$, and $\cos\frac{\pi}{3\times5\times2^k}$, $k = 0, 1, 2, \ldots$ [3, 6]. Explicit values for $n = 5, 8, 12, 15\ldots$ have been given by Vidūnas [11, 12]. We adopt his notation:

$$\phi \equiv 1+\sqrt{5}; \quad \phi^* \equiv 1-\sqrt{5}; \quad \psi \equiv \sqrt{5+2\sqrt{5}}; \quad \psi^* \equiv \sqrt{5-2\sqrt{5}}. \quad (9)$$

The values of ψ and ψ^* are entries A019970 and A019934 of the OEIS [10]. All but the first of the decimal digits of ϕ and ϕ^* are those of entry A002163. Other ways to combine these square roots are known [4, 5].

2. Table

$n = 2$

$$\Re e^{2i\varphi} = 2x^2 - 1; \quad \cos\frac{\pi}{2} = 0.$$

$n=3$

$$\Re e^{3i\varphi} = 4x^3 - 3x;\quad 2\cos\frac{\pi}{3} - 1 = 0.$$

$$\cos\frac{\pi}{3} = \frac{1}{2}.$$

$n=4$

$$\Re e^{4i\varphi} = 8x^4 - 8x^2 + 1;\quad 2\cos^2\frac{\pi}{4} - 1 = 0.$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

is entry A010503 in [10].

$n=5$

$$\Re e^{5i\varphi} = 16x^5 - 20x^3 + 5x;\quad 4\cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5} - 1 = 0.$$

$$\cos\frac{\pi}{5} = \frac{\phi}{4}$$

is entry A019863 in [10].

$n=6$

$$\Re e^{6i\varphi} = 32x^6 - 48x^4 + 18x^2 - 1;\quad 4\cos^2\frac{\pi}{6} - 3 = 0.$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

is entry A010527 in [10].

$n=7$

$$\Re e^{7i\varphi} = 64x^7 - 112x^5 + 56x^3 - 7x;\quad 8\cos^3\frac{\pi}{7} - 4\cos^2\frac{\pi}{7} - 4\cos\frac{\pi}{7} + 1 = 0. \tag{10}$$

$\cos\frac{\pi}{7}$ is entry A073052 in [10].

$n=8$

$$\Re e^{8i\varphi} = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1;\quad 8\cos^4\frac{\pi}{8} - 8\cos^2\frac{\pi}{8} + 1 = 0.$$

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

is entry A144981 in [10].

$n=9$

$$\Re e^{9i\varphi} = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x;\quad 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1 = 0. \tag{11}$$

$\cos\frac{\pi}{9}$ is entry A019879 in [10].

$n=10$

$$\begin{aligned} \Re e^{10i\varphi} &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1; \\ & 16\cos^4\frac{\pi}{10} - 20\cos^2\frac{\pi}{10} + 5 = 0. \end{aligned} \tag{12}$$

$$\cos\frac{\pi}{10} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$$

is entry A019881 in [10].

$n=11$

$$\begin{aligned} \Re e^{11i\varphi} &= 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x; \\ & 32\cos^5\frac{\pi}{11} - 16\cos^4\frac{\pi}{11} - 32\cos^3\frac{\pi}{11} + 12\cos^2\frac{\pi}{11} + 6\cos\frac{\pi}{11} - 1 = 0. \end{aligned}$$

$n=12$

$$\Re e^{12i\varphi} = 2048x^{12}-6144x^{10}+6912x^8-3584x^6+840x^4-72x^2+1;$$
$$16\cos^4\frac{\pi}{12}-16\cos^2\frac{\pi}{12}+1=0. \quad (13)$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

is entry A019884 in [10].

$n=13$

$$\Re e^{13i\varphi} = 4096x^{13}-13312x^{11}+16640x^9-9984x^7+2912x^5-364x^3+13x;$$
$$64\cos^6\frac{\pi}{13}-32\cos^5\frac{\pi}{13}-80\cos^4\frac{\pi}{13}+32\cos^3\frac{\pi}{13}+24\cos^2\frac{\pi}{13}-6\cos\frac{\pi}{13}-1=0.$$

$n=14$

$$\Re e^{14i\varphi} = 8192x^{14}-28672x^{12}+39424x^{10}-26880x^8+9408x^6-1568x^4+98x^2-1;$$
$$64\cos^6\frac{\pi}{14}-112\cos^4\frac{\pi}{14}+56\cos^2\frac{\pi}{14}-7=0. \quad (14)$$

$n=15$

$$\Re e^{15i\varphi} = 16384x^{15}-61440x^{13}+92160x^{11}-70400x^9+28800x^7-6048x^5+560x^3-15x;$$
$$16\cos^4\frac{\pi}{15}+8\cos^3\frac{\pi}{15}-16\cos^2\frac{\pi}{15}-8\cos\frac{\pi}{15}+1=0.$$

$$\cos\frac{\pi}{15}=\phi^*\frac{\sqrt{3}\psi+1}{8\sqrt{5}}$$

is entry A019887 in [10].

$n=16$

$$\Re e^{16i\varphi} = 32768x^{16}-131072x^{14}+212992x^{12}-180224x^{10}+84480x^8-21504x^6+2688x^4-128x^2+1;$$
$$128\cos^8\frac{\pi}{16}-256\cos^6\frac{\pi}{16}+160\cos^4\frac{\pi}{16}-32\cos^2\frac{\pi}{16}+1=0.$$

$$\cos\frac{\pi}{16}=\sqrt{\frac{1}{2}+\frac{1}{4}\sqrt{2+\sqrt{2}}}.$$

$n=17$

$$\Re e^{17i\varphi} = 65536x^{17}-278528x^{15}+487424x^{13}-452608x^{11}+239360x^9-71808x^7+11424x^5 -816x^3+17x;$$
$$256\cos^8\frac{\pi}{17}-128\cos^7\frac{\pi}{17}-448\cos^6\frac{\pi}{17}+192\cos^5\frac{\pi}{17}+240\cos^4\frac{\pi}{17}-80\cos^3\frac{\pi}{17} -40\cos^2\frac{\pi}{17}+8\cos\frac{\pi}{17}+1=0.$$

$n=18$

$$\Re e^{18i\varphi} = 131072x^{18}-589824x^{16}+1105920x^{14}-1118208x^{12}+658944x^{10}-228096x^8 +44352x^6-4320x^4+162x^2-1;$$
$$64\cos^6\frac{\pi}{18}-96\cos^4\frac{\pi}{18}+36\cos^2\frac{\pi}{18}-3=0.$$

$\cos\frac{\pi}{18}$ is entry A019889 in [10].

$n = 19$

$$\Re e^{19i\varphi} = 262144x^{19} - 1245184x^{17} + 2490368x^{15} - 2723840x^{13} + 1770496x^{11} - 695552x^{9} + 160512x^{7} - 20064x^{5} + 1140x^{3} - 19x;$$

$$512\cos^9\frac{\pi}{19} - 256\cos^8\frac{\pi}{19} - 1024\cos^7\frac{\pi}{19} + 448\cos^6\frac{\pi}{19} + 672\cos^5\frac{\pi}{19} - 240\cos^4\frac{\pi}{19} - 160\cos^3\frac{\pi}{19} + 40\cos^2\frac{\pi}{19} + 10\cos\frac{\pi}{19} - 1 = 0.$$

$n = 20$

$$\Re e^{20i\varphi} = 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} - 2050048x^{10} + 549120x^{8} - 84480x^{6} + 6600x^{4} - 200x^{2} + 1;$$

$$256\cos^8\frac{\pi}{20} - 512\cos^6\frac{\pi}{20} + 304\cos^4\frac{\pi}{20} - 48\cos^2\frac{\pi}{20} + 1 = 0.$$

$$\cos\frac{\pi}{20} = \sqrt{\phi^*}\frac{\psi + \sqrt{5}}{4\sqrt{5}}$$

is entry A019890 in [10].

$n = 21$

$$\Re e^{21i\varphi} = 1048576x^{21} - 5505024x^{19} + 12386304x^{17} - 15597568x^{15} + 12042240x^{13} - 5870592x^{11} + 1793792x^{9} - 329472x^{7} + 33264x^{5} - 1540x^{3} + 21x;$$

$$64\cos^6\frac{\pi}{21} + 32\cos^5\frac{\pi}{21} - 96\cos^4\frac{\pi}{21} - 48\cos^3\frac{\pi}{21} + 32\cos^2\frac{\pi}{21} + 16\cos\frac{\pi}{21} + 1 = 0.$$

$n = 22$

$$\Re e^{22i\varphi} = 2097152x^{22} - 11534336x^{20} + 27394048x^{18} - 36765696x^{16} + 30638080x^{14} - 16400384x^{12} + 5637632x^{10} - 1208064x^{8} + 151008x^{6} - 9680x^{4} + 242x^{2} - 1;$$

$$1024\cos^{10}\frac{\pi}{22} - 2816\cos^8\frac{\pi}{22} + 2816\cos^6\frac{\pi}{22} - 1232\cos^4\frac{\pi}{22} + 220\cos^2\frac{\pi}{22} - 11 = 0.$$

$n = 23$

$$\Re e^{23i\varphi} = 4194304x^{23} - 24117248x^{21} + 60293120x^{19} - 85917696x^{17} + 76873728x^{15} - 44843008x^{13} + 17145856x^{11} - 4209920x^{9} + 631488x^{7} - 52624x^{5} + 2024x^{3} - 23x;$$

$$2048\cos^{11}\frac{\pi}{23} - 1024\cos^{10}\frac{\pi}{23} - 5120\cos^9\frac{\pi}{23} + 2304\cos^8\frac{\pi}{23} + 4608\cos^7\frac{\pi}{23} - 1792\cos^6\frac{\pi}{23} - 1792\cos^5\frac{\pi}{23} + 560\cos^4\frac{\pi}{23} + 280\cos^3\frac{\pi}{23} - 60\cos^2\frac{\pi}{23} - 12\cos\frac{\pi}{23} + 1 = 0.$$

$n = 24$

$$\Re e^{24i\varphi} = 8388608x^{24} - 50331648x^{22} + 132120576x^{20} - 199229440x^{18} + 190513152x^{16} - 120324096x^{14} + 50692096x^{12} - 14057472x^{10} + 2471040x^{8} - 256256x^{6} + 13728x^{4} - 288x^{2} + 1;$$

$$256\cos^8\frac{\pi}{24} - 512\cos^6\frac{\pi}{24} + 320\cos^4\frac{\pi}{24} - 64\cos^2\frac{\pi}{24} + 1 = 0.$$

$$\cos\frac{\pi}{24} = \frac{\sqrt{2\sqrt{2} + \sqrt{3} + 1}}{2^{5/4}}$$

is entry A144982 in [10].

$n = 25$

$$\Re e^{25i\varphi} = 16777216x^{25} - 104857600x^{23} + 288358400x^{21} - 458752000x^{19} + 466944000x^{17} - 317521920x^{15} + 146227200x^{13} - 45260800x^{11} + 9152000x^9 - 1144000x^7 + 80080x^5 - 2600x^3 + 25x;$$

$$1024\cos^{10}\frac{\pi}{25} - 2560\cos^8\frac{\pi}{25} + 2240\cos^6\frac{\pi}{25} - 32\cos^5\frac{\pi}{25} - 800\cos^4\frac{\pi}{25} + 40\cos^3\frac{\pi}{25} + 100\cos^2\frac{\pi}{25} - 10\cos\frac{\pi}{25} - 1 = 0.$$

$n = 26$

$$\Re e^{26i\varphi} = 33554432x^{26} - 218103808x^{24} + 627048448x^{22} - 1049624576x^{20} + 1133117440x^{18} - 825556992x^{16} + 412778496x^{14} - 141213696x^{12} + 32361472x^{10} - 4759040x^8 + 416416x^6 - 18928x^4 + 338x^2 - 1;$$

$$4096\cos^{12}\frac{\pi}{26} - 13312\cos^{10}\frac{\pi}{26} + 16640\cos^8\frac{\pi}{26} - 9984\cos^6\frac{\pi}{26} + 2912\cos^4\frac{\pi}{26} - 364\cos^2\frac{\pi}{26} + 13 = 0.$$

$n = 27$

$$\Re e^{27i\varphi} = 67108864x^{27} - 452984832x^{25} + 1358954496x^{23} - 2387607552x^{21} + 2724986880x^{19} - 2118057984x^{17} + 1143078912x^{15} - 428654592x^{13} + 109983744x^{11} - 18670080x^9 + 1976832x^7 - 117936x^5 + 3276x^3 - 27x;$$

$$512\cos^9\frac{\pi}{27} - 1152\cos^7\frac{\pi}{27} + 864\cos^5\frac{\pi}{27} - 240\cos^3\frac{\pi}{27} + 18\cos\frac{\pi}{27} - 1 = 0.$$

$n = 28$

$$\Re e^{28i\varphi} = 134217728x^{28} - 939524096x^{26} + 2936012800x^{24} - 5402263552x^{22} + 6499598336x^{20} - 5369233408x^{18} + 3111714816x^{16} - 1270087680x^{14} + 361181184x^{12} - 69701632x^{10} + 8712704x^8 - 652288x^6 + 25480x^4 - 392x^2 + 1;$$

$$4096\cos^{12}\frac{\pi}{28} - 12288\cos^{10}\frac{\pi}{28} + 13568\cos^8\frac{\pi}{28} - 6656\cos^6\frac{\pi}{28} + 1376\cos^4\frac{\pi}{28} - 96\cos^2\frac{\pi}{28} + 1 = 0.$$

$n = 29$

$$\Re e^{29i\varphi} = 268435456x^{29} - 1946157056x^{27} + 6325010432x^{25} - 12163481600x^{23} + 15386804224x^{21} - 13463453696x^{19} + 8341487616x^{17} - 3683254272x^{15} + 1151016960x^{13} - 249387008x^{11} + 36095488x^9 - 3281408x^7 + 168896x^5 - 4060x^3 + 29x;$$

$$16384\cos^{14}\frac{\pi}{29} - 8192\cos^{13}\frac{\pi}{29} - 53248\cos^{12}\frac{\pi}{29} + 24576\cos^{11}\frac{\pi}{29} + 67584\cos^{10}\frac{\pi}{29} - 28160\cos^9\frac{\pi}{29} - 42240\cos^8\frac{\pi}{29} + 15360\cos^7\frac{\pi}{29} + 13440\cos^6\frac{\pi}{29} - 4032\cos^5\frac{\pi}{29} - 2016\cos^4\frac{\pi}{29} + 448\cos^3\frac{\pi}{29} + 112\cos^2\frac{\pi}{29} - 14\cos\frac{\pi}{29} - 1 = 0.$$

$n = 30$

$$\Re e^{30i\varphi} = 536870912x^{30} - 4026531840x^{28} + 13589544960x^{26} - 27262976000x^{24} + 36175872000x^{22} - 33426505728x^{20} + 22052208640x^{18} - 10478223360x^{16} + 3572121600x^{14} - 859955200x^{12} + 141892608x^{10} - 15275520x^8 + 990080x^6 - 33600x^4 + 450x^2 - 1;$$

$$256\cos^8\frac{\pi}{30} - 448\cos^6\frac{\pi}{30} + 224\cos^4\frac{\pi}{30} - 32\cos^2\frac{\pi}{30} + 1 = 0.$$

$\cos\frac{\pi}{30}$ is entry A019893 in [10].

$n = 31$

$$\Re e^{31i\varphi} = 1073741824x^{31} - 8321499136x^{29} + 29125246976x^{27} - 60850962432x^{25} + 84515225600x^{23} - 82239815680x^{21} + 57567870976x^{19} - 29297934336x^{17} + 10827497472x^{15} - 2870927360x^{13} + 533172224x^{11} - 66646528x^{9} + 5261568x^{7} - 236096x^{5} + 4960x^{3} - 31x;$$

$$32768\cos^{15}\frac{\pi}{31} - 16384\cos^{14}\frac{\pi}{31} - 114688\cos^{13}\frac{\pi}{31} + 53248\cos^{12}\frac{\pi}{31} + 159744\cos^{11}\frac{\pi}{31} - 67584\cos^{10}\frac{\pi}{31} - 112640\cos^{9}\frac{\pi}{31} + 42240\cos^{8}\frac{\pi}{31} + 42240\cos^{7}\frac{\pi}{31} - 13440\cos^{6}\frac{\pi}{31} - 8064\cos^{5}\frac{\pi}{31} + 2016\cos^{4}\frac{\pi}{31} + 672\cos^{3}\frac{\pi}{31} - 112\cos^{2}\frac{\pi}{31} - 16\cos\frac{\pi}{31} + 1 = 0.$$

$n = 32$

$$\Re e^{32i\varphi} = 2147483648x^{32} - 17179869184x^{30} + 62277025792x^{28} - 135291469824x^{26} + 196293427200x^{24} - 200655503360x^{22} + 148562247680x^{20} - 80648077312x^{18} + 32133218304x^{16} - 9313976320x^{14} + 1926299648x^{12} - 275185664x^{10} + 25798656x^{8} - 1462272x^{6} + 43520x^{4} - 512x^{2} + 1;$$

$$32768\cos^{16}\frac{\pi}{32} - 131072\cos^{14}\frac{\pi}{32} + 212992\cos^{12}\frac{\pi}{32} - 180224\cos^{10}\frac{\pi}{32} + 84480\cos^{8}\frac{\pi}{32} - 21504\cos^{6}\frac{\pi}{32} + 2688\cos^{4}\frac{\pi}{32} - 128\cos^{2}\frac{\pi}{32} + 1 = 0.$$

$n = 33$

$$\Re e^{33i\varphi} = 4294967296x^{33} - 35433480192x^{31} + 132875550720x^{29} - 299708186624x^{27} + 453437816832x^{25} - 485826232320x^{23} + 379364311040x^{21} - 218864025600x^{19} + 93564370944x^{17} - 29455450112x^{15} + 6723526656x^{13} - 1083543552x^{11} + 118243840x^{9} - 8186112x^{7} + 323136x^{5} - 5984x^{3} + 33x;$$

$$1024\cos^{10}\frac{\pi}{33} + 512\cos^{9}\frac{\pi}{33} - 2560\cos^{8}\frac{\pi}{33} - 1280\cos^{7}\frac{\pi}{33} + 2176\cos^{6}\frac{\pi}{33} + 1088\cos^{5}\frac{\pi}{33} - 688\cos^{4}\frac{\pi}{33} - 344\cos^{3}\frac{\pi}{33} + 48\cos^{2}\frac{\pi}{33} + 24\cos\frac{\pi}{33} + 1 = 0.$$

$n = 34$

$$\Re e^{34i\varphi} = 8589934592x^{34} - 73014444032x^{32} + 282930970624x^{30} - 661693399040x^{28} + 1042167103488x^{26} - 1167945891840x^{24} + 959384125440x^{22} - 586290298880x^{20} + 267776819200x^{18} - 91044118528x^{16} + 22761029632x^{14} - 4093386752x^{12} + 511673344x^{10} - 42170880x^{8} + 2108544x^{6} - 55488x^{4} + 578x^{2} - 1;$$

$$65536\cos^{16}\frac{\pi}{34} - 278528\cos^{14}\frac{\pi}{34} + 487424\cos^{12}\frac{\pi}{34} - 452608\cos^{10}\frac{\pi}{34} + 239360\cos^{8}\frac{\pi}{34} - 71808\cos^{6}\frac{\pi}{34} + 11424\cos^{4}\frac{\pi}{34} - 816\cos^{2}\frac{\pi}{34} + 17 = 0.$$

References

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642 (29 #4914)
2. Richard Barakat, *Evaluation of the incomplete gamma function of imaginary argument by Chebyshev polynomials*, Math. Comp. **15** (1961), no. 73, 7–11, E: The factors $(iq)^\nu$ in (5), (9), (13) and (14) should read $(ix)^\nu$. The values for the Newmann's factor after (6) need to be swapped. MR MR0128058 (23 #B1103)

3. Paul Bracken and Jiri Čižek, *Evaluation of quantum mechanical perturbative sums in terms of quadratic surds and their use in approximation of* $\zeta(3)/\pi^3$, Int. J. Quant. Chem. **90** (2002), no. 1, 42–53.
4. John H. Conway, Charles Radin, and Lorenzo Sadun, *On angles whose squared trigonometric functions are rational*, arXiv:math-ph/9812019 (1998).
5. ———, *On angles whose squared trigonometric functions are rational*, Disc. Comput. Geom. (1999), no. 22, 321–332.
6. Kurt Girstmair, *Some linear relations between values of trigonometric functions at* $k\pi/n$, Acta Arithm. **81** (1997), 387–398. MR 1472818 (98h:11133)
7. I. Gradstein and I. Ryshik, *Summen-, Produkt- und Integraltafeln*, 1st ed., Harri Deutsch, Thun, 1981. MR 0671418 (83i:00012)
8. S. Gurak, *On the minimal polynomial of gauss periods for prime powers*, Math. Comp. **75** (2006), no. 256, 2021–2035. MR 2240647 (2007c:11088)
9. L. D. Servi, *Nested square roots of 2*, Am. Math. Monthly **110** (2003), no. 4, 326–330. MR 1984573
10. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://www.research.att.com/˜njas/sequences/. MR 1992789 (2004f:11151)
11. Raimundas Vidūnas, *Expressions for values of the gamma function*, arXiv:math.CA/0403510 (2004).
12. ———, *Expressions for values of the gamma function*, Kyushu J. Math **59** (2005), no. 2, 267–283. MR 2188592 (2007b:33005)

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