New Principles of Differential Equations VI

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Abstract

This paper uses Z transformations to get the general solutions of many second-order, third-order and fourth-order linear PDEs for the first time, and uses the general solutions to obtain the exact solutions of many typical definite solution problems. We present the Z_4 transformation for the first time and use it to solve a specific case. We successfully get the Fourier series solution by the series general solution of the one-dimensional homogeneous wave equation, which successfully solves a famed unresolved debate in the history of mathematics.

Keywords: Z_1 transformation; Z_3 transformation; Z_4 transformation; general solutions; exact solution of definite solution problem; general solution and series solution of wave equation.

1. Introduction

Since the characteristic equation method was discovered and perfected, the study on general solutions of PDEs has been very slow and almost stagnant.[1] Later, numerical methods[2−6] and qualitative theory[7−14] increasingly became an important research direction of PDEs, and the use of various analytical methods to get exact solutions of PDEs has always shown vitality.[15−18] In particular, the solitary wave has been attracting attention in the research results of nonlinear PDEs.[19,20]

The Z_1, Z_2 and Z_3 transformations presented in our previous papers[21,22] can efficiently get general solutions or analytical solutions of many n-ary m-order PDEs. For some first-order linear PDEs which cannot be solved by the characteristic equation method, the general solutions can be get by using Z transformations, and the exact solutions of some typical definite solution problems could also be effectively get.

In this paper, we will continue to research the laws and applications of general solutions of linear PDEs. In Section 2, we will use Z transformations to get general solutions for many second-order, third-order and fourth-order linear PDEs for the first time. In the third section, we will use the results get before to research the relationship between the general solution and the Fourier series solution of the one-dimensional homogeneous wave equation, and successfully solve a famed debate in the history of mathematics about the relationship between the general solution and the series solution. In Section 4, we will use general solutions to get exact solutions of many typical definite solution problems of first-, second-, third-, and fourth-order linear PDEs. In Section V we present the Z_4 transformation for the first time, and summarize this paper in Section VI.

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2. General solutions of some typical linear partial differential equations

In this section, if there is no especial interpretation, a_i are arbitrary known constants, c_i, k_i, l_i, C_i and C are random constants, f and f_i are random smooth functions $(i = 1, 2, \ldots)$.

Before using the Z transformations to obtain general solutions for some typical linear PDEs, we first get two new algebraic theorems.

Theorem 1. If $a_i \neq 0, (i = 1, 2, 3), k_j \neq 0, (j = 1, 2, 3, 4)$, and

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 = 0, \tag{2.1}$$

$$a_1 k_3^2 + a_2 k_4^2 + a_3 k_3 k_4 = 0, \tag{2.2}$$

$$2a_1 k_1 k_3 + 2a_2 k_2 k_4 + a_3 (k_1 k_4 + k_2 k_3) = 0. \tag{2.3}$$

Then

$$\frac{k_1}{k_2} = \frac{k_3}{k_4}. \tag{2.4}$$

Prove. Set

$$k_2 = c_1 k_1, k_4 = c_2 k_3.$$

Then

$$a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 = a_1 k_1^2 + a_2 c_1^2 k_1^2 + a_3 c_1 k_1^2 = 0.$$

We get

$$c_1 = \frac{-a_3 \pm \sqrt{a_3^2 - 4a_1 a_2}}{2a_2}.$$

The same can be obtained

$$c_2 = \frac{-a_3 \pm \sqrt{a_3^2 - 4a_1 a_2}}{2a_2}.$$

So

$$2a_1 k_1 k_3 + 2a_2 k_2 k_4 + a_3 (k_1 k_4 + k_2 k_3) = 2a_1 k_1 k_3 + 2a_2 c_1 c_2 k_1 k_3 + a_3 (c_1 + c_2) k_1 k_3 = 0.$$

Namely

$$2a_1 + 2a_2 c_1 c_2 + a_3 (c_1 + c_2) = 0.$$

Set

$$c_1 = \frac{-a_3 + \sqrt{a_3^2 - 4a_1 a_2}}{2a_2}, c_2 = \frac{-a_3 - \sqrt{a_3^2 - 4a_1 a_2}}{2a_2}.$$

Then

$$2a_1 + 2a_2 c_1 c_2 + a_3 (c_1 + c_2) = 4a_1 - \frac{a_3^2}{a_2} = 0.$$

Namely

$$a_3^2 - 4a_1 a_2 = 0, c_1 = c_2,$$

so the theorem is proved. □

Theorem 2. If $a_i \neq 0, (i = 1, 2), k_j \neq 0, (j = 1, 2, 3, 4)$, and

$$a_1 {k_1}^2 + a_2 {k_2}^2 = 0, \tag{2.5}$$

$$a_1 k_1 k_3 + a_2 k_2 k_4 = 0. \tag{2.6}$$

Then

$$\frac{k_1}{k_3} = \frac{k_2}{k_4}. \tag{2.7}$$

Prove. Set

$$k_1 = c_1 k_3, k_2 = c_2 k_4.$$

Then

$$a_1 c_1 k_3^2 + a_2 c_2 k_4^2 = 0,$$
$$a_1 c_2 k_3^2 + a_2 c_2 k_4^2 = 0.$$

We get

$$a_1 (c_1 - c_2) {k_3}^2 = 0.$$

Therefore

$$c_1 = c_2.$$

So the theorem is proved. □

Next we present Theorem 3.

Theorem 3. *In* $\mathbb{R}^2$,

$$a_1 u_{tt} + a_2 u_{xx} + a_3 u_{tx} + a_4 u_t + a_5 u_x = A\left(t, x\right), \tag{2.8}$$

the general solution of Eq. (2.8) is

$$u =$$
$$f\left(q\right) + \mathrm{e}^{\frac{-(a_4 k_1 + a_5 k_2) q}{2a_1 k_1 k_3 + 2a_2 k_2 k_4 + a_3 (k_1 k_4 + k_2 k_3)}} \left(\int g\left(p\right) dp + \frac{\iint \mathrm{e}^{\frac{(a_4 k_1 + a_5 k_2) q}{2a_1 k_1 k_3 + 2a_2 k_2 k_4 + a_3 (k_1 k_4 + k_2 k_3)}} A\left(p, q\right) dq dp}{2a_1 k_1 k_3 + 2a_2 k_2 k_4 + a_3\left(k_1 k_4 + k_2 k_3\right)} \right), \tag{2.9}$$

where $A(t,x)$ is any known function, f is an random second differentiable function, g is an random first differentiable function, and

$$p = k_1 t + k_2 x, q = k_3 t + k_4 x, \tag{2.10}$$

$$k_1 k_4 - k_2 k_3 \neq 0, \tag{2.11}$$

$$a_1 {k_1}^2 + a_2 {k_2}^2 + a_3 k_1 k_2 = 0, \tag{2.12}$$

$$a_1 {k_3}^2 + a_2 {k_4}^2 + a_3 k_3 k_4 = 0, \tag{2.13}$$

$$a_4 k_3 + a_5 k_4 = 0. \tag{2.14}$$

Prove. By Z_1 transformation, set

$$u(t, x) = u(p, q), \tag{2.15}$$

$$p = k_1 t + k_2 x, q = k_3 t + k_4 x,$$

and

$$t = \frac{p k_4 - q k_2}{k_1 k_4 - k_2 k_3}, x = \frac{q k_1 - p k_3}{k_1 k_4 - k_2 k_3},$$
$$k_1 k_4 - k_2 k_3 \neq 0.$$

Then

$$\begin{aligned}
&a_1u_{tt}+a_2u_{xx}+a_3u_{tx}+a_4u_t+a_5u_x\\
&=a_1\left(k_1^2u_{pp}+k_3^2u_{qq}+2k_1k_3u_{pq}\right)+a_2\left(k_2^2u_{pp}+k_4^2u_{qq}+2k_2k_4u_{pq}\right)\\
&+a_3\left(k_1k_2u_{pp}+k_3k_4u_{qq}+\left(k_1k_4+k_2k_3\right)u_{pq}\right)+a_4\left(k_1u_p+k_3u_q\right)+a_5\left(k_2u_p+k_4u_q\right).
\end{aligned}$$

Namely

$$\begin{aligned}
&\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)u_{pp}+\left(a_1k_3^2+a_2k_4^2+a_3k_3k_4\right)u_{qq}\\
&+\left(2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)\right)u_{pq}+\left(a_4k_1+a_5k_2\right)u_p+\left(a_4k_3+a_5k_4\right)u_q\\
&=A\left(p,q\right).
\end{aligned}\tag{2.16}$$

Set

$$a_1{k_1}^2+a_2{k_2}^2+a_3k_1k_2=0,$$
$$a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4=0,$$
$$a_4k_3+a_5k_4=0.$$

So

$$\begin{aligned}
&a_1u_{tt}+a_2u_{xx}+a_3u_{tx}+a_4u_t+a_5u_x\\
&=\left(2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)\right)u_{pq}+\left(a_4k_1+a_5k_2\right)u_p=A\left(p,q\right).
\end{aligned}$$

Set

$$u_p=w,$$

hence

$$\left(2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)\right)w_q+\left(a_4k_1+a_5k_2\right)w=A\left(p,q\right).\tag{2.17}$$

The solution of Eq. (2.17) is

$$w=\mathrm{e}^{\frac{-(a_4k_1+a_5k_2)q}{2a_1k_1k_3+2a_2k_2k_4+a_3(k_1k_4+k_2k_3)}}\left(g\left(p\right)+\frac{\int\mathrm{e}^{\frac{(a_4k_1+a_5k_2)q}{2a_1k_1k_3+2a_2k_2k_4+a_3(k_1k_4+k_2k_3)}}A\left(p,q\right)dq}{2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)}\right).$$

Thereupon

$$\begin{aligned}
&u=f\left(q\right)+\int wdp\\
&=f\left(q\right)+\mathrm{e}^{\frac{-(a_4k_1+a_5k_2)q}{2a_1k_1k_3+2a_2k_2k_4+a_3(k_1k_4+k_2k_3)}}\left(\int g\left(p\right)dp+\frac{\iint\mathrm{e}^{\frac{(a_4k_1+a_5k_2)q}{2a_1k_1k_3+2a_2k_2k_4+a_3(k_1k_4+k_2k_3)}}A(p,q)dqdp}{2a_1k_1k_3+2a_2k_2k_4+a_3(k_1k_4+k_2k_3)}\right).
\end{aligned}$$

So the theorem is proved. □

For (2.16), according to Theorem 1 we cannot set

$$a_1{k_1}^2+a_2{k_2}^2+a_3k_1k_2=0,$$
$$a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4=0,$$
$$2a_1k_1k_3+2a_2k_2k_4+a_3(k_1k_4+k_2k_3)=0.$$

Otherwise p and q are functional dependence. Also cannot be set

$$a_4k_1+a_5k_2=0,$$

$$a_4k_3+a_5k_4=0.$$

Otherwise p and q will also be meaningless or functionally dependent .

In (2.16), if set

$$a_1k_1^2 + a_2k_2^2 + a_3k_1k_2 = 0,$$

$$2a_1k_1k_3 + 2a_2k_2k_4 + a_3(k_1k_4 + k_2k_3) = 0,$$

$$a_4k_1 + a_5k_2 = 0.$$

Then

$$\left(a_1k_3^2 + a_2k_4^2 + a_3k_3k_4\right)u_{qq} + (a_4k_3 + a_5k_4)u_q = A(p,q). \tag{2.18}$$

Set

$$u_q = w.$$

So

$$\left(a_1k_3^2 + a_2k_4^2 + a_3k_3k_4\right)w_q + (a_4k_3 + a_5k_4)w = A(p,q). \tag{2.19}$$

The solution of Eq. (2.19) is

$$w = \mathrm{e}^{\frac{-(a_4k_3+a_5k_4)q}{a_1k_3^2+a_2k_4^2+a_3k_3k_4}}\left(g(p) + \frac{\int \mathrm{e}^{\frac{(a_4k_3+a_5k_4)q}{a_1k_3^2+a_2k_4^2+a_3k_3k_4}}A(p,q)\,dq}{a_1k_3^2 + a_2k_4^2 + a_3k_3k_4}\right).$$

Whereupon

$$\begin{aligned} u &= f(p) + \int w dq \\ &= f(p) + \mathrm{e}^{\frac{-(a_4k_3+a_5k_4)q}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}}g(p) \\ &+ \frac{\int\left(\mathrm{e}^{\frac{-(a_4k_3+a_5k_4)q}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}}\int \mathrm{e}^{\frac{(a_4k_3+a_5k_4)q}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}}A(p,q)dq\right)dq}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}. \end{aligned}$$

So we can get Theorem 4.

Theorem 4. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{tt} + a_2u_{xx} + a_3u_{tx} + a_4u_t + a_5u_x = A(t,x),$$

is

$$u = f(p) + \mathrm{e}^{\frac{-(a_4k_3+a_5k_4)q}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}}g(p) + \frac{\int\left(\mathrm{e}^{\frac{-(a_4k_3+a_5k_4)q}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}}\int \mathrm{e}^{\frac{(a_4k_3+a_5k_4)q}{a_1{k_3}^2+a_2{k_4}^2+a_3k_3k_4}}A(p,q)\,dq\right)dq}{a_1{k_3}^2 + a_2{k_4}^2 + a_3k_3k_4}. \tag{2.20}$$

where A(t,x) is any known function, f and g are random second differentiable functions, and

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

$$k_1k_4 - k_2k_3 \neq 0,$$

$$a_1{k_1}^2 + a_2{k_2}^2 + a_3k_1k_2 = 0,$$

$$2a_1k_1k_3 + 2a_2k_2k_4 + a_3(k_1k_4 + k_2k_3) = 0, \tag{2.21}$$

$$a_4k_1 + a_5k_2 = 0. \tag{2.22}$$

For

$$a_1u_{tt} + a_2u_{xx} + a_3u_{tx} + a_4u_t = A(v), \tag{2.23}$$

where $v = k_1 t + k_2 x + k_3$, set

$$u = f(v).$$

Then

$$a_1 u_{tt} + a_2 u_{xx} + a_3 u_{tx} + a_4 u_t = a_1 {k_1}^2 f'' + a_2 {k_2}^2 f'' + a_3 k_1 k_2 f'' + a_4 k_1 f' = A(v).$$

Set

$$w = f',$$

namely

$$\left(a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2\right) w' + a_4 k_1 w = A(v). \tag{2.24}$$

The solution of Eq. (2.24) is

$$w = \mathrm{e}^{\frac{-a_4 k_1 v}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2}} \left(C + \frac{\int \mathrm{e}^{\frac{a_4 k_1 v}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2}} A(v)\, dv}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2} \right).$$

So the particular solution of Eq. (2.23) is

$$u = \int w dv = \int \mathrm{e}^{\frac{-a_4 k_1 v}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2}} \left(C + \frac{\int \mathrm{e}^{\frac{a_4 k_1 v}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2}} A(v)\, dv}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2} \right) dv. \tag{2.25}$$

In Theorem 3, if

$$a_3 = a_4 = a_5 = 0,$$

then

$$a_1 u_{tt} + a_2 u_{xx} = \left(a_1 k_1^2 + a_2 k_2^2\right) u_{pp} + \left(a_1 k_3^2 + a_2 k_4^2\right) u_{qq} + \left(2a_1 k_1 k_3 + 2a_2 k_2 k_4\right) u_{pq} = A(p,q). \tag{2.26}$$

Set

$$a_1 k_1^2 + a_2 k_2^2 = a_1 k_3^2 + a_2 k_4^2 = 0.$$

Namely

$$k_1 = \pm\sqrt{-\frac{a_2}{a_1}} k_2, k_3 = \pm\sqrt{-\frac{a_2}{a_1}} k_4.$$

Set

$$k_1 = \sqrt{-\frac{a_2}{a_1}} k_2, k_3 = -\sqrt{-\frac{a_2}{a_1}} k_4, (k_2, k_4 \neq 0). \tag{2.27}$$

So

$$a_1 u_{tt} + a_2 u_{xx} = 2\left(a_1 k_1 k_3 + a_2 k_2 k_4\right) u_{pq} = 4 a_2 k_2 k_4 u_{pq} = A(p,q). \tag{2.28}$$

The general solution of Eq. (2.28) is

$$u = f(p) + g(q) + \frac{\iint A(p,q)\, dp dq}{4 a_2 k_2 k_4}. \tag{2.29}$$

Set $k_2 = k_4 = 1$, then

$$p = k_1 x_1 + k_2 x_2 = \sqrt{-\frac{a_2}{a_1}} t + x,$$

$$q = k_3 x_1 + k_4 x_2 = -\sqrt{-\frac{a_2}{a_1}} t + x,$$

$$t=\frac{k_4p-k_2q}{k_1k_4-k_2k_3}=\sqrt{-\frac{a_1}{a_2}}\frac{p-q}{2},$$

$$x=\frac{-k_3p+k_1q}{k_1k_4-k_2k_3}=\frac{p+q}{2}.$$

So (2.29) can be written as

$$u=f\left(\sqrt{-\frac{a_2}{a_1}}t+x\right)+g\left(-\sqrt{-\frac{a_2}{a_1}}t+x\right)+\frac{1}{4a_2}\iint A\left(\sqrt{-\frac{a_1}{a_2}}\frac{p-q}{2},\frac{p+q}{2}\right)dpdq.$$

So we can get Theorem 5.

Theorem 5. *In* $\mathbb{R}^2$,

$$a_1u_{tt}+a_2u_{xx}=A\left(t,x\right), \tag{2.30}$$

the general solution of Eq. (2.30) is

$$u=f\left(\sqrt{-\frac{a_2}{a_1}}t+x\right)+g\left(-\sqrt{-\frac{a_2}{a_1}}t+x\right)+\frac{1}{4a_2}\iint A\left(\sqrt{-\frac{a_1}{a_2}}\frac{p-q}{2},\frac{p+q}{2}\right)dpdq, \tag{2.31}$$

where $A(t,x)$ is any known function, f and g are random second differentiable function, and

$$p=\sqrt{-\frac{a_2}{a_1}}t+x, q=-\sqrt{-\frac{a_2}{a_1}}t+x. \tag{2.32}$$

Note that according to Theorem 2, if set $a_1{k_1}^2+a_2{k_2}^2=0, 2a_1k_1k_3+2a_2k_2k_4=0$, or $a_1{k_3}^2+a_2{k_4}^2=0, 2a_1k_1k_3+2a_2k_2k_4=0$, the general solution of (2.30) cannot be get.

In Theorem 3, if

$$a_4=a_5=0.$$

Then

$$\begin{aligned}&a_1u_{tt}+a_2u_{xx}+a_3u_{tx}\\&=\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)u_{pp}+\left(a_1k_3^2+a_2k_4^2+a_3k_3k_4\right)u_{qq}\\&+\left(2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)\right)u_{pq}=A\left(p,q\right).\end{aligned}$$

Set

$$a_1k_1^2+a_2k_2^2+a_3k_1k_2=a_1k_3^2+a_2k_4^2+a_3k_3k_4=0.$$

Therefore

$$\left(2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)\right)u_{pq}=A\left(p,q\right).$$

So the general solution of $a_1u_{tt}+a_2u_{xx}+a_3u_{tx}=A(t,x)$ is

$$u=f\left(p\right)+g\left(q\right)+\frac{\iint A\left(p,q\right)dpdq}{2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)}.$$

If set

$$a_1k_1^2+a_2k_2^2+a_3k_1k_2=2a_1k_1k_3+2a_2k_2k_4+a_3\left(k_1k_4+k_2k_3\right)=0.$$

Then

$$\left(a_1k_3^2+a_2k_4^2+a_3k_3k_4\right)u_{qq}=A\left(p,q\right).$$

So the general solution of $a_1u_{tt}+a_2u_{xx}+a_3u_{tx}=A(t,x)$ is

$$u=f\left(p\right)+qg\left(p\right)+\frac{\iint A\left(p,q\right)dqdq}{a_1k_3^2+a_2k_4^2+a_3k_3k_4}.$$

So we can get Theorems 6 and 7.

Theorem 6. *In* $\mathbb{R}^2$,

$$a_1u_{tt} + a_2u_{xx} + a_3u_{tx} = A(t,x), \tag{2.33}$$

the general solution of Eq. (2.33) is

$$u = f(p) + g(q) + \frac{\iint A(p,q)\,dpdq}{2a_1k_1k_3 + 2a_2k_2k_4 + a_3(k_1k_4 + k_2k_3)}, \tag{2.34}$$

where $A(t,x)$ *is any known function,* f *and* g *are random second differentiable functions, and*

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

$$k_1k_4 - k_2k_3 \neq 0,$$
$$a_1k_1^2 + a_2k_2^2 + a_3k_1k_2 = a_1k_3^2 + a_2k_4^2 + a_3k_3k_4 = 0. \tag{2.35}$$

Theorem 7. *In* $\mathbb{R}^2$, *the general solution of*

$$a_1u_{tt} + a_2u_{xx} + a_3u_{tx} = A(t,x)$$

is

$$u = f(p) + qg(p) + \frac{\iint A(p,q)\,dqdq}{a_1k_3^2 + a_2k_4^2 + a_3k_3k_4}, \tag{2.36}$$

where $A(t,x)$ *is any known function,* f *and* g *are random second differentiable functions, and*

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

$$k_1k_4 - k_2k_3 \neq 0,$$
$$a_1k_1^2 + a_2k_2^2 + a_3k_1k_2 = 2a_1k_1k_3 + 2a_2k_2k_4 + a_3(k_1k_4 + k_2k_3) = 0. \tag{2.37}$$

According to Theorems 3-7, the general solutions for their corresponding homogeneous equations may be obtained directly, or they may be get again by using Z_1 transformation. Readers may try it by themselves.

Next we propose Theorem 8.

Theorem 8. *In* $\mathbb{R}^2$,

$$a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = 0, \tag{2.38}$$

the general solution of Eq. (2.38) is

$$u = \left(C_1e^{\lambda_1 v} + C_2e^{\lambda_2 v}\right)(h_1(l_1x + l_2y + l_3) + h_2(l_4x + l_5y + l_6)), \tag{2.39}$$

where h_1 *and* h_2 *are random second differentiable functions, and*

$$v = k_1x + k_2y + k_3, \tag{2.40}$$

$$\lambda_1 = \frac{-a_4k_1 - a_5k_2 + \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}, \tag{2.41}$$

$$\lambda_2 = \frac{-a_4k_1 - a_5k_2 - \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}, \tag{2.42}$$

where C_1, C_2, k_3, l_3 and l_6 are random constants; k_1, k_2, l_1, l_2, l_4 and l_5 are constants which satisfy

$$l_1 = \frac{-a_3l_2 + \sqrt{a_3^2l_2^2 - 4a_1a_2l_2^2}}{2a_1}, l_4 = \frac{-a_3l_5 - \sqrt{a_3^2l_5^2 - 4a_1a_2l_5^2}}{2a_1}, \tag{2.43}$$

$$k_1\lambda_1\left(2a_1l_1 + a_3l_2\right) + k_2\lambda_1\left(2a_2l_2 + a_3l_1\right) + a_4l_1 + a_5l_2 = 0, \tag{2.44}$$

$$k_1\lambda_2\left(2a_1l_1 + a_3l_2\right) + k_2\lambda_2\left(2a_2l_2 + a_3l_1\right) + a_4l_1 + a_5l_2 = 0, \tag{2.45}$$

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) > 0. \tag{2.46}$$

Prove. By Z_1 transformation, set

$$u(x, y) = f(v) = f(k_1x + k_2y + k_3),$$

where $v(x, y) = k_1x + k_2y + k_3$; k_1, k_2 and k_3 are undetermined constants, f is an undetermined second differentiable function, so

$$\begin{aligned} &a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u \\ &= \left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) f_v^{''} + (a_4k_1 + a_5k_2) f_v^{'} + a_6f = 0. \end{aligned} \tag{2.47}$$

The characteristic equation of Eq. (2.47) is

$$\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)\lambda^2 + (a_4k_1 + a_5k_2)\lambda + a_6 = 0. \tag{2.48}$$

If

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) > 0,$$

the particular solution of Eq. (2.38) is

$$u = f = C_1e^{\lambda_1v} + C_2e^{\lambda_2v}, \tag{2.49}$$

where C_1 and C_2 are arbitrary constants, and

$$\lambda_1 = \frac{-a_4k_1 - a_5k_2 + \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)},$$

$$\lambda_2 = \frac{-a_4k_1 - a_5k_2 - \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}.$$

For getting the general solution of Eq. (2.38), by Z_3 transformation, we set

$$u(x, y) = gh(w) = g(x, y)\, h(l_1x + l_2y + l_3), \tag{2.50}$$

where $w(x, y) = l_1x + l_2y + l_3$; l_1, l_2 and l_3 are undetermined constants, h and g are undetermined second differentiable functions, so

$$\begin{aligned} &a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u \\ &= a_1\left(g_{xx}h + 2l_1g_xh_w^{'} + l_1^2gh_w^{''}\right) + a_2\left(g_{yy}h + 2l_2g_yh_w^{'} + l_2^2gh_w^{''}\right) \\ &+a_3\left(g_{xy}h + l_2g_xh_w^{'} + l_1g_yh_w^{'} + l_1l_2gh_w^{''}\right) + a_4\left(g_xh + l_1gh_w^{'}\right) + a_5\left(g_yh + l_2gh_w^{'}\right) + a_6gh. \end{aligned}$$

Namely

$$\left(a_1l_1^2+a_2l_2^2+a_3l_1l_2\right)gh_w^{''}+\left(\left(2a_1l_1+a_3l_2\right)g_x+\left(2a_2l_2+a_3l_1\right)g_y+\left(a_4l_1+a_5l_2\right)g\right)h_w^{'}$$
$$+\left(a_1g_{xx}+a_2g_{yy}+a_3g_{xy}+a_4g_x+a_5g_y+a_6g\right)h=0. \tag{2.51}$$

Set $h(w)$ is an random second differentiable function, and

$$a_1l_1^2+a_3l_2l_1+a_2l_2^2=0\Longrightarrow l_1=\frac{-a_3l_2\pm\sqrt{a_3^2l_2^2-4a_1a_2l_2^2}}{2a_1}, \tag{2.52}$$

$$\left(2a_1l_1+a_3l_2\right)g_x+\left(2a_2l_2+a_3l_1\right)g_y+\left(a_4l_1+a_5l_2\right)g=0, \tag{2.53}$$

$$a_1g_{xx}+a_2g_{yy}+a_3g_{xy}+a_4g_x+a_5g_y+a_6g=0. \tag{2.54}$$

By Eq. (2.49), the particular solution of Eq. (2.54) is

$$g=C_1\mathrm{e}^{\lambda_1v}+C_2\mathrm{e}^{\lambda_2v}. \tag{2.55}$$

Substituting (2.55) into (2.53) we get

$$\begin{aligned}&\left(2a_1l_1+a_3l_2\right)g_x+\left(2a_2l_2+a_3l_1\right)g_y+\left(a_4l_1+a_5l_2\right)g\\&=\left(C_1k_1\lambda_1\left(2a_1l_1+a_3l_2\right)+C_1k_2\lambda_1\left(2a_2l_2+a_3l_1\right)+C_1\left(a_4l_1+a_5l_2\right)\right)e^{\lambda_1v}\\&+\left(C_2k_1\lambda_2\left(2a_1l_1+a_3l_2\right)+C_2k_2\lambda_2\left(2a_2l_2+a_3l_1\right)+C_2\left(a_4l_1+a_5l_2\right)\right)e^{\lambda_2v}=0.\end{aligned}$$

Then

$$k_1\lambda_1\left(2a_1l_1+a_3l_2\right)+k_2\lambda_1\left(2a_2l_2+a_3l_1\right)+a_4l_1+a_5l_2=0,$$
$$k_1\lambda_2\left(2a_1l_1+a_3l_2\right)+k_2\lambda_2\left(2a_2l_2+a_3l_1\right)+a_4l_1+a_5l_2=0.$$

So the general solution of Eq. (2.38) is

$$u=g\left(h_1\left(w_1\right)+h_2\left(w_2\right)\right)=\left(C_1\mathrm{e}^{\lambda_1v}+C_2\mathrm{e}^{\lambda_2v}\right)\left(h_1\left(l_1x+l_2y+l_3\right)+h_2\left(l_4x+l_5y+l_6\right)\right),$$

where $v,\lambda_1,\lambda_2,k_1,k_2,l_1,l_2,l_4$ and l_5 satisfy Eqs. (2.40-2.46), so the theorem is proved. □

In Theorem 8, if

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)<0,$$

or

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)=0.$$

By analogous calculation, we may have Theorem 9 and 10.

Theorem 9. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=0,$$

is

$$u=\left(C_1\sin\lambda_2v+C_2\cos\lambda_2v\right)\mathrm{e}^{\lambda_1v}\left(h_1\left(l_1x+l_2y+l_3\right)+h_2\left(l_4x+l_5y+l_6\right)\right), \tag{2.56}$$
$$v=k_1x+k_2y+k_3,$$

where h_1 and h_2 are random second differentiable functions; $C_1,C_2,\lambda_1,\lambda_2,k_1,k_2,l_1,l_2,l_4$ and l_5 are constants which satisfy

$$\lambda_1=\frac{-a_4k_1-a_5k_2}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)},\lambda_2=\frac{\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}, \tag{2.57}$$

$$l_1=\frac{-a_3l_2+\sqrt{a_3^2l_2^2-4a_1a_2l_2^2}}{2a_1}, l_4=\frac{-a_3l_5-\sqrt{a_3^2l_5^2-4a_1a_2l_5^2}}{2a_1},$$

$$(C_1\lambda_1-C_2\lambda_2)\left(2a_1k_1l_1+a_3k_1l_2+2a_2k_2l_2+a_3k_2l_1\right)+C_1\left(a_4l_1+a_5l_2\right)=0, \tag{2.58}$$

$$k_1\left(C_1\lambda_2+C_2\lambda_1\right)\left(2a_1l_1+a_3l_2\right)+k_2\left(C_1\lambda_2-C_2\lambda_1\right)\left(2a_2l_2+a_3l_1\right)+C_2\left(a_4l_1+a_5l_2\right)=0, \tag{2.59}$$

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)<0. \tag{2.60}$$

Theorem 10. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=0,$$

is

$$u=Cv\mathrm{e}^{\lambda v}\left(h_1\left(l_1x+l_2y+l_3\right)+h_2\left(l_4x+l_5y+l_6\right)\right), \tag{2.61}$$

$$v=k_1x+k_2y+k_3,$$

where h_1 and h_2 are random second differentiable functions; $\lambda, k_1, k_2, k_4, k_5, l_1, l_2, l_4$ and l_5 are constants which satisfy

$$\lambda=\frac{-a_4k_1-a_5k_2}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}, \tag{2.62}$$

$$l_1=\frac{-a_3l_2+\sqrt{a_3^2l_2^2-4a_1a_2l_2^2}}{2a_1}, l_4=\frac{-a_3l_5-\sqrt{a_3^2l_5^2-4a_1a_2l_5^2}}{2a_1},$$

$$\left(2a_1l_1+a_3l_2\right)\left(k_1+v\lambda k_1\right)+\left(2a_2l_2+a_3l_1\right)\left(k_2+v\lambda k_2\right)+\left(a_4l_1+a_5l_2\right)v=0, \tag{2.63}$$

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)=0. \tag{2.64}$$

We propose Theorem 11 as follows.

Theorem 11. *In $\mathbb{R}^2$,*

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=A\left(x,y\right), \tag{2.65}$$

the general solution of Eq. (2.65) is

$$u=g\left(f_1\left(p\right)+f_2\left(q\right)+\iint\frac{A\left(p,q\right)dpdq}{\left(2a_1c_1c_3+2a_2c_2c_4+a_3\left(c_1c_4+c_2c_3\right)\right)g}\right), \tag{2.66}$$

where f_1 and f_2 are random second differentiable functions, and

$$p=c_1x+c_2y, q=c_3x+c_4y, \tag{2.67}$$

$$g=C_1\mathrm{e}^{\lambda_1v}+C_2\mathrm{e}^{\lambda_2v}, v\left(x,y\right)=k_1x+k_2y, \tag{2.68}$$

where $\lambda_1, \lambda_2, k_1, k_2, c_1, c_2, c_3, c_4$ are constants which satisfy

$$a_1c_1^2+a_2c_2^2+a_3c_1c_2=a_1c_3^2+a_2c_4^2+a_3c_3c_4=0, \tag{2.69}$$

$$c_1c_4-c_2c_3\neq 0, \tag{2.70}$$

$$\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)>0,$$

$$\lambda_1=\frac{-a_4k_1-a_5k_2+\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)},$$

$$\lambda_2 = \frac{-a_4k_1 - a_5k_2 - \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)},$$

$$\lambda_1k_1\left(2a_1c_1 + a_3c_2\right) + \lambda_1k_2\left(2a_2c_2 + a_3c_1\right) + a_4c_1 + a_5c_2 = 0, \tag{2.71}$$

$$\lambda_2k_1\left(2a_1c_1 + a_3c_2\right) + \lambda_2k_2\left(2a_2c_2 + a_3c_1\right) + a_4c_1 + a_5c_2 = 0, \tag{2.72}$$

$$\lambda_1k_1\left(2a_1c_3 + a_3c_4\right) + \lambda_1k_2\left(2a_2c_4 + a_3c_3\right) + a_4c_3 + a_5c_4 = 0, \tag{2.73}$$

$$\lambda_2k_1\left(2a_1c_3 + a_3c_4\right) + \lambda_2k_2\left(2a_2c_4 + a_3c_3\right) + a_4c_3 + a_5c_4 = 0, \tag{2.74}$$

Prove. By Z_3 transformation, we set

$$u(x,y) = g(x,y)h(p,q), \tag{2.75}$$

$$p = c_1x + c_2y, q = c_3x + c_4y,$$

where $c_1 - c_4$ are undetermined constants, g and h are undetermined second differentiable function, and

$$x = \frac{pc_4 - qc_2}{c_1c_4 - c_2c_3}, y = \frac{qc_1 - pc_3}{c_1c_4 - c_2c_3}, \tag{2.76}$$
$$c_1c_4 - c_2c_3 \neq 0.$$

By Eq. (2.75), we get

$$\begin{aligned}
&a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u \\
&= a_1\left(hg_{xx} + 2g_x\left(c_1h_p + c_3h_q\right) + g\left(c_1^2h_{pp} + c_3^2h_{qq} + 2c_1c_3h_{pq}\right)\right) \\
&+a_2\left(hg_{yy} + 2g_y\left(c_2h_p + c_4h_q\right) + g\left(c_2^2h_{pp} + c_4^2h_{qq} + 2c_2c_4h_{pq}\right)\right) \\
&+a_3\left(hg_{xy} + g_x\left(c_2h_p + c_4h_q\right) + g_y\left(c_1h_p + c_3h_q\right) + g\left(c_1c_2h_{pp} + c_3c_4h_{qq} + \left(c_1c_4 + c_2c_3\right)h_{pq}\right)\right) \\
&+a_4\left(g_xh + g\left(c_1h_p + c_3h_q\right)\right) + a_5\left(g_yh + g\left(c_2h_p + c_4h_q\right)\right) + a_6gh \\
&= \left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right)gh_{pp} + \left(a_1c_3^2 + a_2c_4^2 + a_3c_3c_4\right)gh_{qq} \\
&+ \left(2a_1c_1c_3 + 2a_2c_2c_4 + a_3\left(c_1c_4 + c_2c_3\right)\right)gh_{pq} \\
&+ \left(\left(2a_1c_1 + a_3c_2\right)g_x + \left(2a_2c_2 + a_3c_1\right)g_y + \left(a_4c_1 + a_5c_2\right)g\right)h_p \\
&+ \left(\left(2a_1c_3 + a_3c_4\right)g_x + \left(2a_2c_4 + a_3c_3\right)g_y + \left(a_4c_3 + a_5c_4\right)g\right)h_q \\
&+ \left(a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g\right)h = A\left(x,y\right).
\end{aligned}$$

Set

$$a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g = 0,$$

$$\left(2a_1c_1 + a_3c_2\right)g_x + \left(2a_2c_2 + a_3c_1\right)g_y + \left(a_4c_1 + a_5c_2\right)g = 0, \tag{2.77}$$

$$\left(2a_1c_3 + a_3c_4\right)g_x + \left(2a_2c_4 + a_3c_3\right)g_y + \left(a_4c_3 + a_5c_4\right)g = 0. \tag{2.78}$$

Set $g = g(v)$ and $v(x,y) = k_1x + k_2y$, then

$$\begin{aligned}
&a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g \\
&= \left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)g_v^{''} + \left(a_4k_1 + a_5k_2\right)g_v^{'} + a_6g = 0.
\end{aligned} \tag{2.51}$$

If

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) > 0,$$

the particular solution of $a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g = 0$ is

$$g = C_1\mathrm{e}^{\lambda_1v} + C_2\mathrm{e}^{\lambda_2v},$$

where C_1 and C_2 are arbitrary constants, and

$$\lambda_1 = \frac{-a_4k_1 - a_5k_2 + \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)},$$

$$\lambda_2 = \frac{-a_4k_1 - a_5k_2 - \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}.$$

Substituting $g = C_1\mathrm{e}^{\lambda_1 v} + C_2\mathrm{e}^{\lambda_2 v}$ into (2.77) we get

$$\begin{aligned}
&(2a_1c_1 + a_3c_2)\, g_x + (2a_2c_2 + a_3c_1)\, g_y + (a_4c_1 + a_5c_2)\, g\\
&= (2a_1c_1 + a_3c_2)\left(C_1\lambda_1k_1e^{\lambda_1 v} + C_2\lambda_2k_1e^{\lambda_2 v}\right)\\
&+ (2a_2c_2 + a_3c_1)\left(C_1\lambda_1k_2e^{\lambda_1 v} + C_2\lambda_2k_2e^{\lambda_2 v}\right) + (a_4c_1 + a_5c_2)\left(C_1e^{\lambda_1 v} + C_2e^{\lambda_2 v}\right)\\
&= (\lambda_1k_1\,(2a_1c_1 + a_3c_2) + \lambda_1k_2\,(2a_2c_2 + a_3c_1) + a_4c_1 + a_5c_2)\, C_1e^{\lambda_1 v}\\
&+ (\lambda_2k_1\,(2a_1c_1 + a_3c_2) + \lambda_2k_2\,(2a_2c_2 + a_3c_1) + a_4c_1 + a_5c_2)\, C_2e^{\lambda_2 v} = 0.
\end{aligned}$$

So

$$\lambda_1k_1\,(2a_1c_1 + a_3c_2) + \lambda_1k_2\,(2a_2c_2 + a_3c_1) + a_4c_1 + a_5c_2 = 0,$$

$$\lambda_2k_1\,(2a_1c_1 + a_3c_2) + \lambda_2k_2\,(2a_2c_2 + a_3c_1) + a_4c_1 + a_5c_2 = 0.$$

Substituting $g = C_1\mathrm{e}^{\lambda_1 v} + C_2\mathrm{e}^{\lambda_2 v}$ into (2.78) we obtain

$$\begin{aligned}
&(2a_1c_3 + a_3c_4)\, g_x + (2a_2c_4 + a_3c_3)\, g_y + (a_4c_3 + a_5c_4)\, g\\
&= (2a_1c_3 + a_3c_4)\left(C_1\lambda_1k_1e^{\lambda_1 v} + C_2\lambda_2k_1e^{\lambda_2 v}\right)\\
&+ (2a_2c_4 + a_3c_3)\left(C_1\lambda_1k_2e^{\lambda_1 v} + C_2\lambda_2k_2e^{\lambda_2 v}\right) + (a_4c_3 + a_5c_4)\left(C_1e^{\lambda_1 v} + C_2e^{\lambda_2 v}\right)\\
&= (\lambda_1k_1\,(2a_1c_3 + a_3c_4) + \lambda_1k_2\,(2a_2c_4 + a_3c_3) + a_4c_3 + a_5c_4)\, C_1e^{\lambda_1 v}\\
&+ (\lambda_2k_1\,(2a_1c_3 + a_3c_4) + \lambda_2k_2\,(2a_2c_4 + a_3c_3) + a_4c_3 + a_5c_4)\, C_2e^{\lambda_2 v} = 0.
\end{aligned}$$

Then

$$\lambda_1k_1\,(2a_1c_3 + a_3c_4) + \lambda_1k_2\,(2a_2c_4 + a_3c_3) + a_4c_3 + a_5c_4 = 0,$$

$$\lambda_2k_1\,(2a_1c_3 + a_3c_4) + \lambda_2k_2\,(2a_2c_4 + a_3c_3) + a_4c_3 + a_5c_4 = 0.$$

So

$$\begin{aligned}
&a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u\\
&= \left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right) gh_{pp} + \left(a_1c_3^2 + a_2c_4^2 + a_3c_3c_4\right) gh_{qq}\\
&+ (2a_1c_1c_3 + 2a_2c_2c_4 + a_3\,(c_1c_4 + c_2c_3))\, gh_{pq} = A\,(x, y)\,.
\end{aligned} \tag{2.80}$$

Set

$$a_1c_1^2 + a_2c_2^2 + a_3c_1c_2 = a_1c_3^2 + a_2c_4^2 + a_3c_3c_4 = 0.$$

Then

$$(2a_1c_1c_3 + 2a_2c_2c_4 + a_3\,(c_1c_4 + c_2c_3))\, gh_{pq} = A\,(x, y)\,, \tag{2.81}$$

the general solution of (2.81) is

$$h = f_1\,(p) + f_2\,(q) + \iint \frac{A\,(p, q)\, dpdq}{(2a_1c_1c_3 + 2a_2c_2c_4 + a_3\,(c_1c_4 + c_2c_3))\, g}. \tag{2.82}$$

So the general solution of Eq. (2.65) is

$$u = g\left(f_1\,(p) + f_2\,(q) + \iint \frac{A\,(p, q)\, dpdq}{(2a_1c_1c_3 + 2a_2c_2c_4 + a_3\,(c_1c_4 + c_2c_3))\, g}\right).$$

So the theorem is proved. □

In (2.80), if set

$$a_1c_3^2 + a_2c_4^2 + a_3c_3c_4 = 2a_1c_1c_3 + 2a_2c_2c_4 + a_3\,(c_1c_4 + c_2c_3) = 0.$$

Then

$$\left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right) gh_{pp} = A(x, y), \tag{2.83}$$

the general solution of (2.83) is

$$h = f_1(q) + pf_2(q) + \iint \frac{A(p, q)\, dpdp}{\left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right) g}. \tag{2.84}$$

So the general solution of Eq. (2.65) is

$$u = g\left(f_1(q) + pf_2(q) + \iint \frac{A(p, q)\, dpdp}{\left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right) g}\right).$$

So we can get Theorems 12.

Theorem 12. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = A(x, y),$$

is

$$u = g\left(f_1(q) + pf_2(q) + \iint \frac{A(p, q)\, dpdp}{\left(a_1c_1^2 + a_2c_2^2 + a_3c_1c_2\right) g}\right), \tag{2.85}$$

where f_1 and f_2 are arbitrary second differentiable functions, and

$$p = c_1x + c_2y, q = c_3x + c_4y,$$

$$g = C_1\mathrm{e}^{\lambda_1 v} + C_2\mathrm{e}^{\lambda_2 v}, v(x, y) = k_1x + k_2y,$$

where $k_1, k_2, \lambda_1, \lambda_2, c_1, c_2, c_3$ and c_4 are constants which satisfy (2.41,2.42,2.46,2.70-2.74) and

$$a_1c_3^2 + a_2c_4^2 + a_3c_3c_4 = 2a_1c_1c_3 + 2a_2c_2c_4 + a_3(c_1c_4 + c_2c_3) = 0. \tag{2.86}$$

In Theorem 11, if

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) < 0,$$

or

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) = 0.$$

By analogous calculation, we may get Theorem 13 and 14.

Theorem 13. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = A(x, y),$$

is

$$u = g\left(f_1(p) + f_2(q) + \iint \frac{A(p, q)\, dpdq}{\left(2a_1c_1c_3 + 2a_2c_2c_4 + a_3(c_1c_4 + c_2c_3)\right) g}\right),$$

where f_1 and f_2 are arbitrary second differentiable functions, and

$$p = c_1x + c_2y, q = c_3x + c_4y,$$

$$g = (C_1\sin\lambda_2 v + C_2\cos\lambda_2 v)\,\mathrm{e}^{\lambda_1 v}, v(x, y) = k_1x + k_2y,$$

where $C_1, C_2, \lambda_1, \lambda_2, k_1, k_2, c_1, c_2, c_3, c_4$ are constants which satisfy

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) < 0,$$

$$\lambda_1 = \frac{-a_4k_1 - a_5k_2}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}, \lambda_2 = \frac{\sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)}}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)},$$

$$a_1c_1^2 + a_2c_2^2 + a_3c_1c_2 = a_1c_3^2 + a_2c_4^2 + a_3c_3c_4 = 0,$$

$$c_1c_4 - c_2c_3 \neq 0,$$

$$(C_1\lambda_1k_1 - C_2\lambda_2k_1)(2a_1c_1 + a_3c_2) + (C_1\lambda_1k_2 - C_2\lambda_2k_2)(2a_2c_2 + a_3c_1) + C_1(a_4c_1 + a_5c_2) = 0, \tag{2.87}$$

$$(C_2\lambda_1k_1 + C_1\lambda_2k_1)(2a_1c_1 + a_3c_2) + (C_2\lambda_1k_2 + C_1\lambda_2k_2)(2a_2c_2 + a_3c_1) + C_2(a_4c_1 + a_5c_2) = 0, \tag{2.88}$$

$$(C_1\lambda_1k_1 - C_2\lambda_2k_1)(2a_1c_3 + a_3c_4) + (C_1\lambda_1k_2 - C_2\lambda_2k_2)(2a_2c_4 + a_3c_3) + C_1(a_4c_3 + a_5c_4) = 0, \tag{2.89}$$

$$(C_2\lambda_1k_1 + C_1\lambda_2k_1)(2a_1c_3 + a_3c_4) + (C_2\lambda_1k_2 + C_1\lambda_2k_2)(2a_2c_4 + a_3c_3) + C_2(a_4c_3 + a_5c_4) = 0. \tag{2.90}$$

Theorem 14. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = A(x, y),$$

is

$$u = g\left(f_1(p) + f_2(q) + \iint \frac{A(p, q)\,dpdq}{(2a_1c_1c_3 + 2a_2c_2c_4 + a_3(c_1c_4 + c_2c_3))\,g}\right),$$

where f_1 and f_2 are arbitrary second differentiable functions, and

$$p = c_1x + c_2y, q = c_3x + c_4y,$$

$$g = Cv\mathrm{e}^{\lambda v}, v(x, y) = k_1x + k_2y,$$

where $k_1, k_2, \lambda, c_1, c_2, c_3, c_4$ are constants which satisfy

$$(a_4k_1 + a_5k_2)^2 - 4a_6\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right) = 0,$$

$$\lambda = \frac{-a_4k_1 - a_5k_2}{2\left(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2\right)},$$

$$a_1c_1^2 + a_2c_2^2 + a_3c_1c_2 = a_1c_3^2 + a_2c_4^2 + a_3c_3c_4 = 0,$$

$$c_1c_4 - c_2c_3 \neq 0,$$

$$2a_1c_1k_1 + a_3c_2k_1 + a_3c_1k_2 + 2a_2c_2k_2 = 0, \tag{2.91}$$

$$a_4c_1k_1 + a_5c_2k_1 + 2\lambda a_1c_1k_1^2 + \lambda a_3c_2k_1^2 + \lambda a_3c_1k_1k_2 + 2\lambda a_2c_2k_1k_2 = 0, \tag{2.92}$$

$$a_4c_1k_2 + a_5c_2k_2 + 2\lambda a_1c_1k_1k_2 + \lambda a_3c_2k_1k_2 + \lambda a_3c_1k_2^2 + 2\lambda a_2c_2k_2^2 = 0, \tag{2.93}$$

$$2a_1c_3k_1 + a_3c_4k_1 + a_3c_3k_2 + 2a_2c_4k_2 = 0, \tag{2.94}$$

$$a_4c_3k_1 + a_5c_4k_1 + 2\lambda a_1c_3k_1^2 + \lambda a_3c_4k_1^2 + \lambda a_3c_3k_1k_2 + 2\lambda a_2c_4k_1k_2 = 0, \tag{2.95}$$

$$a_4c_3k_2 + a_5c_4k_2 + 2\lambda a_1c_3k_1k_2 + \lambda a_3c_4k_1k_2 + \lambda a_3c_3k_2^2 + 2\lambda a_2c_4k_2^2 = 0. \tag{2.96}$$

Similar to Theorem 12, we can propose Theorems 15 and 16.

Theorem 15. *In* $\mathbb{R}^2$*, the general solution of*

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=A\left(x,y\right),$$

is

$$u=g\left(f_1\left(q\right)+pf_2\left(q\right)+\iint\frac{A\left(p,q\right)dpdp}{\left(a_1c_1^2+a_2c_2^2+a_3c_1c_2\right)g}\right),$$

where f_1 *and* f_2 *are arbitrary second differentiable functions, and*

$$p=c_1x+c_2y,q=c_3x+c_4y,$$

$$g=\left(C_1\mathrm{sin}\lambda_2v+C_2\mathrm{cos}\lambda_2v\right)\mathrm{e}^{\lambda_1v},v(x,y)=k_1x+k_2y,$$

$$\lambda_1=\frac{-a_4k_1-a_5k_2}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)},\lambda_2=\frac{\sqrt{\left(a_4k_1+a_5k_2\right)^2-4a_6\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)}}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)},$$

where $C_1,C_2,\lambda_1,\lambda_2,k_1,k_2,c_1,c_2,c_3,c_4$ *are constants which satisfy (2.60, 2.70,2.86-2.90).*

Theorem 16. *In* $\mathbb{R}^2$*, the general solution of*

$$a_1u_{xx}+a_2u_{yy}+a_3u_{xy}+a_4u_x+a_5u_y+a_6u=A\left(x,y\right),$$

is

$$u=g\left(f_1\left(q\right)+pf_2\left(q\right)+\iint\frac{A\left(p,q\right)dpdp}{\left(a_1c_1^2+a_2c_2^2+a_3c_1c_2\right)g}\right),$$

where f_1 *and* f_2 *are arbitrary second differentiable functions, and*

$$p=c_1x+c_2y,q=c_3x+c_4y,$$

$$g=Cv\mathrm{e}^{\lambda v},v(x,y)=k_1x+k_2y,$$

$$\lambda=\frac{-a_4k_1-a_5k_2}{2\left(a_1k_1^2+a_2k_2^2+a_3k_1k_2\right)},$$

where $k_1,k_2,\lambda,c_1,c_2,c_3,c_4$ *are constants which satisfy (2.64,2.70,2.86,2.91-2.96).*

Eqs. (2.38, 2.65) are important linear PDEs. One-dimensional homogeneous and non-homogeneous wave equations, heat equations, two-dimensional reaction-diffusion-convection equation[23–25] and Helmholtz equations, etc. are all special cases of them.

Next we propose Theorem 17.

Theorem 17. *In* $\mathbb{R}^3$*,*

$$a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y=A\left(t,x,y\right),\tag{2.97}$$

the general solution of Eq. (2.97) is

$$u=f\left(q,r\right)+\mathrm{e}^{\frac{-\kappa q}{\lambda}}\left(\int g\left(p,r\right)dp+\frac{1}{\lambda}\iint\mathrm{e}^{\frac{\kappa q}{\lambda}}A\left(p,q,r\right)dpdq\right),\tag{2.98}$$

where $A(t,x,y)$ is any known function, f is a random second differentiable function, g is a random first differentiable function, and

$$p=l_1t+l_2x+l_3y, q=l_4t+l_5x+l_6y, r=l_7t+l_8x+l_9y, \tag{2.99}$$

$$-l_3l_5l_7+l_2l_6l_7+l_3l_4l_8-l_1l_6l_8-l_2l_4l_9+l_1l_5l_9\neq 0, \tag{2.100}$$

$$\kappa=a_7l_1+a_8l_2+a_9l_3, \tag{2.101}$$

$$\lambda=2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right), \tag{2.102}$$

$$a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3=0, \tag{2.103}$$

$$a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6=0, \tag{2.104}$$

$$a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9=0, \tag{2.105}$$

$$2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)=0, \tag{2.106}$$

$$2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)=0, \tag{2.107}$$

$$a_7l_4+a_8l_5+a_9l_6=0, \tag{2.108}$$

$$a_7l_7+a_8l_8+a_9l_9=0. \tag{2.109}$$

Prove. By Z_1 transformation, we set

$$u=u(p,q,r), A(t,x,y)=A(p,q,r), \tag{2.110}$$

and

$$p=l_1t+l_2x+l_3y, q=l_4t+l_5x+l_6y, r=l_7t+l_8x+l_9y,$$

where $l_1, l_2, \ldots l_9$ are undetermined constants, and

$$\frac{\partial\left(p,q,r\right)}{\partial\left(t,x,y\right)}=-l_3l_5l_7+l_2l_6l_7+l_3l_4l_8-l_1l_6l_8-l_2l_4l_9+l_1l_5l_9\neq 0.$$

By Eq. (2.99), we get

$$t=-\frac{-rl_3l_5+rl_2l_6+ql_3l_8-pl_6l_8-ql_2l_9+pl_5l_9}{l_3l_5l_7-l_2l_6l_7-l_3l_4l_8+l_1l_6l_8+l_2l_4l_9-l_1l_5l_9}, \tag{2.111}$$

$$x=-\frac{rl_3l_4-rl_1l_6-ql_3l_7+pl_6l_7+ql_1l_9-pl_4l_9}{l_3l_5l_7-l_2l_6l_7-l_3l_4l_8+l_1l_6l_8+l_2l_4l_9-l_1l_5l_9}, \tag{2.112}$$

$$y=\frac{rl_2l_4-rl_1l_5-ql_2l_7+pl_5l_7+ql_1l_8-pl_4l_8}{l_3l_5l_7-l_2l_6l_7-l_3l_4l_8+l_1l_6l_8+l_2l_4l_9-l_1l_5l_9}. \tag{2.113}$$

So

$$
\begin{aligned}
& a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y \\
& =a_1\left(l_1^2u_{pp}+l_4^2u_{qq}+l_7^2u_{rr}+2l_1l_4u_{pq}+2l_1l_7u_{pr}+2l_4l_7u_{qr}\right) \\
& +a_2\left(l_2^2u_{pp}+l_5^2u_{qq}+l_8^2u_{rr}+2l_2l_5u_{pq}+2l_2l_8u_{pr}+2l_5l_8u_{qr}\right) \\
& +a_3\left(l_3^2u_{pp}+l_6^2u_{qq}+l_9^2u_{rr}+2l_3l_6u_{pq}+2l_3l_9u_{pr}+2l_6l_9u_{qr}\right) \\
& +a_4\left(l_1l_2u_{pp}+l_4l_5u_{qq}+l_7l_8u_{rr}+\left(l_1l_5+l_2l_4\right)u_{pq}+\left(l_1l_8+l_2l_7\right)u_{pr}+\left(l_4l_8+l_5l_7\right)u_{qr}\right) \\
& +a_5\left(l_1l_3u_{pp}+l_4l_6u_{qq}+l_7l_9u_{rr}+\left(l_1l_6+l_3l_4\right)u_{pq}+\left(l_1l_9+l_3l_7\right)u_{pr}+\left(l_4l_9+l_6l_7\right)u_{qr}\right) \\
& +a_6\left(l_2l_3u_{pp}+l_5l_6u_{qq}+l_8l_9u_{rr}+\left(l_2l_6+l_3l_5\right)u_{pq}+\left(l_2l_9+l_3l_8\right)u_{pr}+\left(l_5l_9+l_6l_8\right)u_{qr}\right) \\
& +a_7\left(l_1u_p+l_4u_q+l_7u_r\right)+a_8\left(l_2u_p+l_5u_q+l_8u_r\right)+a_9\left(l_3u_p+l_6u_q+l_9u_r\right) \\
& =\left(a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3\right)u_{pp} \\
& +\left(a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6\right)u_{qq} \\
& +\left(a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9\right)u_{rr} \\
& +\left(2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right)\right)u_{pq} \\
& +\left(2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)\right)u_{pr} \\
& +\left(2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)\right)u_{qr} \\
& +\left(a_7l_1+a_8l_2+a_9l_3\right)u_p+\left(a_7l_4+a_8l_5+a_9l_6\right)u_q+\left(a_7l_7+a_8l_8+a_9l_9\right)u_r=A\left(p,q,r\right).
\end{aligned}
\tag{2.114}
$$

Set

$$
\begin{gathered}
a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3=0, \\
a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6=0, \\
a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9=0, \\
2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)=0, \\
2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)=0, \\
a_7l_4+a_8l_5+a_9l_6=0, \\
a_7l_7+a_8l_8+a_9l_9=0.
\end{gathered}
$$

Then

$$
\begin{aligned}
& a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y \\
& =\left(2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right)\right)u_{pq} \\
& +\left(a_7l_1+a_8l_2+a_9l_3\right)u_p=A\left(p,q,r\right).
\end{aligned}
$$

Set

$$u_p=w.$$

So

$$
\begin{aligned}
& \left(2a_1l_1l_4+2a_2l_2l_5+2a_3l_3l_6+a_4\left(l_1l_5+l_2l_4\right)+a_5\left(l_1l_6+l_3l_4\right)+a_6\left(l_2l_6+l_3l_5\right)\right)w_q \\
& +\left(a_7l_1+a_8l_2+a_9l_3\right)w=A\left(p,q,r\right).
\end{aligned}
\tag{2.115}
$$

The solution of Eq. (2.115) is

$$w=\mathrm{e}^{\frac{-\kappa q}{\lambda}}\left(g\left(p,r\right)+\frac{1}{\lambda}\int\mathrm{e}^{\frac{\kappa q}{\lambda}}A\left(p,q,r\right)dq\right),$$

$$\kappa=a_7l_1+a_8l_2+a_9l_3,$$

$$\lambda = 2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5).$$

Therefore

$$u = f(q,r) + \int w dp = f(q,r) + \mathrm{e}^{\frac{-\kappa q}{\lambda}}\left(\int g(p,r)\,dp + \frac{1}{\lambda}\iint \mathrm{e}^{\frac{\kappa q}{\lambda}} A(p,q,r)\,dpdq\right).$$

So the theorem is proved. □

In (2.114), if set

$$a_1l_4^2 + a_2l_5^2 + a_3l_6^2 + a_4l_4l_5 + a_5l_4l_6 + a_6l_5l_6 = 0,$$

$$a_1l_7^2 + a_2l_8^2 + a_3l_9^2 + a_4l_7l_8 + a_5l_7l_9 + a_6l_8l_9 = 0,$$

$$2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5) = 0,$$

$$2a_1l_1l_7 + 2a_2l_2l_8 + 2a_3l_3l_9 + a_4(l_1l_8 + l_2l_7) + a_5(l_1l_9 + l_3l_7) + a_6(l_2l_9 + l_3l_8) = 0,$$

$$2a_1l_4l_7 + 2a_2l_5l_8 + 2a_3l_6l_9 + a_4(l_4l_8 + l_5l_7) + a_5(l_4l_9 + l_6l_7) + a_6(l_5l_9 + l_6l_8) = 0,$$

$$a_7l_4 + a_8l_5 + a_9l_6 = 0,$$

$$a_7l_7 + a_8l_8 + a_9l_9 = 0.$$

Then

$$\begin{aligned}&a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y\\ &= \left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3\right)u_{pp} + (a_7l_1 + a_8l_2 + a_9l_3)\,u_p\\ &= A(p,q,r).\end{aligned}$$

Set

$$u_p = w.$$

So

$$\left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3\right)w_p + (a_7l_1 + a_8l_2 + a_9l_3)\,w = A(p,q,r). \quad (2.116)$$

The solution of Eq. (2.116) is

$$w = \mathrm{e}^{\frac{-\kappa p}{\tau}}\left(g(q,r) + \frac{1}{\tau}\int \mathrm{e}^{\frac{\kappa p}{\tau}} A(p,q,r)\,dp\right),$$

$$\kappa = a_7l_1 + a_8l_2 + a_9l_3,$$

$$\tau = a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3.$$

Thereupon

$$\begin{aligned}u &= f(q,r) + \int w dp = f(q,r) - \frac{\tau}{\kappa}\mathrm{e}^{\frac{-\kappa p}{\tau}} g(q,r) + \frac{1}{\tau}\int\left(\mathrm{e}^{\frac{-\kappa p}{\tau}}\int \mathrm{e}^{\frac{\kappa p}{\tau}} A(p,q,r)\,dp\right)dp\\ &= f(q,r) + \mathrm{e}^{\frac{-\kappa p}{\tau}} g(q,r) + \frac{1}{\tau}\int\left(\mathrm{e}^{\frac{-\kappa p}{\tau}}\int \mathrm{e}^{\frac{\kappa p}{\tau}} A(p,q,r)\,dp\right)dp.\end{aligned}$$

Note that $g(q,r)$ is a random second differentiable function, then $-\frac{\tau}{\kappa}\mathrm{e}^{\frac{-\kappa p}{\tau}} g(q,r) = \mathrm{e}^{\frac{-\kappa p}{\tau}} g(q,r)$. So we can get Theorem 18.

Theorem 18. *In $\mathbb{R}^3$, the general solution of*

$$a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y = A(t,x,y),$$

is

$$u = f(q,r) + \mathrm{e}^{\frac{-\kappa p}{\tau}} g(q,r) + \frac{1}{\tau}\int \left(\mathrm{e}^{\frac{-\kappa p}{\tau}} \int \mathrm{e}^{\frac{\kappa p}{\tau}} A(p,q,r)\,dp\right) dp, \tag{2.117}$$

where f and g are random second differentiable functions, and

$$p = l_1 t + l_2 x + l_3 y, q = l_4 t + l_5 x + l_6 y, r = l_7 t + l_8 x + l_9 y,$$

$$-l_3 l_5 l_7 + l_2 l_6 l_7 + l_3 l_4 l_8 - l_1 l_6 l_8 - l_2 l_4 l_9 + l_1 l_5 l_9 \neq 0,$$

$$\kappa = a_7 l_1 + a_8 l_2 + a_9 l_3,$$

$$\tau = a_1 l_1^2 + a_2 l_2^2 + a_3 l_3^2 + a_4 l_1 l_2 + a_5 l_1 l_3 + a_6 l_2 l_3,$$

$$a_1 l_4^2 + a_2 l_5^2 + a_3 l_6^2 + a_4 l_4 l_5 + a_5 l_4 l_6 + a_6 l_5 l_6 = 0,$$

$$a_1 l_7^2 + a_2 l_8^2 + a_3 l_9^2 + a_4 l_7 l_8 + a_5 l_7 l_9 + a_6 l_8 l_9 = 0,$$

$$2a_1 l_1 l_4 + 2a_2 l_2 l_5 + 2a_3 l_3 l_6 + a_4 (l_1 l_5 + l_2 l_4) + a_5 (l_1 l_6 + l_3 l_4) + a_6 (l_2 l_6 + l_3 l_5) = 0,$$

$$2a_1 l_1 l_7 + 2a_2 l_2 l_8 + 2a_3 l_3 l_9 + a_4 (l_1 l_8 + l_2 l_7) + a_5 (l_1 l_9 + l_3 l_7) + a_6 (l_2 l_9 + l_3 l_8) = 0,$$

$$2a_1 l_4 l_7 + 2a_2 l_5 l_8 + 2a_3 l_6 l_9 + a_4 (l_4 l_8 + l_5 l_7) + a_5 (l_4 l_9 + l_6 l_7) + a_6 (l_5 l_9 + l_6 l_8) = 0,$$

$$a_7 l_4 + a_8 l_5 + a_9 l_6 = 0,$$

$$a_7 l_7 + a_8 l_8 + a_9 l_9 = 0.$$

Next we propose Theorem 19.

Theorem 19. *In $\mathbb{R}^3$,*

$$a_1 u_{tt} + a_2 u_{xx} + a_3 u_{yy} + a_4 u_{tx} + a_5 u_{ty} + a_6 u_{xy} + a_7 u_t + a_8 u_x + a_9 u_y + a_{10} u = A(t,x,y), \tag{2.118}$$

the general solution of Eq. (2.118) is

$$u = g(h_1(p,r) + h_2(q,r)) + \frac{g}{B}\iint \frac{A(p,q,r)\,dpdq}{g}, \tag{2.119}$$

where h_1 and h_2 are random second differentiable functions, and

$$p = l_1 t + l_2 x + l_3 y, q = l_4 t + l_5 x + l_6 y, r = l_7 t + l_8 x + l_9 y,$$

$$g = C_1 \mathrm{e}^{\lambda_1 v} + C_2 \mathrm{e}^{\lambda_2 v}, v = k_1 t + k_2 x + k_3 y + k_4, \tag{2.120}$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4aa_{10}}}{2a}, \lambda_2 = \frac{-b - \sqrt{b^2 - 4aa_{10}}}{2a}, \tag{2.121}$$

$$a = a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_1 k_2 + a_5 k_1 k_3 + a_6 k_2 k_3, \tag{2.122}$$

$$b = a_7 k_1 + a_8 k_2 + a_9 k_3, \tag{2.123}$$

$$B = 2a_1 l_1 l_4 + 2a_2 l_2 l_5 + 2a_3 l_3 l_6 + a_4 (l_1 l_5 + l_2 l_4) + a_5 (l_1 l_6 + l_3 l_4) + a_6 (l_2 l_6 + l_3 l_5), \tag{2.124}$$

where $k_1 - k_3$ and $l_1 - l_9$ are constants which satisfy

$$b^2 - 4aa_{10} > 0, \tag{2.125}$$

$$-l_3 l_5 l_7 + l_2 l_6 l_7 + l_3 l_4 l_8 - l_1 l_6 l_8 - l_2 l_4 l_9 + l_1 l_5 l_9 \neq 0,$$

$$k_1\lambda_1\left(2a_1l_1+a_4l_2+a_5l_3\right)+k_2\lambda_1\left(2a_2l_2+a_4l_1+a_6l_3\right)+k_3\lambda_1\left(2a_3l_3+a_5l_1+a_6l_2\right) \\ +a_7l_1+a_8l_2+a_9l_3=0, \tag{2.126}$$

$$k_1\lambda_2\left(2a_1l_1+a_4l_2+a_5l_3\right)+k_2\lambda_2\left(2a_2l_2+a_4l_1+a_6l_3\right)+k_3\lambda_2\left(2a_3l_3+a_5l_1+a_6l_2\right) \\ +a_7l_1+a_8l_2+a_9l_3=0, \tag{2.127}$$

$$k_1\lambda_1\left(2a_1l_4+a_4l_5+a_5l_6\right)+k_2\lambda_1\left(2a_2l_5+a_4l_4+a_6l_6\right)+k_3\lambda_1\left(2a_3l_6+a_5l_4+a_6l_5\right) \\ +a_7l_4+a_8l_5+a_9l_6=0, \tag{2.128}$$

$$k_1\lambda_2\left(2a_1l_4+a_4l_5+a_5l_6\right)+k_2\lambda_2\left(2a_2l_5+a_4l_4+a_6l_6\right)+k_3\lambda_2\left(2a_3l_6+a_5l_4+a_6l_5\right) \\ +a_7l_4+a_8l_5+a_9l_6=0, \tag{2.129}$$

$$k_1\lambda_1\left(2a_1l_7+a_4l_8+a_5l_9\right)+k_2\lambda_1\left(2a_2l_8+a_4l_7+a_6l_9\right)+k_3\lambda_1\left(2a_3l_9+a_5l_7+a_6l_8\right)) \\ +a_7l_7+a_8l_8+a_9l_9=0, \tag{2.130}$$

$$k_1\lambda_2\left(2a_1l_7+a_4l_8+a_5l_9\right)+k_2\lambda_2\left(2a_2l_8+a_4l_7+a_6l_9\right)+k_3\lambda_2\left(2a_3l_9+a_5l_7+a_6l_8\right) \\ +a_7l_7+a_8l_8+a_9l_9=0, \tag{2.131}$$

$$a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3=0, \tag{2.132}$$

$$a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6=0, \tag{2.133}$$

$$a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9=0, \tag{2.134}$$

$$2a_1l_1l_7+2a_2l_2l_8+2a_3l_3l_9+a_4\left(l_1l_8+l_2l_7\right)+a_5\left(l_1l_9+l_3l_7\right)+a_6\left(l_2l_9+l_3l_8\right)=0, \tag{2.135}$$

$$2a_1l_4l_7+2a_2l_5l_8+2a_3l_6l_9+a_4\left(l_4l_8+l_5l_7\right)+a_5\left(l_4l_9+l_6l_7\right)+a_6\left(l_5l_9+l_6l_8\right)=0. \tag{2.136}$$

Prove. By Z_1 transformation, we set

$$u\left(t,x,y\right)=f\left(v\right)=f\left(k_1t+k_2x+k_3y+k_4\right),$$

where $v(t,x,y)=k_1t+k_2x+k_3y+k_4$; $k_1,k_2,\ldots k_4$ are undetermined constants, f is an undetermined second differentiable function, so

$$a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{tx}+a_5u_{ty}+a_6u_{xy}+a_7u_t+a_8u_x+a_9u_y+a_{10}u \\ =\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3\right)f_v^{''}+\left(a_7k_1+a_8k_2+a_9k_3\right)f_v^{'}+a_{10}f=0. \tag{2.137}$$

The characteristic equation of Eq. (2.137) is

$$\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3\right)\lambda^2+\left(a_7k_1+a_8k_2+a_9k_3\right)\lambda+a_{10}=0.$$

So

$$\lambda=\frac{-b\pm\sqrt{b^2-4aa_{10}}}{2a},$$
$$a=a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_1k_2+a_5k_1k_3+a_6k_2k_3,$$
$$b=a_7k_1+a_8k_2+a_9k_3.$$

If

$$b^2-4aa_{10}>0,$$

the particular solution of Eq. (2.118) is

$$u=f=C_1\mathrm{e}^{\lambda_1v}+C_2\mathrm{e}^{\lambda_2v},$$

where

$$\lambda_1 = \frac{-b+\sqrt{b^2-4aa_{10}}}{2a}\lambda_2 = \frac{-b-\sqrt{b^2-4aa_{10}}}{2a}. \tag{2.138}$$

For getting the general solution of Eq. (2.118), by Z_3 transformation, set

$$u = g(t,x,y)h(p,q,r),$$

and

$$p = l_1t + l_2x + l_3y, q = l_4t + l_5x + l_6y, r = l_7t + l_8x + l_9y,$$

where $l_1 - l_9$ are undetermined constants, h and g are undetermined second differentiable functions, and

$$\frac{\partial(p,q,r)}{\partial(t,x,y)} = -l_3l_5l_7 + l_2l_6l_7 + l_3l_4l_8 - l_1l_6l_8 - l_2l_4l_9 + l_1l_5l_9 \neq 0.$$

Then

$$\begin{aligned}
&a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u\\
&= a_1\left(g_{tt}h + 2g_t\left(l_1h_p + l_4h_q + l_7h_r\right) + g\left(l_1^2h_{pp} + l_4^2h_{qq} + l_7^2h_{rr} + 2l_1l_4h_{pq} + 2l_1l_7h_{pr} + 2l_4l_7h_{qr}\right)\right)\\
&+ a_2\left(g_{xx}h + 2g_x\left(l_2h_p + l_5h_q + l_8h_r\right) + g\left(l_2^2h_{pp} + l_5^2h_{qq} + l_8^2h_{rr} + 2l_2l_5h_{pq} + 2l_2l_8h_{pr} + 2l_5l_8h_{qr}\right)\right)\\
&+ a_3\left(g_{yy}h + 2g_y\left(l_3h_p + l_6h_q + l_9h_r\right) + g\left(l_3^2h_{pp} + l_6^2h_{qq} + l_9^2h_{rr} + 2l_3l_6h_{pq} + 2l_3l_9h_{pr} + 2l_6l_9h_{qr}\right)\right)\\
&+ a_4\left(g_{tx}h + g_t\left(l_2h_p + l_5h_q + l_8h_r\right) + g_x\left(l_1h_p + l_4h_q + l_7h_r\right)\right))\\
&+ a_4g\left(l_1l_2h_{pp} + l_4l_5h_{qq} + l_7l_8h_{rr} + \left(l_1l_5 + l_2l_4\right)h_{pq} + \left(l_1l_8 + l_2l_7\right)h_{pr} + \left(l_4l_8 + l_5l_7\right)h_{qr}\right)\\
&+ a_5\left(g_{ty}h + g_t\left(l_3h_p + l_6h_q + l_9h_r\right) + g_y\left(l_1h_p + l_4h_q + l_7h_r\right)\right)\\
&+ a_5g\left(l_1l_3h_{pp} + l_4l_6h_{qq} + l_7l_9h_{rr} + \left(l_1l_6 + l_3l_4\right)h_{pq} + \left(l_1l_9 + l_3l_7\right)h_{pr} + \left(l_4l_9 + l_6l_7\right)h_{qr}\right)\\
&+ a_6\left(g_{xy}h + g_x\left(l_3h_p + l_6h_q + l_9h_r\right) + g_y\left(l_2h_p + l_5h_q + l_8h_r\right)\right)\\
&+ a_6g\left(l_2l_3h_{pp} + l_5l_6h_{qq} + l_8l_9h_{rr} + \left(l_2l_6 + l_3l_5\right)h_{pq} + \left(l_2l_9 + l_3l_8\right)h_{pr} + \left(l_5l_9 + l_6l_8\right)h_{qr}\right)\\
&+ a_7\left(g_th + g\left(l_1h_p + l_4h_q + l_7h_r\right)\right) + a_8\left(g_xh + g\left(l_2h_p + l_5h_q + l_8h_r\right)\right)\\
&+ a_9\left(g_yh + g\left(l_3h_p + l_6h_q + l_9h_r\right)\right) + a_{10}gh\\
&= \left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3\right)gh_{pp}\\
&+ \left(a_1l_4^2 + a_2l_5^2 + a_3l_6^2 + a_4l_4l_5 + a_5l_4l_6 + a_6l_5l_6\right)gh_{qq}\\
&+ \left(a_1l_7^2 + a_2l_8^2 + a_3l_9^2 + a_4l_7l_8 + a_5l_7l_9 + a_6l_8l_9\right)gh_{rr}\\
&+ \left(2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4\left(l_1l_5 + l_2l_4\right) + a_5\left(l_1l_6 + l_3l_4\right) + a_6\left(l_2l_6 + l_3l_5\right)\right)gh_{pq}\\
&+ \left(2a_1l_1l_7 + 2a_2l_2l_8 + 2a_3l_3l_9 + a_4\left(l_1l_8 + l_2l_7\right) + a_5\left(l_1l_9 + l_3l_7\right) + a_6\left(l_2l_9 + l_3l_8\right)\right)gh_{pr}\\
&+ \left(2a_1l_4l_7 + 2a_2l_5l_8 + 2a_3l_6l_9 + a_4\left(l_4l_8 + l_5l_7\right) + a_5\left(l_4l_9 + l_6l_7\right) + a_6\left(l_5l_9 + l_6l_8\right)\right)gh_{qr}\\
&+ \left(\left(2a_1l_1 + a_4l_2 + a_5l_3\right)g_t + \left(2a_2l_2 + a_4l_1 + a_6l_3\right)g_x + \left(2a_3l_3 + a_5l_1 + a_6l_2\right)g_y\right)h_p\\
&+ (a_7l_1 + a_8l_2 + a_9l_3)gh_p\\
&+ \left(\left(2a_1l_4 + a_4l_5 + a_5l_6\right)g_t + \left(2a_2l_5 + a_4l_4 + a_6l_6\right)g_x + \left(2a_3l_6 + a_5l_4 + a_6l_5\right)g_y\right)h_q\\
&+ \left(a_7l_4 + a_8l_5 + a_9l_6\right)gh_q\\
&+ \left(\left(2a_1l_7 + a_4l_8 + a_5l_9\right)g_t + \left(2a_2l_8 + a_4l_7 + a_6l_9\right)g_x + \left(2a_3l_9 + a_5l_7 + a_6l_8\right)g_y\right)h_r\\
&+ \left(a_7l_7 + a_8l_8 + a_9l_9\right)gh_r\\
&+ \left(a_1g_{tt} + a_2g_{xx} + a_3g_{yy} + a_4g_{tx} + a_5g_{ty} + a_6g_{xy} + a_7g_t + a_8g_x + a_9g_y + a_{10}g\right)h.
\end{aligned} \tag{2.139}$$

Set

$$a_1g_{tt} + a_2g_{xx} + a_3g_{yy} + a_4g_{tx} + a_5g_{ty} + a_6g_{xy} + a_7g_t + a_8g_x + a_9g_y + a_{10}g = 0, \tag{2.140}$$

$$\begin{aligned}&(2a_1l_1+a_4l_2+a_5l_3)\,g_t+(2a_2l_2+a_4l_1+a_6l_3)\,g_x+(2a_3l_3+a_5l_1+a_6l_2)\,g_y\\&+(a_7l_1+a_8l_2+a_9l_3)\,g=0,\end{aligned}\tag{2.141}$$

$$\begin{aligned}&(2a_1l_4+a_4l_5+a_5l_6)\,g_t+(2a_2l_5+a_4l_4+a_6l_6)\,g_x+(2a_3l_6+a_5l_4+a_6l_5)\,g_y\\&+(a_7l_4+a_8l_5+a_9l_6)\,g=0,\end{aligned}\tag{2.142}$$

$$\begin{aligned}&(2a_1l_7+a_4l_8+a_5l_9)\,g_t+(2a_2l_8+a_4l_7+a_6l_9)\,g_x+(2a_3l_9+a_5l_7+a_6l_8)\,g_y\\&+(a_7l_7+a_8l_8+a_9l_9)\,g=0.\end{aligned}\tag{2.143}$$

Set $g=g(v), v=k_1t+k_2x+k_3y+k_4$, the particular solution of Eq. (2.140) is

$$g=C_1\mathrm{e}^{\lambda_1 v}+C_2\mathrm{e}^{\lambda_2 v},\tag{2.144}$$

where λ_1 and λ_2 satisfy Eq. (2.138). Substituting (2.144) into (2.141) we have

$$\begin{aligned}&(2a_1l_1+a_4l_2+a_5l_3)\,g_t+(2a_2l_2+a_4l_1+a_6l_3)\,g_x+(2a_3l_3+a_5l_1+a_6l_2)\,g_y\\&+(a_7l_1+a_8l_2+a_9l_3)\,g\\&=(2a_1l_1+a_4l_2+a_5l_3)\left(C_1k_1\lambda_1\mathrm{e}^{\lambda_1 v}+C_2k_1\lambda_2\mathrm{e}^{\lambda_2 v}\right)\\&+(2a_2l_2+a_4l_1+a_6l_3)\left(C_1k_2\lambda_1\mathrm{e}^{\lambda_1 v}+C_2k_2\lambda_2\mathrm{e}^{\lambda_2 v}\right)\\&+(2a_3l_3+a_5l_1+a_6l_2)\left(C_1k_3\lambda_1\mathrm{e}^{\lambda_1 v}+C_2k_3\lambda_2\mathrm{e}^{\lambda_2 v}\right)\\&+(a_7l_1+a_8l_2+a_9l_3)\left(C_1\mathrm{e}^{\lambda_1 v}+C_2\mathrm{e}^{\lambda_2 v}\right)\\&=k_1\lambda_1C_1\mathrm{e}^{\lambda_1 v}(2a_1l_1+a_4l_2+a_5l_3)+k_2\lambda_1C_1\mathrm{e}^{\lambda_1 v}(2a_2l_2+a_4l_1+a_6l_3)\\&+k_3\lambda_1C_1\mathrm{e}^{\lambda_1 v}(2a_3l_3+a_5l_1+a_6l_2)+C_1\mathrm{e}^{\lambda_1 v}(a_7l_1+a_8l_2+a_9l_3)\\&+k_1\lambda_2C_2\mathrm{e}^{\lambda_2 v}(2a_1l_1+a_4l_2+a_5l_3)+k_2\lambda_2C_2\mathrm{e}^{\lambda_2 v}(2a_2l_2+a_4l_1+a_6l_3)\\&+k_3\lambda_2C_2\mathrm{e}^{\lambda_2 v}(2a_3l_3+a_5l_1+a_6l_2)+C_2\mathrm{e}^{\lambda_2 v}(a_7l_1+a_8l_2+a_9l_3)=0.\end{aligned}$$

Namely

$$\begin{aligned}&k_1\lambda_1(2a_1l_1+a_4l_2+a_5l_3)+k_2\lambda_1(2a_2l_2+a_4l_1+a_6l_3)+k_3\lambda_1(2a_3l_3+a_5l_1+a_6l_2)\\&+a_7l_1+a_8l_2+a_9l_3=0\end{aligned}$$

$$\begin{aligned}&k_1\lambda_2(2a_1l_1+a_4l_2+a_5l_3)+k_2\lambda_2(2a_2l_2+a_4l_1+a_6l_3)+k_3\lambda_2(2a_3l_3+a_5l_1+a_6l_2)\\&+a_7l_1+a_8l_2+a_9l_3=0.\end{aligned}$$

Substituting (2.144) into (2.142,2.143) respectively, we have

$$\begin{aligned}&k_1\lambda_1(2a_1l_4+a_4l_5+a_5l_6)+k_2\lambda_1(2a_2l_5+a_4l_4+a_6l_6)+k_3\lambda_1(2a_3l_6+a_5l_4+a_6l_5)\\&+a_7l_4+a_8l_5+a_9l_6=0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_2(2a_1l_4+a_4l_5+a_5l_6)+k_2\lambda_2(2a_2l_5+a_4l_4+a_6l_6)+k_3\lambda_2(2a_3l_6+a_5l_4+a_6l_5)\\&+a_7l_4+a_8l_5+a_9l_6=0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_1(2a_1l_7+a_4l_8+a_5l_9)+k_2\lambda_1(2a_2l_8+a_4l_7+a_6l_9)+k_3\lambda_1(2a_3l_9+a_5l_7+a_6l_8)\\&+a_7l_7+a_8l_8+a_9l_9=0,\end{aligned}$$

$$\begin{aligned}&k_1\lambda_2(2a_1l_7+a_4l_8+a_5l_9)+k_2\lambda_2(2a_2l_8+a_4l_7+a_6l_9)+k_3\lambda_2(2a_3l_9+a_5l_7+a_6l_8)\\&+a_7l_7+a_8l_8+a_9l_9=0.\end{aligned}$$

Case 1:
In (2.139), set

$$\begin{aligned}a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_1l_2+a_5l_1l_3+a_6l_2l_3&=0,\\a_1l_4^2+a_2l_5^2+a_3l_6^2+a_4l_4l_5+a_5l_4l_6+a_6l_5l_6&=0,\\a_1l_7^2+a_2l_8^2+a_3l_9^2+a_4l_7l_8+a_5l_7l_9+a_6l_8l_9&=0,\end{aligned}$$

$$2a_1l_1l_7 + 2a_2l_2l_8 + 2a_3l_3l_9 + a_4(l_1l_8 + l_2l_7) + a_5(l_1l_9 + l_3l_7) + a_6(l_2l_9 + l_3l_8) = 0,$$

$$2a_1l_4l_7 + 2a_2l_5l_8 + 2a_3l_6l_9 + a_4(l_4l_8 + l_5l_7) + a_5(l_4l_9 + l_6l_7) + a_6(l_5l_9 + l_6l_8) = 0.$$

Then

$$\begin{aligned}&a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u \\ &= (2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5))\, gh_{pq} \\ &= A(p,q,r).\end{aligned}$$

That is

$$h_{pq} = \frac{A(p,q,r)}{(2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5))\, g}, \tag{2.145}$$

the general solution of Eq. (2.145) is

$$h = h_1(p,r) + h_2(q,r) + \iint \frac{A(p,q,r)\, dpdq}{Bg},$$

where

$$B = 2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5).$$

So the general solution of Eq. (2.118) is

$$u = gh = g(h_1(p,r) + h_2(q,r)) + \frac{g}{B}\iint \frac{A(p,q,r)\, dpdq}{g}.$$

Whereupon the theorem is proved. □

Case 2:
In (2.139), set

$$a_1l_4^2 + a_2l_5^2 + a_3l_6^2 + a_4l_4l_5 + a_5l_4l_6 + a_6l_5l_6 = 0,$$

$$a_1l_7^2 + a_2l_8^2 + a_3l_9^2 + a_4l_7l_8 + a_5l_7l_9 + a_6l_8l_9 = 0,$$

$$2a_1l_1l_7 + 2a_2l_2l_8 + 2a_3l_3l_9 + a_4(l_1l_8 + l_2l_7) + a_5(l_1l_9 + l_3l_7) + a_6(l_2l_9 + l_3l_8) = 0,$$

$$2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5) = 0,$$

$$2a_1l_4l_7 + 2a_2l_5l_8 + 2a_3l_6l_9 + a_4(l_4l_8 + l_5l_7) + a_5(l_4l_9 + l_6l_7) + a_6(l_5l_9 + l_6l_8) = 0.$$

Then

$$\begin{aligned}&a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u \\ &= \left(a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3\right) gh_{pp} = A(p,q,r).\end{aligned}$$

Namely

$$h_{pp} = \frac{A(p,q,r)}{Eg}, \tag{2.146}$$

$$E = a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3. \tag{2.147}$$

The general solution of Eq. (2.146) is

$$h = h_1(q,r) + ph_2(q,r) + \iint \frac{A(p,q,r)\, dpdp}{Eg}.$$

Then the general solution of Eq. (2.118) is

$$u = gh = g\left(h_1(q,r) + ph_2(q,r)\right) + \frac{g}{E}\iint \frac{A(p,q,r)\,dpdp}{g}.$$

So we can get Theorem 20.

Theorem 20. *In $\mathbb{R}^3$, the general solution of*

$$a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u = A(t,x,y),$$

is

$$u = g\left(h_1(q,r) + ph_2(q,r)\right) + \frac{g}{E}\iint \frac{A(p,q,r)\,dpdp}{g}, \tag{2.148}$$

where h_1 and h_2 are random second differentiable functions, and

$$g = C_1\mathrm{e}^{\lambda_1 v} + C_2\mathrm{e}^{\lambda_2 v}, v = k_1t + k_2x + k_3y + k_4,$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4aa_{10}}}{2a}, \lambda_2 = \frac{-b - \sqrt{b^2 - 4aa_{10}}}{2a},$$

$$a = a_1k_1^2 + a_2k_2^2 + a_3k_3^2 + a_4k_1k_2 + a_5k_1k_3 + a_6k_2k_3,$$

$$b = a_7k_1 + a_8k_2 + a_9k_3,$$

$$p = l_1t + l_2x + l_3y, q = l_4t + l_5x + l_6y, r = l_7t + l_8x + l_9y,$$

$$E = a_1l_1^2 + a_2l_2^2 + a_3l_3^2 + a_4l_1l_2 + a_5l_1l_3 + a_6l_2l_3,$$

where $k_1 - k_3$ and $l_1 - l_9$ are constants which satisfy

$$b^2 - 4aa_{10} > 0,$$

$$-l_3l_5l_7 + l_2l_6l_7 + l_3l_4l_8 - l_1l_6l_8 - l_2l_4l_9 + l_1l_5l_9 \neq 0,$$

$$k_1\lambda_1(2a_1l_1 + a_4l_2 + a_5l_3) + k_2\lambda_1(2a_2l_2 + a_4l_1 + a_6l_3) + k_3\lambda_1(2a_3l_3 + a_5l_1 + a_6l_2) + a_7l_1 + a_8l_2 + a_9l_3 = 0,$$

$$k_1\lambda_2(2a_1l_1 + a_4l_2 + a_5l_3) + k_2\lambda_2(2a_2l_2 + a_4l_1 + a_6l_3) + k_3\lambda_2(2a_3l_3 + a_5l_1 + a_6l_2) + a_7l_1 + a_8l_2 + a_9l_3 = 0,$$

$$k_1\lambda_1(2a_1l_4 + a_4l_5 + a_5l_6) + k_2\lambda_1(2a_2l_5 + a_4l_4 + a_6l_6) + k_3\lambda_1(2a_3l_6 + a_5l_4 + a_6l_5) + a_7l_4 + a_8l_5 + a_9l_6 = 0,$$

$$k_1\lambda_2(2a_1l_4 + a_4l_5 + a_5l_6) + k_2\lambda_2(2a_2l_5 + a_4l_4 + a_6l_6) + k_3\lambda_2(2a_3l_6 + a_5l_4 + a_6l_5) + a_7l_4 + a_8l_5 + a_9l_6 = 0,$$

$$k_1\lambda_1(2a_1l_7 + a_4l_8 + a_5l_9) + k_2\lambda_1(2a_2l_8 + a_4l_7 + a_6l_9) + k_3\lambda_1(2a_3l_9 + a_5l_7 + a_6l_8) + a_7l_7 + a_8l_8 + a_9l_9 = 0,$$

$$k_1\lambda_2(2a_1l_7 + a_4l_8 + a_5l_9) + k_2\lambda_2(2a_2l_8 + a_4l_7 + a_6l_9) + k_3\lambda_2(2a_3l_9 + a_5l_7 + a_6l_8) + a_7l_7 + a_8l_8 + a_9l_9 = 0,$$

$$a_1l_4^2 + a_2l_5^2 + a_3l_6^2 + a_4l_4l_5 + a_5l_4l_6 + a_6l_5l_6 = 0,$$

$$a_1l_7^2 + a_2l_8^2 + a_3l_9^2 + a_4l_7l_8 + a_5l_7l_9 + a_6l_8l_9 = 0,$$

$$2a_1l_1l_7 + 2a_2l_2l_8 + 2a_3l_3l_9 + a_4(l_1l_8 + l_2l_7) + a_5(l_1l_9 + l_3l_7) + a_6(l_2l_9 + l_3l_8) = 0,$$

$$2a_1l_1l_4 + 2a_2l_2l_5 + 2a_3l_3l_6 + a_4(l_1l_5 + l_2l_4) + a_5(l_1l_6 + l_3l_4) + a_6(l_2l_6 + l_3l_5) = 0,$$
$$2a_1l_4l_7 + 2a_2l_5l_8 + 2a_3l_6l_9 + a_4(l_4l_8 + l_5l_7) + a_5(l_4l_9 + l_6l_7) + a_6(l_5l_9 + l_6l_8) = 0.$$

For other cases and $b^2 - 4aa_{10} = 0, b^2 - 4aa_{10} < 0$, similar calculations could be done.

Eq. (2.118) is a significant linear PDEs. Two-dimensional heat equation, wave equation, Fokker-Planck Equation,[26−28] Telegraph Equation,[29−31] etc. are all special cases of them.

Next we propose Theorem 21,22.

Theorem 21. *In* $\mathbb{R}^2$,

$$a_1u_{ttt} + a_2u_{xxx} + a_3u_{ttx} + a_4u_{txx} = A(t,x), \tag{2.149}$$

the general solution of Eq. (2.149) is

$$u = f(q) + pg(q) + p^2h(q) + \frac{\iiint A(p,q)\,dpdpdp}{a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2}, \tag{2.150}$$

where f, g *and* h *are random third differentiable functions, and*

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

$$k_1k_4 - k_2k_3 \neq 0,$$

$$a_1k_3^3 + a_2k_4^3 + a_3k_3^2k_4 + a_4k_3k_4^2 = 0, \tag{2.151}$$

$$3a_1k_1k_3^2 + 3a_2k_2k_4^2 + a_3(k_2k_3^2 + 2k_1k_3k_4) + a_4(k_1k_4^2 + 2k_2k_3k_4) = 0, \tag{2.152}$$

$$3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3(k_1^2k_4 + 2k_1k_2k_3) + a_4(k_2^2k_3 + 2k_1k_2k_4) = 0. \tag{2.153}$$

Prove. By Z_1 transformation, we set

$$u(t,x) = u(p,q),$$

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

and

$$t = \frac{pk_4 - qk_2}{k_1k_4 - k_2k_3}, x = \frac{qk_1 - pk_3}{k_1k_4 - k_2k_3},$$
$$k_1k_4 - k_2k_3 \neq 0.$$

Namely

$$\begin{aligned}
&a_1u_{ttt} + a_2u_{xxx} + a_3u_{ttx} + a_4u_{txx}\\
&= a_1\left(k_1^3u_{ppp} + k_3^3u_{qqq} + 3k_1k_3^2u_{pqq} + 3k_1^2k_3u_{ppq}\right) + a_2\left(k_2^3u_{ppp} + k_4^3u_{qqq} + 3k_2k_4^2u_{pqq} + 3k_2^2k_4u_{ppq}\right)\\
&+ a_3\left(k_1^2k_2u_{ppp} + k_3^2k_4u_{qqq} + \left(k_2k_3^2 + 2k_1k_3k_4\right)u_{pqq} + \left(k_1^2k_4 + 2k_1k_2k_3\right)u_{ppq}\right)\\
&+ a_4\left(k_1k_2^2u_{ppp} + k_3k_4^2u_{qqq} + \left(k_1k_4^2 + 2k_2k_3k_4\right)u_{pqq} + \left(k_2^2k_3 + 2k_1k_2k_4\right)u_{ppq}\right)\\
&= \left(a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2\right)u_{ppp} + \left(a_1k_3^3 + a_2k_4^3 + a_3k_3^2k_4 + a_4k_3k_4^2\right)u_{qqq}\\
&+ \left(3a_1k_1k_3^2 + 3a_2k_2k_4^2 + a_3\left(k_2k_3^2 + 2k_1k_3k_4\right) + a_4\left(k_1k_4^2 + 2k_2k_3k_4\right)\right)u_{pqq}\\
&+ \left(3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3\left(k_1^2k_4 + 2k_1k_2k_3\right) + a_4\left(k_2^2k_3 + 2k_1k_2k_4\right)\right)u_{ppq}\\
&= A(p,q).
\end{aligned} \tag{2.154}$$

Set

$$a_1k_3^3 + a_2k_4^3 + a_3k_3^2k_4 + a_4k_3k_4^2 = 0,$$
$$3a_1k_1k_3^2 + 3a_2k_2k_4^2 + a_3\left(k_2k_3^2 + 2k_1k_3k_4\right) + a_4\left(k_1k_4^2 + 2k_2k_3k_4\right) = 0,$$
$$3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3\left(k_1^2k_4 + 2k_1k_2k_3\right) + a_4\left(k_2^2k_3 + 2k_1k_2k_4\right) = 0.$$

So

$$a_1u_{ttt} + a_2u_{xxx} + a_3u_{ttx} + a_4u_{txx} = \left(a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2\right)u_{ppp} = A(p,q). \quad (2.155)$$

The general solution of Eq. (2.155) is

$$u = f(q) + pg(q) + p^2h(q) + \frac{\iiint A(p,q)\,dpdpdp}{a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2}.$$

So the theorem is proved. □

In (2.154), if set

$$a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2 = 0,$$
$$a_1k_3^3 + a_2k_4^3 + a_3k_3^2k_4 + a_4k_3k_4^2 = 0,$$
$$3a_1k_1k_3^2 + 3a_2k_2k_4^2 + a_3\left(k_2k_3^2 + 2k_1k_3k_4\right) + a_4\left(k_1k_4^2 + 2k_2k_3k_4\right) = 0.$$

Then

$$\begin{aligned} &a_1u_{ttt} + a_2u_{xxx} + a_3u_{ttx} + a_4u_{txx} \\ &= \left(3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3\left(k_1^2k_4 + 2k_1k_2k_3\right) + a_4\left(k_2^2k_3 + 2k_1k_2k_4\right)\right)u_{ppq} = A(p,q). \end{aligned} \quad (2.156)$$

The general solution of Eq. (2.156) is

$$u = f(p) + g(q) + ph(q) + \frac{\iiint A(p,q)\,dpdpdp}{3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3\left(k_1^2k_4 + 2k_1k_2k_3\right) + a_4\left(k_2^2k_3 + 2k_1k_2k_4\right)}.$$

So we can present Theorem 22.

Theorem 22. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{ttt} + a_2u_{xxx} + a_3u_{ttx} + a_4u_{txx} = A(t,x),$$

is

$$u = f(p) + g(q) + ph(q) + \frac{\iiint A(p,q)\,dpdpdp}{3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3\left(k_1^2k_4 + 2k_1k_2k_3\right) + a_4\left(k_2^2k_3 + 2k_1k_2k_4\right)}, \quad (2.157)$$

where f, g and h are random third differentiable functions, and

$$p = k_1t + k_2x, q = k_3t + k_4x,$$

$$k_1k_4 - k_2k_3 \neq 0,$$
$$a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2 = 0,$$
$$a_1k_3^3 + a_2k_4^3 + a_3k_3^2k_4 + a_4k_3k_4^2 = 0,$$
$$3a_1k_1k_3^2 + 3a_2k_2k_4^2 + a_3\left(k_2k_3^2 + 2k_1k_3k_4\right) + a_4\left(k_1k_4^2 + 2k_2k_3k_4\right) = 0.$$

Next we propose Theorem 23.

Theorem 23. *In* $\mathbb{R}^3$*,*

$$a_1u_{ttt}+a_2u_{xxx}+a_3u_{ttx}+a_4u_{txx}+a_5u_{tt}+a_6u_{xx}+a_7u_{tx}+a_8u_t+a_9u_x=A\left(t,x\right), \quad (2.158)$$

the general solution of Eq. (2.158) is

$$\begin{aligned} u=&f\left(q\right)+g\left(q\right)\int\psi_1\left(p\right)dp+h\left(q\right)\int\psi_2\left(p\right)dp\\ &+\int\left(\int_{p_0}^{p}\frac{\psi_1\left(s\right)\psi_2\left(p\right)-\psi_1\left(p\right)\psi_2\left(s\right)}{\psi_1\left(s\right)\psi_2'\left(s\right)-\psi_1\left(s\right)\psi_1'\left(s\right)}A\left(s,q\right)ds\right)dp, \end{aligned} \quad (2.159)$$

where f,g *and* h *are random third differentiable functions, and*

$$p=k_1t+k_2x,q=k_3t+k_4x,$$

$$k_1k_4-k_2k_3\neq0,$$

$$a_1k_3^3+a_2k_4^3+a_3k_3^2k_4+a_4k_3k_4^2=0, \quad (2.160)$$

$$3a_1k_1k_3^2+3a_2k_2k_4^2+a_3\left(k_2k_3^2+2k_1k_3k_4\right)+a_4\left(k_1k_4^2+2k_2k_3k_4\right)=0, \quad (2.161)$$

$$3a_1k_1^2k_3+3a_2k_2^2k_4+a_3\left(k_1^2k_4+2k_1k_2k_3\right)+a_4\left(k_2^2k_3+2k_1k_2k_4\right)=0, \quad (2.162)$$

$$a_5k_3^2+a_6k_4^2+a_7k_3k_4=0, \quad (2.163)$$

$$2a_5k_1k_3+2a_6k_2k_4+a_7\left(k_1k_4+k_2k_3\right)=0, \quad (2.164)$$

$$a_8k_3+a_9k_4=0, \quad (2.165)$$

$\psi_1(p)$ *and* $\psi_2(p)$ *are two linearly independent particular solutions for*

$$\left(a_1k_1^3+a_2k_2^3+a_3k_1^2k_2+a_4k_1k_2^2\right)w_{pp}+\left(a_5k_1^2+a_6k_2^2+a_7k_1k_2\right)w_p+\left(a_8k_1+a_9k_2\right)w=0$$

.

Prove. By Z_1 transformation, set

$$u(t,x)=u(p,q),$$

$$p=k_1t+k_2x,q=k_3t+k_4x,$$

and

$$k_1k_4-k_2k_3\neq0.$$

Then

$$\begin{aligned} &a_1u_{ttt}+a_2u_{xxx}+a_3u_{ttx}+a_4u_{txx}+a_5u_{tt}+a_6u_{xx}+a_7u_{tx}+a_8u_t+a_9u_x\\ &=a_1\left(k_1^3u_{ppp}+k_3^3u_{qqq}+3k_1k_3^2u_{pqq}+3k_1^2k_3u_{ppq}\right)\\ &+a_2\left(k_2^3u_{ppp}+k_4^3u_{qqq}+3k_2k_4^2u_{pqq}+3k_2^2k_4u_{ppq}\right)\\ &+a_3\left(k_1^2k_2u_{ppp}+k_3^2k_4u_{qqq}+\left(k_2k_3^2+2k_1k_3k_4\right)u_{pqq}+\left(k_1^2k_4+2k_1k_2k_3\right)u_{ppq}\right)\\ &+a_4\left(k_1k_2^2u_{ppp}+k_3k_4^2u_{qqq}+\left(k_1k_4^2+2k_2k_3k_4\right)u_{pqq}+\left(k_2^2k_3+2k_1k_2k_4\right)u_{ppq}\right)\\ &+a_5\left(k_1^2u_{pp}+k_3^2u_{qq}+2k_1k_3u_{pq}\right)+a_6\left(k_2^2u_{pp}+k_4^2u_{qq}+2k_2k_4u_{pq}\right)\\ &+a_7\left(k_1k_2u_{pp}+k_3k_4u_{qq}+\left(k_1k_4+k_2k_3\right)u_{pq}\right)+a_8\left(k_1u_p+k_3u_q\right)+a_9\left(k_2u_p+k_4u_q\right)\\ &=\left(a_1k_1^3+a_2k_2^3+a_3k_1^2k_2+a_4k_1k_2^2\right)u_{ppp}+\left(a_1k_3^3+a_2k_4^3+a_3k_3^2k_4+a_4k_3k_4^2\right)u_{qqq}\\ &+\left(3a_1k_1k_3^2+3a_2k_2k_4^2+a_3\left(k_2k_3^2+2k_1k_3k_4\right)+a_4\left(k_1k_4^2+2k_2k_3k_4\right)\right)u_{pqq}\\ &+\left(3a_1k_1^2k_3+3a_2k_2^2k_4+a_3\left(k_1^2k_4+2k_1k_2k_3\right)+a_4\left(k_2^2k_3+2k_1k_2k_4\right)\right)u_{ppq}\\ &+\left(a_5k_1^2+a_6k_2^2+a_7k_1k_2\right)u_{pp}+\left(a_5k_3^2+a_6k_4^2+a_7k_3k_4\right)u_{qq}\\ &+\left(2a_5k_1k_3+2a_6k_2k_4+a_7\left(k_1k_4+k_2k_3\right)\right)u_{pq}+\left(a_8k_1+a_9k_2\right)u_p+\left(a_8k_3+a_9k_4\right)u_q\\ &=A(p,q). \end{aligned}$$

Set

$$a_1k_3^3 + a_2k_4^3 + a_3k_3^2k_4 + a_4k_3k_4^2 = 0,$$
$$3a_1k_1k_3^2 + 3a_2k_2k_4^2 + a_3\left(k_2k_3^2 + 2k_1k_3k_4\right) + a_4\left(k_1k_4^2 + 2k_2k_3k_4\right) = 0,$$
$$3a_1k_1^2k_3 + 3a_2k_2^2k_4 + a_3\left(k_1^2k_4 + 2k_1k_2k_3\right) + a_4\left(k_2^2k_3 + 2k_1k_2k_4\right) = 0,$$
$$a_5k_3^2 + a_6k_4^2 + a_7k_3k_4 = 0,$$
$$2a_5k_1k_3 + 2a_6k_2k_4 + a_7\left(k_1k_4 + k_2k_3\right) = 0,$$
$$a_8k_3 + a_9k_4 = 0.$$

So

$$\begin{aligned}&a_1u_{ttt} + a_2u_{xxx} + a_3u_{ttx} + a_4u_{txx} + a_5u_{tt} + a_6u_{xx} + a_7u_{tx} + a_8u_t + a_9u_x\\&= \left(a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2\right)u_{ppp} + \left(a_5k_1^2 + a_6k_2^2 + a_7k_1k_2\right)u_{pp} + \left(a_8k_1 + a_9k_2\right)u_p\\&= A(p,q).\end{aligned} \tag{2.166}$$

Set

$$w = u_p.$$

Then

$$\begin{aligned}&\left(a_1k_1^3 + a_2k_2^3 + a_3k_1^2k_2 + a_4k_1k_2^2\right)w_{pp} + \left(a_5k_1^2 + a_6k_2^2 + a_7k_1k_2\right)w_p + \left(a_8k_1 + a_9k_2\right)w\\&= A(p,q).\end{aligned} \tag{2.167}$$

The solution of Eq. (2.167) is

$$w = g\left(q\right)\psi_1\left(p\right) + h\left(q\right)\psi_2\left(p\right) + \int_{p_0}^{p}\frac{\psi_1\left(s\right)\psi_2\left(p\right) - \psi_1\left(p\right)\psi_2\left(s\right)}{\psi_1\left(s\right)\psi_2'\left(s\right) - \psi_1\left(s\right)\psi_1'\left(s\right)}A\left(s,q\right)ds,$$

$\psi_1(p)$ and $\psi_2(p)$ are two linearly independent particular solutions for the homogeneous equation of (2.167), and $g(q), h(q)$ are random unary functions. then

$$\begin{aligned}&u = f\left(q\right) + \int wdp\\&= f\left(q\right) + g\left(q\right)\int\psi_1\left(p\right)dp + h\left(q\right)\int\psi_2\left(p\right)dp + \int\left(\int_{p_0}^{p}\tfrac{\psi_1(s)\psi_2(p) - \psi_1(p)\psi_2(s)}{\psi_1(s)\psi_2'(s) - \psi_1(s)\psi_1'(s)}A\left(s,q\right)ds\right)dp.\end{aligned}$$

So the theorem is proved. □

We propose Theorem 24 as follows.

Theorem 24. *In* $\mathbb{R}^3$*,*

$$\begin{aligned}&a_1u_{ttt} + a_2u_{xxx} + a_3u_{yyy} + a_4u_{ttx} + a_5u_{tty} + a_6u_{txx} + a_7u_{txy} + a_8u_{tyy} + a_9u_{xxy} + a_{10}u_{xyy}\\&= A(t,x,y),\end{aligned} \tag{2.168}$$

the general solution of Eq. (2.168) is

$$u = f\left(p,q\right) + g\left(p,r\right) + h\left(q,r\right) + \frac{1}{B}\iiint A\left(p,q,r\right)dpdqdr, \tag{2.169}$$

where f, g *and* h *are random third differentiable functions, and*

$$p = l_1t + l_2x + l_3y, q = l_4t + l_5x + l_6y, r = l_7t + l_8x + l_9y,$$

$$-l_3l_5l_7 + l_2l_6l_7 + l_3l_4l_8 - l_1l_6l_8 - l_2l_4l_9 + l_1l_5l_9 \neq 0,$$

$$\begin{aligned} B = {} & 6a_1l_1l_4l_7 + 6a_2l_2l_5l_8 + 6a_3l_3l_6l_9 + 2a_4\left(l_1l_4l_8 + l_1l_5l_7 + l_2l_4l_7\right) + 2a_5\left(l_1l_4l_9 + l_1l_6l_7 + l_3l_4l_7\right) \\ & + 2a_6\left(l_1l_5l_8 + l_2l_4l_8 + l_2l_5l_7\right) + a_7\left(l_3\left(l_4l_8 + l_5l_7\right) + l_6\left(l_1l_8 + l_2l_7\right) + l_9\left(l_1l_5 + l_2l_4\right)\right) \\ & + 2a_8\left(l_1l_6l_9 + l_3l_4l_9 + l_3l_6l_7\right) + 2a_9\left(l_2l_5l_9 + l_2l_6l_8 + l_3l_5l_8\right) + a_{10}, \end{aligned} \tag{2.170}$$

where $l_1 - l_9$ are constants which satisfy

$$a_1l_1^3 + a_2l_2^3 + a_3l_3^3 + a_4l_1^2l_2 + a_5l_1^2l_3 + a_6l_1l_2^2 + a_7l_1l_2l_3 + a_8l_1l_3^2 + a_9l_2^2l_3 + a_{10}l_2l_3^2 = 0, \tag{2.171}$$

$$a_1l_4^3 + a_2l_5^3 + a_3l_6^3 + a_4l_4^2l_5 + a_5l_4^2l_6 + a_6l_4l_5^2 + a_7l_4l_5l_6 + a_8l_4l_6^2 + a_9l_5^2l_6 + a_{10}l_5l_6^2 = 0, \tag{2.172}$$

$$a_1l_7^3 + a_2l_8^3 + a_3l_9^3 + a_4l_7^2l_8 + a_5l_7^2l_9 + a_6l_7l_8^2 + a_7l_7l_8l_9 + a_8l_7l_9^2 + a_9l_8^2l_9 + a_{10}l_8l_9^2 = 0, \tag{2.173}$$

$$\begin{aligned} & 3a_1l_1^2l_4 + 3a_2l_2^2l_5 + 3a_3l_3^2l_6 + a_4\left(l_1^2l_5 + 2l_1l_2l_4\right) + a_5\left(l_1^2l_6 + 2l_1l_3l_4\right) + a_6\left(l_2^2l_4 + 2l_1l_2l_5\right) \\ & + a_7\left(l_1l_2l_6 + l_3\left(l_1l_5 + l_2l_4\right)\right) + a_8\left(l_3^2l_4 + 2l_1l_3l_6\right) + a_9\left(l_2^2l_6 + 2l_2l_3l_5\right) + a_{10}\left(l_3^2l_5 + 2l_2l_3l_6\right) = 0, \end{aligned} \tag{2.174}$$

$$\begin{aligned} & 3a_1l_1^2l_7 + 3a_2l_2^2l_8 + 3a_3l_3^2l_9 + a_4\left(l_1^2l_8 + 2l_1l_2l_7\right) + a_5\left(l_1^2l_9 + 2l_1l_3l_7\right) + a_6\left(l_2^2l_7 + 2l_1l_2l_8\right) \\ & + a_7\left(l_1l_2l_9 + l_3\left(l_1l_8 + l_2l_7\right)\right) + a_8\left(l_3^2l_7 + 2l_1l_3l_9\right) + a_9\left(l_2^2l_9 + 2l_2l_3l_8\right) + a_{10}\left(l_3^2l_8 + 2l_2l_3l_9\right) = 0, \end{aligned} \tag{2.175}$$

$$\begin{aligned} & 3a_1l_1l_4^2 + 3a_2l_2l_5^2 + 3a_3l_3l_6^2 + a_4\left(l_1l_4^2 + 2l_1l_4l_5\right) + a_5\left(l_3l_4^2 + 2l_1l_4l_6\right) + a_6\left(l_1l_5^2 + 2l_2l_4l_5\right) \\ & + a_7\left(l_3l_4l_5 + l_6\left(l_1l_5 + l_2l_4\right)\right) + a_8\left(l_1l_6^2 + 2l_3l_4l_6\right) + a_9\left(l_3l_5^2 + 2l_2l_5l_6\right) + a_{10}\left(l_2l_6^2 + 2l_3l_5l_6\right) = 0, \end{aligned} \tag{2.176}$$

$$\begin{aligned} & 3a_1l_1l_7^2 + 3a_2l_2l_8^2 + 3a_3l_3l_9^2 + a_4\left(l_2l_7^2 + 2l_1l_7l_8\right) + a_5\left(l_3l_7^2 + 2l_1l_7l_9\right) + a_6\left(l_1l_8^2 + 2l_2l_7l_8\right) \\ & + a_7\left(l_3l_7l_8 + l_9\left(l_1l_8 + l_2l_7\right)\right) + a_8\left(l_1l_9^2 + 2l_3l_7l_9\right) + a_9\left(l_3l_8^2 + 2l_2l_8l_9\right) + a_{10}\left(l_2l_9^2 + 2l_3l_8l_9\right), \end{aligned} \tag{2.177}$$

$$\begin{aligned} & 3a_1l_4^2l_7 + 3a_2l_5^2l_8 + 3a_3l_6^2l_9 + a_4\left(l_4^2l_8 + 2l_4l_5l_7\right) + a_5\left(l_4^2l_9 + 2l_4l_6l_7\right) + a_6\left(l_5^2l_7 + 2l_4l_5l_8\right) \\ & + a_7\left(l_4l_5l_9 + l_6\left(l_4l_8 + l_5l_7\right)\right) + a_8\left(l_6^2l_7 + 2l_4l_6l_9\right) + a_9\left(l_5^2l_9 + 2l_5l_6l_8\right) + a_{10}\left(l_6^2l_8 + 2l_5l_6l_9\right) = 0, \end{aligned} \tag{2.178}$$

$$\begin{aligned} & 3a_1l_4l_7^2 + 3a_2l_5l_8^2 + 3a_3l_6l_9^2 + a_4\left(l_5l_7^2 + 2l_4l_7l_8\right) + a_5\left(l_6l_7^2 + 2l_4l_7l_9\right) + a_6\left(l_4l_8^2 + 2l_5l_7l_8\right) \\ & + a_7\left(l_6l_7l_8 + l_9\left(l_4l_8 + l_5l_7\right)\right) + a_8\left(l_4l_9^2 + 2l_6l_7l_9\right) + a_9\left(l_6l_8^2 + 2l_5l_8l_9\right) + a_{10}\left(l_5l_9^2 + 2l_6l_8l_9\right) = 0. \end{aligned} \tag{2.179}$$

Prove. By Z_1 transformation, set

$$u = u(p,q,r), A(t,x,y) = A(p,q,r),$$

and

$$p = l_1t + l_2x + l_3y, q = l_4t + l_5x + l_6y, r = l_7t + l_8x + l_9y,$$

where $l_1 - l_9$ are undetermined constants, and

$$\frac{\partial\left(p,q,r\right)}{\partial\left(t,x,y\right)} = -l_3l_5l_7 + l_2l_6l_7 + l_3l_4l_8 - l_1l_6l_8 - l_2l_4l_9 + l_1l_5l_9 \neq 0.$$

So

$$
\begin{aligned}
& a_1 u_{ttt} + a_2 u_{xxx} + a_3 u_{yyy} + a_4 u_{ttx} + a_5 u_{tty} + a_6 u_{txx} + a_7 u_{txy} + a_8 u_{tyy} + a_9 u_{xxy} + a_{10} u_{xyy} \\
& = a_1 \left(l_1^3 u_{ppp} + l_4^3 u_{qqq} + l_7^3 u_{rrr} \right) + a_1 \left(3 l_1^2 l_4 u_{ppq} + 3 l_1^2 l_7 u_{ppr} + 3 l_1 l_4^2 u_{pqq} \right) \\
& + a_1 \left(3 l_1 l_7^2 u_{prr} + 3 l_4^2 l_7 u_{qqr} + 3 l_4 l_7^2 u_{qrr} + 6 l_1 l_4 l_7 u_{pqr} \right) + a_2 \left(l_2^3 u_{ppp} + l_5^3 u_{qqq} + l_8^3 u_{rrr} \right) \\
& + a_2 \left(3 l_2^2 l_5 u_{ppq} + 3 l_2^2 l_8 u_{ppr} + 3 l_2 l_5^2 u_{pqq} \right) + a_2 \left(3 l_2 l_8^2 u_{prr} + 3 l_5^2 l_8 u_{qqr} + 3 l_5 l_8^2 u_{qrr} + 6 l_2 l_5 l_8 u_{pqr} \right) \\
& + a_3 \left(l_3^3 u_{ppp} + l_6^3 u_{qqq} + l_9^3 u_{rrr} \right) + a_3 \left(3 l_3^2 l_6 u_{ppq} + 3 l_3^2 l_9 u_{ppr} + 3 l_3 l_6^2 u_{pqq} \right) \\
& + a_3 \left(3 l_3 l_9^2 u_{prr} + 3 l_6^2 l_9 u_{qqr} + 3 l_6 l_9^2 u_{qrr} + 6 l_3 l_6 l_9 u_{pqr} \right) + a_4 \left(l_1^2 l_2 u_{ppp} + l_4^2 l_5 u_{qqq} + l_7^2 l_8 u_{rrr} \right) \\
& + a_4 \left(\left(l_1^2 l_5 + 2 l_1 l_2 l_4 \right) u_{ppq} + \left(l_1^2 l_8 + 2 l_1 l_2 l_7 \right) u_{ppr} + \left(l_1 l_4^2 + 2 l_1 l_4 l_5 \right) u_{pqq} + \left(l_4^2 l_8 + 2 l_4 l_5 l_7 \right) u_{qqr} \right) \\
& + a_4 \left(\left(l_2 l_7^2 + 2 l_1 l_7 l_8 \right) u_{prr} + \left(l_5 l_7^2 + 2 l_4 l_7 l_8 \right) u_{qrr} + \left(2 l_1 l_4 l_8 + 2 l_1 l_5 l_7 + 2 l_2 l_4 l_7 \right) u_{pqr} \right) \\
& + a_5 \left(l_1^2 l_3 u_{ppp} + l_4^2 l_6 u_{qqq} + l_7^2 l_9 u_{rrr} \right) \\
& + a_5 \left(\left(l_1^2 l_6 + 2 l_1 l_3 l_4 \right) u_{ppq} + \left(l_1^2 l_9 + 2 l_1 l_3 l_7 \right) u_{ppr} + \left(l_3 l_4^2 + 2 l_1 l_4 l_6 \right) u_{pqq} + \left(l_4^2 l_9 + 2 l_4 l_6 l_7 \right) u_{qqr} \right) \\
& + a_5 \left(\left(l_3 l_7^2 + 2 l_1 l_7 l_9 \right) u_{prr} + \left(l_6 l_7^2 + 2 l_4 l_7 l_9 \right) u_{qrr} + \left(2 l_1 l_4 l_9 + 2 l_1 l_6 l_7 + 2 l_3 l_4 l_7 \right) u_{pqr} \right) \\
& + a_6 \left(l_1 l_2^2 u_{ppp} + l_4 l_5^2 u_{qqq} + l_7 l_8^2 u_{rrr} \right) \\
& + a_6 \left(\left(l_1 l_2 l_5 + l_2 \left(l_1 l_5 + l_2 l_4 \right) \right) u_{ppq} + \left(l_1 l_2 l_8 + l_2 \left(l_1 l_8 + l_2 l_7 \right) \right) u_{ppr} + \left(l_2 l_4 l_5 + l_5 \left(l_1 l_5 + l_2 l_4 \right) \right) u_{pqq} \right) \\
& + a_6 \left(\left(l_4 l_5 l_8 + l_5 \left(l_4 l_8 + l_5 l_7 \right) \right) u_{qqr} + \left(l_2 l_7 l_8 + l_8 \left(l_1 l_8 + l_2 l_7 \right) \right) u_{prr} + \left(l_5 l_7 l_8 + l_8 \left(l_4 l_8 + l_5 l_7 \right) \right) u_{qrr} \right) \\
& + a_6 \left(\left(l_2 \left(l_4 l_8 + l_5 l_7 \right) + l_5 \left(l_1 l_8 + l_2 l_7 \right) + l_8 \left(l_1 l_5 + l_2 l_4 \right) \right) u_{pqr} \right) \\
& + a_7 \left(l_1 l_2 l_3 u_{ppp} + l_4 l_5 l_6 u_{qqq} + l_7 l_8 l_9 u_{rrr} \right) \\
& + a_7 \left(\left(l_1 l_2 l_6 + l_3 \left(l_1 l_5 + l_2 l_4 \right) \right) u_{ppq} + \left(l_1 l_2 l_9 + l_3 \left(l_1 l_8 + l_2 l_7 \right) \right) u_{ppr} + \left(l_3 l_4 l_5 + l_6 \left(l_1 l_5 + l_2 l_4 \right) \right) u_{pqq} \right) \\
& + a_7 \left(\left(l_4 l_5 l_9 + l_6 \left(l_4 l_8 + l_5 l_7 \right) \right) u_{qqr} + \left(l_3 l_7 l_8 + l_9 \left(l_1 l_8 + l_2 l_7 \right) \right) u_{prr} + \left(l_6 l_7 l_8 + l_9 \left(l_4 l_8 + l_5 l_7 \right) \right) u_{qrr} \right) \\
& + a_7 \left(\left(l_3 \left(l_4 l_8 + l_5 l_7 \right) + l_6 \left(l_1 l_8 + l_2 l_7 \right) + l_9 \left(l_1 l_5 + l_2 l_4 \right) \right) u_{pqr} \right) \\
& + a_8 \left(l_1 l_3^2 u_{ppp} + l_4 l_6^2 u_{qqq} + l_7 l_9^2 u_{rrr} \right) \\
& + a_8 \left(\left(l_1 l_3 l_6 + l_3 \left(l_1 l_6 + l_3 l_4 \right) \right) u_{ppq} + \left(l_1 l_3 l_9 + l_3 \left(l_1 l_9 + l_3 l_7 \right) \right) u_{ppr} + \left(l_3 l_4 l_6 + l_6 \left(l_1 l_6 + l_3 l_4 \right) \right) u_{pqq} \right) \\
& + a_8 \left(\left(l_4 l_6 l_9 + l_6 \left(l_4 l_9 + l_6 l_7 \right) \right) u_{qqr} + \left(l_3 l_7 l_9 + l_9 \left(l_1 l_9 + l_3 l_7 \right) \right) u_{prr} + \left(l_6 l_7 l_9 + l_9 \left(l_4 l_9 + l_6 l_7 \right) \right) u_{qrr} \right) \\
& + a_8 \left(\left(l_9 \left(l_1 l_6 + l_3 l_4 \right) + l_6 \left(l_1 l_9 + l_3 l_7 \right) + l_3 \left(l_4 l_9 + l_6 l_7 \right) \right) u_{pqr} \right) \\
& + a_9 \left(l_2^2 l_3 u_{ppp} + l_5^2 l_6 u_{qqq} + l_8^2 l_9 u_{rrr} \right) \\
& + a_9 \left(\left(l_2^2 l_6 + 2 l_2 l_3 l_5 \right) u_{ppq} + \left(l_2^2 l_9 + 2 l_2 l_3 l_8 \right) u_{ppr} + \left(l_3 l_5^2 + 2 l_2 l_5 l_6 \right) u_{pqq} + \left(l_5^2 l_9 + 2 l_5 l_6 l_8 \right) u_{qqr} \right) \\
& + a_9 \left(\left(l_3 l_8^2 + 2 l_2 l_8 l_9 \right) u_{prr} + \left(l_6 l_8^2 + 2 l_5 l_8 l_9 \right) u_{qrr} + \left(2 l_2 l_5 l_9 + 2 l_2 l_6 l_8 + 2 l_3 l_5 l_8 \right) u_{pqr} \right) \\
& + a_{10} \left(l_2 l_3^2 u_{ppp} + l_5 l_6^2 u_{qqq} + l_8 l_9^2 u_{rrr} \right) \\
& + a_{10} \left(\left(l_2 l_3 l_6 + l_3 \left(l_2 l_6 + l_3 l_5 \right) \right) u_{ppq} + \left(l_2 l_3 l_9 + l_3 \left(l_2 l_9 + l_3 l_8 \right) \right) u_{ppr} + \left(l_3 l_5 l_6 + l_6 \left(l_2 l_6 + l_3 l_5 \right) \right) u_{pqq} \right) \\
& + a_{10} \left(\left(l_5 l_6 l_9 + l_6 \left(l_5 l_9 + l_6 l_8 \right) \right) u_{qqr} + \left(l_3 l_8 l_9 + l_9 \left(l_2 l_9 + l_3 l_8 \right) \right) u_{prr} + \left(l_6 l_8 l_9 + l_9 \left(l_5 l_9 + l_6 l_8 \right) \right) u_{qrr} \right) \\
& + a_{10} \left(\left(l_3 \left(l_5 l_9 + l_6 l_8 \right) + l_6 \left(l_2 l_9 + l_3 l_8 \right) + l_9 \left(l_2 l_6 + l_3 l_5 \right) \right) u_{pqr} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(a_1l_1^3 + a_2l_2^3 + a_3l_3^3 + a_4l_1^2l_2 + a_5l_1^2l_3 + a_6l_1l_2^2 + a_7l_1l_2l_3 + a_8l_1l_3^2 + a_9l_2^2l_3 + a_{10}l_2l_3^2\right) u_{ppp} \\
&+ \left(a_1l_4^3 + a_2l_5^3 + a_3l_6^3 + a_4l_4^2l_5 + a_5l_4^2l_6 + a_6l_4l_5^2 + a_7l_4l_5l_6 + a_8l_4l_6^2 + a_9l_5^2l_6 + a_{10}l_5l_6^2\right) u_{qqq} \\
&+ \left(a_1l_7^3 + a_2l_8^3 + a_3l_9^3 + a_4l_7^2l_8 + a_5l_7^2l_9 + a_6l_7l_8^2 + a_7l_7l_8l_9 + a_8l_7l_9^2 + a_9l_8^2l_9 + a_{10}l_8l_9^2\right) u_{rrr} \\
&+ \left(3a_1l_1^2l_4 + 3a_2l_2^2l_5 + 3a_3l_3^2l_6 + a_4\left(l_1^2l_5 + 2l_1l_2l_4\right)\right) u_{ppq} \\
&+ \left(a_5\left(l_1^2l_6 + 2l_1l_3l_4\right) + a_6\left(l_2^2l_4 + 2l_1l_2l_5\right) + a_7\left(l_1l_2l_6 + l_3\left(l_1l_5 + l_2l_4\right)\right)\right) u_{ppq} \\
&+ \left(a_8\left(l_3^2l_4 + 2l_1l_3l_6\right) + a_9\left(l_2^2l_6 + 2l_2l_3l_5\right) + a_{10}\left(l_3^2l_5 + 2l_2l_3l_6\right)\right) u_{ppq} \\
&+ \left(3a_1l_1^2l_7 + 3a_2l_2^2l_8 + 3a_3l_3^2l_9 + a_4\left(l_1^2l_8 + 2l_1l_2l_7\right)\right) u_{ppr} \\
&+ \left(a_5\left(l_1^2l_9 + 2l_1l_3l_7\right) + a_6\left(l_2^2l_7 + 2l_1l_2l_8\right) + a_7\left(l_1l_2l_9 + l_3\left(l_1l_8 + l_2l_7\right)\right)\right) u_{ppr} \\
&+ \left(a_8\left(l_3^2l_7 + 2l_1l_3l_9\right) + a_9\left(l_2^2l_9 + 2l_2l_3l_8\right) + a_{10}\left(l_3^2l_8 + 2l_2l_3l_9\right)\right) u_{ppr} \\
&+ \left(3a_1l_1l_4^2 + 3a_2l_2l_5^2 + 3a_3l_3l_6^2 + a_4\left(l_1l_4^2 + 2l_1l_4l_5\right)\right) u_{pqq} \\
&+ \left(a_5\left(l_3l_4^2 + 2l_1l_4l_6\right) + a_6\left(l_1l_5^2 + 2l_2l_4l_5\right) + a_7\left(l_3l_4l_5 + l_6\left(l_1l_5 + l_2l_4\right)\right)\right) u_{pqq} \\
&+ \left(a_8\left(l_1l_6^2 + 2l_3l_4l_6\right) + a_9\left(l_3l_5^2 + 2l_2l_5l_6\right) + a_{10}\left(l_2l_6^2 + 2l_3l_5l_6\right)\right) u_{pqq} \\
&+ \left(3a_1l_1l_7^2 + 3a_2l_2l_8^2 + 3a_3l_3l_9^2 + a_4\left(l_2l_7^2 + 2l_1l_7l_8\right)\right) u_{prr} \\
&+ \left(a_5\left(l_3l_7^2 + 2l_1l_7l_9\right) + a_6\left(l_1l_8^2 + 2l_2l_7l_8\right) + a_7\left(l_3l_7l_8 + l_9\left(l_1l_8 + l_2l_7\right)\right)\right) u_{prr} \\
&+ \left(a_8\left(l_1l_9^2 + 2l_3l_7l_9\right) + a_9\left(l_3l_8^2 + 2l_2l_8l_9\right) + a_{10}\left(l_2l_9^2 + 2l_3l_8l_9\right)\right) u_{prr} \\
&+ \left(3a_1l_4^2l_7 + 3a_2l_5^2l_8 + 3a_3l_6^2l_9 + a_4\left(l_4^2l_8 + 2l_4l_5l_7\right)\right) u_{qqr} \\
&+ \left(a_5\left(l_4^2l_9 + 2l_4l_6l_7\right) + a_6\left(l_5^2l_7 + 2l_4l_5l_8\right) + a_7\left(l_4l_5l_9 + l_6\left(l_4l_8 + l_5l_7\right)\right)\right) u_{qqr} \\
&+ \left(a_8\left(l_6^2l_7 + 2l_4l_6l_9\right) + a_9\left(l_5^2l_9 + 2l_5l_6l_8\right) + a_{10}\left(l_6^2l_8 + 2l_5l_6l_9\right)\right) u_{qqr} \\
&+ \left(3a_1l_4l_7^2 + 3a_2l_5l_8^2 + 3a_3l_6l_9^2 + a_4\left(l_5l_7^2 + 2l_4l_7l_8\right)\right) u_{qrr} \\
&+ \left(a_5\left(l_6l_7^2 + 2l_4l_7l_9\right) + a_6\left(l_4l_8^2 + 2l_5l_7l_8\right) + a_7\left(l_6l_7l_8 + l_9\left(l_4l_8 + l_5l_7\right)\right)\right) u_{qrr} \\
&+ \left(a_8\left(l_4l_9^2 + 2l_6l_7l_9\right) + a_9\left(l_6l_8^2 + 2l_5l_8l_9\right) + a_{10}\left(l_5l_9^2 + 2l_6l_8l_9\right)\right) u_{qrr} \\
&+ \left(6a_1l_1l_4l_7 + 6a_2l_2l_5l_8 + 6a_3l_3l_6l_9 + 2a_4\left(l_1l_4l_8 + l_1l_5l_7 + l_2l_4l_7\right)\right) u_{pqr} \\
&+ \left(2a_5\left(l_1l_4l_9 + l_1l_6l_7 + l_3l_4l_7\right) + 2a_6\left(l_1l_5l_8 + l_2l_4l_8 + l_2l_5l_7\right)\right) u_{pqr} \\
&+ \left(a_7\left(l_3\left(l_4l_8 + l_5l_7\right) + l_6\left(l_1l_8 + l_2l_7\right) + l_9\left(l_1l_5 + l_2l_4\right)\right)\right) u_{pqr} \\
&+ \left(2a_8\left(l_1l_6l_9 + l_3l_4l_9 + l_3l_6l_7\right) + 2a_9\left(l_2l_5l_9 + l_2l_6l_8 + l_3l_5l_8\right) + a_{10}\right) u_{pqr} \\
&= A(p,q,r).
\end{aligned}
\tag{2.180}
$$

Set

$$
\begin{aligned}
&a_1l_1^3 + a_2l_2^3 + a_3l_3^3 + a_4l_1^2l_2 + a_5l_1^2l_3 + a_6l_1l_2^2 + a_7l_1l_2l_3 + a_8l_1l_3^2 + a_9l_2^2l_3 + a_{10}l_2l_3^2 = 0, \\
&a_1l_4^3 + a_2l_5^3 + a_3l_6^3 + a_4l_4^2l_5 + a_5l_4^2l_6 + a_6l_4l_5^2 + a_7l_4l_5l_6 + a_8l_4l_6^2 + a_9l_5^2l_6 + a_{10}l_5l_6^2 = 0, \\
&a_1l_7^3 + a_2l_8^3 + a_3l_9^3 + a_4l_7^2l_8 + a_5l_7^2l_9 + a_6l_7l_8^2 + a_7l_7l_8l_9 + a_8l_7l_9^2 + a_9l_8^2l_9 + a_{10}l_8l_9^2 = 0, \\
&3a_1l_1^2l_4 + 3a_2l_2^2l_5 + 3a_3l_3^2l_6 + a_4\left(l_1^2l_5 + 2l_1l_2l_4\right) + a_5\left(l_1^2l_6 + 2l_1l_3l_4\right) + a_6\left(l_2^2l_4 + 2l_1l_2l_5\right) \\
&\quad + a_7\left(l_1l_2l_6 + l_3\left(l_1l_5 + l_2l_4\right)\right) + a_8\left(l_3^2l_4 + 2l_1l_3l_6\right) + a_9\left(l_2^2l_6 + 2l_2l_3l_5\right) + a_{10}\left(l_3^2l_5 + 2l_2l_3l_6\right) = 0, \\
&3a_1l_1^2l_7 + 3a_2l_2^2l_8 + 3a_3l_3^2l_9 + a_4\left(l_1^2l_8 + 2l_1l_2l_7\right) + a_5\left(l_1^2l_9 + 2l_1l_3l_7\right) + a_6\left(l_2^2l_7 + 2l_1l_2l_8\right) \\
&\quad + a_7\left(l_1l_2l_9 + l_3\left(l_1l_8 + l_2l_7\right)\right) + a_8\left(l_3^2l_7 + 2l_1l_3l_9\right) + a_9\left(l_2^2l_9 + 2l_2l_3l_8\right) + a_{10}\left(l_3^2l_8 + 2l_2l_3l_9\right) = 0, \\
&3a_1l_1l_4^2 + 3a_2l_2l_5^2 + 3a_3l_3l_6^2 + a_4\left(l_1l_4^2 + 2l_1l_4l_5\right) + a_5\left(l_3l_4^2 + 2l_1l_4l_6\right) + a_6\left(l_1l_5^2 + 2l_2l_4l_5\right) \\
&\quad + a_7\left(l_3l_4l_5 + l_6\left(l_1l_5 + l_2l_4\right)\right) + a_8\left(l_1l_6^2 + 2l_3l_4l_6\right) + a_9\left(l_3l_5^2 + 2l_2l_5l_6\right) + a_{10}\left(l_2l_6^2 + 2l_3l_5l_6\right) = 0, \\
&3a_1l_1l_7^2 + 3a_2l_2l_8^2 + 3a_3l_3l_9^2 + a_4\left(l_2l_7^2 + 2l_1l_7l_8\right) + a_5\left(l_3l_7^2 + 2l_1l_7l_9\right) + a_6\left(l_1l_8^2 + 2l_2l_7l_8\right) \\
&\quad + a_7\left(l_3l_7l_8 + l_9\left(l_1l_8 + l_2l_7\right)\right) + a_8\left(l_1l_9^2 + 2l_3l_7l_9\right) + a_9\left(l_3l_8^2 + 2l_2l_8l_9\right) + a_{10}\left(l_2l_9^2 + 2l_3l_8l_9\right) = 0,
\end{aligned}
$$

$$3a_1l_4^2l_7+3a_2l_5^2l_8+3a_3l_6^2l_9+a_4\left(l_4^2l_8+2l_4l_5l_7\right)+a_5\left(l_4^2l_9+2l_4l_6l_7\right)+a_6\left(l_5^2l_7+2l_4l_5l_8\right)$$
$$+a_7\left(l_4l_5l_9+l_6\left(l_4l_8+l_5l_7\right)\right)+a_8\left(l_6^2l_7+2l_4l_6l_9\right)+a_9\left(l_5^2l_9+2l_5l_6l_8\right)+a_{10}\left(l_6^2l_8+2l_5l_6l_9\right)=0,$$
$$3a_1l_4l_7^2+3a_2l_5l_8^2+3a_3l_6l_9^2+a_4\left(l_5l_7^2+2l_4l_7l_8\right)+a_5\left(l_6l_7^2+2l_4l_7l_9\right)+a_6\left(l_4l_8^2+2l_5l_7l_8\right)$$
$$+a_7\left(l_6l_7l_8+l_9\left(l_4l_8+l_5l_7\right)\right)+a_8\left(l_4l_9^2+2l_6l_7l_9\right)+a_9\left(l_6l_8^2+2l_5l_8l_9\right)+a_{10}\left(l_5l_9^2+2l_6l_8l_9\right)=0.$$

So

$$\begin{aligned}&a_1u_{ttt}+a_2u_{xxx}+a_3u_{yyy}+a_4u_{ttx}+a_5u_{tty}+a_6u_{txx}+a_7u_{txy}+a_8u_{tyy}+a_9u_{xxy}+a_{10}u_{xyy}\\&=\left(6a_1l_1l_4l_7+6a_2l_2l_5l_8+6a_3l_3l_6l_9+2a_4\left(l_1l_4l_8+l_1l_5l_7+l_2l_4l_7\right)\right)u_{pqr}\\&+\left(2a_5\left(l_1l_4l_9+l_1l_6l_7+l_3l_4l_7\right)+2a_6\left(l_1l_5l_8+l_2l_4l_8+l_2l_5l_7\right)\right)u_{pqr}\\&+\left(a_7\left(l_3\left(l_4l_8+l_5l_7\right)+l_6\left(l_1l_8+l_2l_7\right)+l_9\left(l_1l_5+l_2l_4\right)\right)\right)u_{pqr}\\&+\left(2a_8\left(l_1l_6l_9+l_3l_4l_9+l_3l_6l_7\right)+2a_9\left(l_2l_5l_9+l_2l_6l_8+l_3l_5l_8\right)+a_{10}\right)u_{pqr}\\&=A(p,q,r).\end{aligned}\tag{2.181}$$

The general solution of Eq. (2.181) is

$$u=f\left(p,q\right)+g\left(p,r\right)+h\left(q,r\right)+\frac{1}{B}\iiint A\left(p,q,r\right)dpdqdr.$$

So the theorem is proved. □

Similar to the theorem 3,6,8, in (2.180), if the coefficients of the other 9 terms are set to zero, the general solutions of (2.168) in different conditions will be get, such as

$$u=f\left(q,r\right)+pg\left(q,r\right)+p^2h\left(q,r\right)+\frac{1}{B_1}\iiint A\left(p,q,r\right)dpdpdp,\tag{2.182}$$

$$B_1=a_1l_1^3+a_2l_2^3+a_3l_3^3+a_4l_1^2l_2+a_5l_1^2l_3+a_6l_1l_2^2+a_7l_1l_2l_3+a_8l_1l_3^2+a_9l_2^2l_3+a_{10}l_2l_3^2,\tag{2.183}$$

$$u=f\left(q,r\right)+g\left(p,r\right)+ph\left(q,r\right)+\frac{1}{B_2}\iiint A\left(p,q,r\right)dpdpdq,\tag{2.184}$$

$$\begin{aligned}B_2&=3a_1l_1^2l_4+3a_2l_2^2l_5+3a_3l_3^2l_6+a_4\left(l_1^2l_5+2l_1l_2l_4\right)+a_5\left(l_1^2l_6+2l_1l_3l_4\right)+a_6\left(l_2^2l_4+2l_1l_2l_5\right)\\&+a_7\left(l_1l_2l_6+l_3\left(l_1l_5+l_2l_4\right)\right)+a_8\left(l_3^2l_4+2l_1l_3l_6\right)+a_9\left(l_2^2l_6+2l_2l_3l_5\right)+a_{10}\left(l_3^2l_5+2l_2l_3l_6\right).\end{aligned}\tag{2.185}$$

Readers may try it for themselves.

Next we propose Theorem 25.

Theorem 25. *In* $\mathbb{R}^2$,

$$a_1u_{tttt}+a_2u_{xxxx}+a_3u_{tttx}+a_4u_{ttxx}+a_5u_{txxx}=A\left(t,x\right),\tag{2.186}$$

the general solution of Eq. (2.186) is

$$u=f_1\left(p\right)+f_2\left(q\right)+qf_3\left(p\right)+pf_4\left(q\right)+\frac{1}{B_1}\int\dots\int A\left(p,q\right)dpdpdqdq,\tag{2.187}$$

where f_1, f_2, f_3 *and* f_4 *are random fourth differentiable functions, and*

$$\begin{aligned}B_1&=6a_1k_1^2k_3^2+6a_2k_2^2k_4^2+3a_3\left(k_1^2k_3k_4+k_1k_2k_3^2\right)+a_4\left(k_1^2k_4^2+4k_1k_2k_3k_4+k_2^2k_3^2\right)\\&+3a_5\left(k_2^2k_3k_4+k_1k_2k_4^2\right),\end{aligned}\tag{2.188}$$

$$a_1k_1^4+a_2k_2^4+a_3k_1^3k_2+a_4k_1^2k_2^2+a_5k_1k_2^3=0,\tag{2.189}$$

$$a_1k_3^4 + a_2k_4^4 + a_3k_3^3k_4 + a_4k_3^2k_4^2 + a_5k_3k_4^3 = 0, \tag{2.190}$$

$$4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(k_1^3k_4 + 3k_1^2k_2k_3\right) + 2a_4\left(k_1^2k_2k_4 + k_1k_2^2k_3\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right) = 0, \tag{2.191}$$

$$4a_1k_1k_3^3 + 4a_2k_2k_4^3 + a_3\left(k_2k_3^3 + 3k_1k_3^2k_4\right) + 2a_4\left(k_2k_3^2k_4 + k_1k_3k_4^2\right) + a_5\left(k_1k_4^3 + 3k_2k_3k_4^2\right) = 0. \tag{2.192}$$

Prove. By Z_1 transformation, set

$$u(t,x) = u(p,q),$$
$$p = k_1t + k_2x, q = k_3t + k_4x,$$

and

$$k_1k_4 - k_2k_3 \neq 0.$$

Then

$$\begin{aligned}
&a_1u_{tttt} + a_2u_{xxxx} + a_3u_{tttx} + a_4u_{ttxx} + a_5u_{txxx}\\
&= a_1\left(k_1^4u_{pppp} + k_3^4u_{qqqq} + 4k_1^3k_3u_{pppq} + 4k_1k_3^3u_{pqqq} + 6k_1^2k_3^2u_{ppqq}\right)\\
&+ a_2\left(k_2^4u_{pppp} + k_4^4u_{qqqq} + 4k_2^3k_4u_{pppq} + 4k_2k_4^3u_{pqqq} + 6k_2^2k_4^2u_{ppqq}\right)\\
&+ a_3\left(k_1^3k_2u_{pppp} + k_3^3k_4u_{qqqq} + \left(k_1^3k_4 + 3k_1^2k_2k_3\right)u_{pppq}\right)\\
&+ a_3\left(\left(k_2k_3^3 + 2k_1k_3^2k_4 + k_1k_3^2k_4\right)u_{pqqq} + \left(k_1^2k_3k_4 + 2k_1k_2k_3^2 + k_1k_2k_3^2 + 2k_1^2k_3k_4\right)u_{ppqq}\right)\\
&+ a_4\left(k_1^2k_2^2u_{pppp} + k_3^2k_4^2u_{qqqq} + 2\left(k_1^2k_2k_4 + k_1k_2^2k_3\right)u_{pppq}\right)\\
&+ a_4\left(2\left(k_2k_3^2k_4 + k_1k_3k_4^2\right)u_{pqqq} + \left(k_1^2k_4^2 + 4k_1k_2k_3k_4 + k_2^2k_3^2\right)u_{ppqq}\right)\\
&+ a_5\left(k_1k_2^3u_{pppp} + k_3k_4^3u_{qqqq} + \left(k_2^3k_3 + 3k_1k_2^2k_4\right)u_{pppq}\right)\\
&+ a_5\left(\left(k_1k_4^3 + 3k_2k_3k_4^2\right)u_{pqqq} + 3\left(k_2^2k_3k_4 + k_1k_2k_4^2\right)u_{ppqq}\right)\\
&= \left(a_1k_1^4 + a_2k_2^4 + a_3k_1^3k_2 + a_4k_1^2k_2^2 + a_5k_1k_2^3\right)u_{pppp}\\
&+ \left(a_1k_3^4 + a_2k_4^4 + a_3k_3^3k_4 + a_4k_3^2k_4^2 + a_5k_3k_4^3\right)u_{qqqq}\\
&+ \left(4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(k_1^3k_4 + 3k_1^2k_2k_3\right)\right)u_{pppq}\\
&+ \left(2a_4\left(k_1^2k_2k_4 + k_1k_2^2k_3\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right)\right)u_{pppq}\\
&+ \left(4a_1k_1k_3^3 + 4a_2k_2k_4^3 + a_3\left(k_2k_3^3 + 3k_1k_3^2k_4\right)\right)u_{pqqq}\\
&+ \left(2a_4\left(k_2k_3^2k_4 + k_1k_3k_4^2\right) + a_5\left(k_1k_4^3 + 3k_2k_3k_4^2\right)\right)u_{pqqq}\\
&+ \left(6a_1k_1^2k_3^2 + 6a_2k_2^2k_4^2 + 3a_3\left(k_1^2k_3k_4 + k_1k_2k_3^2\right)\right)u_{ppqq}\\
&+ \left(a_4\left(k_1^2k_4^2 + 4k_1k_2k_3k_4 + k_2^2k_3^2\right) + 3a_5\left(k_2^2k_3k_4 + k_1k_2k_4^2\right)\right)u_{ppqq}\\
&= A\left(p,q\right).
\end{aligned} \tag{2.193}$$

Set

$$a_1k_1^4 + a_2k_2^4 + a_3k_1^3k_2 + a_4k_1^2k_2^2 + a_5k_1k_2^3 = 0,$$
$$a_1k_3^4 + a_2k_4^4 + a_3k_3^3k_4 + a_4k_3^2k_4^2 + a_5k_3k_4^3 = 0,$$
$$4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(k_1^3k_4 + 3k_1^2k_2k_3\right) + 2a_4\left(k_1^2k_2k_4 + k_1k_2^2k_3\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right) = 0,$$
$$4a_1k_1k_3^3 + 4a_2k_2k_4^3 + a_3\left(k_2k_3^3 + 3k_1k_3^2k_4\right) + 2a_4\left(k_2k_3^2k_4 + k_1k_3k_4^2\right) + a_5\left(k_1k_4^3 + 3k_2k_3k_4^2\right) = 0.$$

So

$$\begin{aligned}
&a_1u_{tttt} + a_2u_{xxxx} + a_3u_{tttx} + a_4u_{ttxx} + a_5u_{txxx}\\
&= \left(6a_1k_1^2k_3^2 + 6a_2k_2^2k_4^2 + 3a_3\left(k_1^2k_3k_4 + k_1k_2k_3^2\right)\right)u_{ppqq}\\
&+ \left(a_4\left(k_1^2k_4^2 + 4k_1k_2k_3k_4 + k_2^2k_3^2\right) + 3a_5\left(k_2^2k_3k_4 + k_1k_2k_4^2\right)\right)u_{ppqq} = A\left(p,q\right).
\end{aligned} \tag{2.194}$$

The general solution of Eq. (2.194) is

$$u = f_1(p) + f_2(q) + qf_3(p) + pf_4(q) + \frac{1}{B_1}\int \dots \int A(p,q)\, dpdpdqdq,$$

$$\begin{aligned} B_1 &= 6a_1k_1^2k_3^2 + 6a_2k_2^2k_4^2 + 3a_3\left(k_1^2k_3k_4 + k_1k_2k_3^2\right) + a_4\left(k_1^2k_4^2 + 4k_1k_2k_3k_4 + k_2^2k_3^2\right) \\ &+3a_5\left(k_2^2k_3k_4 + k_1k_2k_4^2\right). \end{aligned}$$

So the theorem is proved. □

In (2.193), if set

$$\begin{gathered} a_1k_3^4 + a_2k_4^4 + a_3k_3^3k_4 + a_4k_3^2k_4^2 + a_5k_3k_4^3 = 0, \\ 4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(k_1^3k_4 + 3k_1^2k_2k_3\right) + 2a_4\left(k_1^2k_2k_4 + k_1k_2^2k_3\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right) = 0, \\ 4a_1k_1k_3^3 + 4a_2k_2k_4^3 + a_3\left(k_2k_3^3 + 3k_1k_3^2k_4\right) + 2a_4\left(k_2k_3^2k_4 + k_1k_3k_4^2\right) + a_5\left(k_1k_4^3 + 3k_2k_3k_4^2\right) = 0, \\ 6a_1k_1^2k_3^2 + 6a_2k_2^2k_4^2 + 3a_3\left(k_1^2k_3k_4 + k_1k_2k_3^2\right) + a_4\left(k_1^2k_4^2 + 4k_1k_2k_3k_4 + k_2^2k_3^2\right) \\ + 3a_5\left(k_2^2k_3k_4 + k_1k_2k_4^2\right) = 0. \end{gathered}$$

Then

$$\begin{aligned} &a_1u_{tttt} + a_2u_{xxxx} + a_3u_{tttx} + a_4u_{ttxx} + a_5u_{txxx} \\ &= \left(a_1k_1^4 + a_2k_2^4 + a_3k_1^3k_2 + a_4k_1^2k_2^2 + a_5k_1k_2^3\right)u_{pppp} \\ &= A(p,q). \end{aligned} \tag{2.195}$$

The general solution of Eq. (2.195) is

$$u = f_1(q) + pf_2(q) + p^2f_3(q) + p^3f_4(q) + \frac{1}{B_2}\int \dots \int A(p,q)\, dpdpdpdp,$$

$$B_2 = a_1k_1^4 + a_2k_2^4 + a_3k_1^3k_2 + a_4k_1^2k_2^2 + a_5k_1k_2^3.$$

In (2.193), if set

$$\begin{gathered} a_1k_1^4 + a_2k_2^4 + a_3k_1^3k_2 + a_4k_1^2k_2^2 + a_5k_1k_2^3 = 0, \\ a_1k_3^4 + a_2k_4^4 + a_3k_3^3k_4 + a_4k_3^2k_4^2 + a_5k_3k_4^3 = 0, \\ 4a_1k_1k_3^3 + 4a_2k_2k_4^3 + a_3\left(k_2k_3^3 + 3k_1k_3^2k_4\right) + 2a_4\left(k_2k_3^2k_4 + k_1k_3k_4^2\right) + a_5\left(k_1k_4^3 + 3k_2k_3k_4^2\right) = 0, \\ 6a_1k_1^2k_3^2 + 6a_2k_2^2k_4^2 + 3a_3\left(k_1^2k_3k_4 + k_1k_2k_3^2\right) + a_4\left(k_1^2k_4^2 + 4k_1k_2k_3k_4 + k_2^2k_3^2\right) \\ + 3a_5\left(k_2^2k_3k_4 + k_1k_2k_4^2\right) = 0. \end{gathered}$$

Then

$$\begin{aligned} &a_1u_{tttt} + a_2u_{xxxx} + a_3u_{tttx} + a_4u_{ttxx} + a_5u_{txxx} \\ &= \left(4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(k_1^3k_4 + 3k_1^2k_2k_3\right)\right)u_{pppq} \\ &+ \left(2a_4\left(k_1^2k_2k_4 + k_1k_2^2k_3\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right)\right)u_{pppq} \\ &= A(p,q). \end{aligned} \tag{2.196}$$

The general solution of Eq. (2.196) is

$$u = f_1(p) + f_2(q) + pf_3(q) + p^2f_4(q) + \frac{1}{B_3}\int \dots \int A(p,q)\, dpdpdpdq,$$

$$B_3 = 4a_1k_1^3k_3 + 4a_2k_2^3k_4 + a_3\left(k_1^3k_4 + 3k_1^2k_2k_3\right) + 2a_4\left(k_1^2k_2k_4 + k_1k_2^2k_3\right) + a_5\left(k_2^3k_3 + 3k_1k_2^2k_4\right).$$

So we can get Theorem 26,27.

Theorem 26. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{tttt}+a_2u_{xxxx}+a_3u_{tttx}+a_4u_{ttxx}+a_5u_{txxx}=A\left(t,x\right),$$

is

$$u=f_1\left(q\right)+pf_2\left(q\right)+p^2f_3\left(q\right)+p^3f_4\left(q\right)+\frac{1}{B_2}\int\cdots\int A\left(p,q\right)dpdpdpdp, \tag{2.197}$$

where f_1, f_2, f_3 and f_4 are random fourth differentiable functions, and

$$B_2=a_1k_1^4+a_2k_2^4+a_3k_1^3k_2+a_4k_1^2k_2^2+a_5k_1k_2^3, \tag{2.198}$$

$$a_1k_3^4+a_2k_4^4+a_3k_3^3k_4+a_4k_3^2k_4^2+a_5k_3k_4^3=0,$$

$$4a_1k_1^3k_3+4a_2k_2^3k_4+a_3\left(k_1^3k_4+3k_1^2k_2k_3\right)+2a_4\left(k_1^2k_2k_4+k_1k_2^2k_3\right)+a_5\left(k_2^3k_3+3k_1k_2^2k_4\right)=0,$$

$$4a_1k_1k_3^3+4a_2k_2k_4^3+a_3\left(k_2k_3^3+3k_1k_3^2k_4\right)+2a_4\left(k_2k_3^2k_4+k_1k_3k_4^2\right)+a_5\left(k_1k_4^3+3k_2k_3k_4^2\right)=0,$$

$$6a_1k_1^2k_3^2+6a_2k_2^2k_4^2+3a_3\left(k_1^2k_3k_4+k_1k_2k_3^2\right)+a_4\left(k_1^2k_4^2+4k_1k_2k_3k_4+k_2^2k_3^2\right)$$
$$+3a_5\left(k_2^2k_3k_4+k_1k_2k_4^2\right)=0.$$

Theorem 27. *In $\mathbb{R}^2$, the general solution of*

$$a_1u_{tttt}+a_2u_{xxxx}+a_3u_{tttx}+a_4u_{ttxx}+a_5u_{txxx}=A\left(t,x\right),$$

is

$$u=f_1\left(p\right)+f_2\left(q\right)+pf_3\left(q\right)+p^2f_4\left(q\right)+\frac{1}{B_3}\int\cdots\int A\left(p,q\right)dpdpdpdq, \tag{2.199}$$

where f_1, f_2, f_3 and f_4 are random fourth differentiable functions, and

$$B_3=4a_1k_1^3k_3+4a_2k_2^3k_4+a_3\left(k_1^3k_4+3k_1^2k_2k_3\right)+2a_4\left(k_1^2k_2k_4+k_1k_2^2k_3\right)+a_5\left(k_2^3k_3+3k_1k_2^2k_4\right), \tag{2.200}$$

$$a_1k_1^4+a_2k_2^4+a_3k_1^3k_2+a_4k_1^2k_2^2+a_5k_1k_2^3=0,$$

$$a_1k_3^4+a_2k_4^4+a_3k_3^3k_4+a_4k_3^2k_4^2+a_5k_3k_4^3=0,$$

$$4a_1k_1k_3^3+4a_2k_2k_4^3+a_3\left(k_2k_3^3+3k_1k_3^2k_4\right)+2a_4\left(k_2k_3^2k_4+k_1k_3k_4^2\right)+a_5\left(k_1k_4^3+3k_2k_3k_4^2\right)=0,$$

$$6a_1k_1^2k_3^2+6a_2k_2^2k_4^2+3a_3\left(k_1^2k_3k_4+k_1k_2k_3^2\right)+a_4\left(k_1^2k_4^2+4k_1k_2k_3k_4+k_2^2k_3^2\right)$$
$$+3a_5\left(k_2^2k_3k_4+k_1k_2k_4^2\right)=0.$$

3. General solution and Fourier series solution of one-dimensional homogeneous wave equation.

The 1D homogeneous wave equation

$$u_{tt}-a^2u_{xx}=0, \tag{3.1}$$

is one of the earliest PDEs to be studied deeply. Almost all current professional books and textbooks have pointed out that the general solution of Eq. (3.1) is

$$u=f_1(x+at)+f_2(x-at). \tag{3.2}$$

Fourier series solution of Eq. (3.1) is

$$u=\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi at}{l}\right)+B_n\sin\left(\frac{n\pi at}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right). \tag{3.3}$$

By (3.2) we could not get (3.3) obviously, there is no answer why the particular solution (3.3) could not be get by the general solution (3.2).

In our previous paper,[21] it was once get

$$a_1u_{x_1}^{(2)}+a_2u_{x_2}^{(2)}+\ldots+a_nu_{x_n}^{(2)}+a_{n+1}u_{x_2x_3}=0, \tag{3.4}$$

the basic general solution and the series general solution for Eq. (3.4) are

$$\begin{aligned}u={}&f_1\left(\left(-\frac{a_2k_2^2+\ldots+a_nk_n^2+a_{n+1}k_2k_3}{a_1}\right)^{\frac{1}{2}}x_1+k_2x_2+\ldots+k_nx_n+k_{n+1}\right)\\&+f_2\left(-\left(-\frac{a_2l_2^2+\ldots+a_nl_n^2+a_{n+1}l_2l_3}{a_1}\right)^{\frac{1}{2}}x_1+l_2x_2+\ldots+l_nx_n+l_{n+1}\right)+c_1v,\end{aligned} \tag{3.5}$$

$$\begin{aligned}u={}&\sum_{i=1}^{s}f_{1_i}\left(\left(-\frac{a_2k_{i_2}^2+\ldots+a_nk_{i_n}^2+a_{n+1}k_{i_2}k_{i_3}}{a_1}\right)^{\frac{1}{2}}x_1+k_{i_2}x_2+\ldots+k_{i_n}x_n+k_{i_{n+1}}\right)\\&+\sum_{i=1}^{s}f_{2_i}\left(-\left(-\frac{a_2l_{i_2}^2+\ldots+a_nl_{i_n}^2+a_{n+1}l_{i_2}l_{i_3}}{a_1}\right)^{\frac{1}{2}}x_1+l_{i_2}x_2+\ldots+l_{i_n}x_n+l_{i_{n+1}}\right)+c_1v,\end{aligned} \tag{3.6}$$

where f_1, f_{1_i}, f_2 and f_{2_i} are random unary 2th-differentiable functions, k_2-k_{n+1} and l_2-l_{n+1} are arbitrary parameters, $k_{i_2}-k_{i_{n+1}}$ and $l_{i_2}-l_{i_{n+1}}$ are random determined parameters.

Eq. (3.1) is a special case of Eq. (3.4), by (3.5) and (3.6), its basic general solution and series general solution may be get respectively

$$u=f_1\left(k_1x+k_1at+k_2\right)+f_2\left(k_3x-k_3at+k_4\right)+k_5x+k_6t+k_7, \tag{3.7}$$

$$u=\sum_{i=1}^{s}\left(f_{1_i}\left(k_{1_i}x+k_{1_i}at+k_{2_i}\right)+f_{2_i}\left(k_{3_i}x-k_{3_i}at+k_{4_i}\right)\right)+k_5x+k_6t+k_7, \tag{3.8}$$

where f_1, f_{1_i}, f_2 and f_{2_i} are random unary second differentiable functions, k_1-k_7 are random parameters, $k_{1_i}-k_{4_i}$ are random determined parameters, $(1\le s\le\infty)$. Of course, the general solution of Eq. (3.1) may also be written as

$$u=f_1\left(k_1x+k_1at+k_2\right)+\sum_{i=1}^{s}f_{2_i}\left(k_{3_i}x-k_{3_i}at+k_{4_i}\right)+k_5x+k_6t+k_7, \tag{3.9}$$

and so on, but in the paper we will not discuss general solutions in special forms.

By the above results we may see that (3.2) is a special case of (3.7) and (3.8), and is not an intact general solution of Eq. (3.1), so by (3.2) we cannot get the Fourier series solution.

Theoretically every specific series solution of Eq. (3.1) may be get by (3.8), as a case, we will get the Fourier series solution (3.3).

Set

$$f_{1_n}\left(k_{1_n}x+k_{1_n}at+k_{2_n}\right)=C_n\sin\left(k_{1_n}x+k_{1_n}at+k_{2_n}\right), \tag{3.10}$$

$$f_{2_n}\left(k_{3_n}x-k_{3_n}at+k_{4_n}\right)=D_n\cos\left(k_{3_n}x-k_{3_n}at+k_{4_n}\right). \tag{3.11}$$

So

$$\begin{aligned}
u &= \sum_{n=1}^{s} \left(f_{1_n}\left(k_{1_n}x + k_{1_n}at + k_{2_n}\right) + f_{2_n}\left(k_{3_n}x - k_{3_n}at + k_{4_n}\right)\right) \\
&= \sum_{n=1}^{s} C_n \left(\sin\left(k_{1_n}x\right)\cos\left(k_{1_n}at\right)\cos k_{2_n} + \cos\left(k_{1_n}x\right)\sin\left(k_{1_n}at\right)\cos k_{2_n}\right) \\
&+ \sum_{n=1}^{s} C_n \left(\cos\left(k_{1_n}x\right)\cos\left(k_{1_n}at\right)\sin k_{2_n} - \sin\left(k_{1_n}x\right)\sin\left(k_{1_n}at\right)\sin k_{2_n}\right) \\
&+ \sum_{n=1}^{s} D_n \left(\cos\left(k_{3_n}x\right)\cos\left(k_{3_n}at\right)\cos k_{4_n} + \sin\left(k_{3_n}x\right)\sin\left(k_{3_n}at\right)\cos k_{4_n}\right) \\
&+ \sum_{n=1}^{s} D_n \left(-\sin\left(k_{3_n}x\right)\cos\left(k_{3_n}at\right)\sin k_{4_n} + \cos\left(k_{3_n}x\right)\sin\left(k_{3_n}at\right)\sin k_{4_n}\right).
\end{aligned}$$

Set $k_{1_n} = k_{3_n} = k_n$, then

$$\begin{aligned}
u &= \sum_{n=1}^{s} \left(C_n\cos k_{2_n} - D_n\sin k_{4_n}\right)\sin\left(k_nx\right)\cos\left(k_nat\right) \\
&+ \sum_{n=1}^{s} \left(C_n\cos k_{2_n} + D_n\sin k_{4_n}\right)\cos\left(k_nx\right)\sin\left(k_nat\right) \\
&+ \sum_{n=1}^{s} \left(C_n\sin k_{2_n} + D_n\cos k_{4_n}\right)\cos\left(k_nx\right)\cos\left(k_nat\right) \\
&+ \sum_{n=1}^{s} \left(-C_n\sin k_{2_n} + D_n\cos k_{4_n}\right)\sin\left(k_nx\right)\sin\left(k_nat\right).
\end{aligned} \tag{3.12}$$

Set

$$C_n\cos k_{2_n} + D_n\sin k_{4_n} = C_n\sin k_{2_n} + D_n\cos k_{4_n} = 0.$$

And set

$$C_n = -D_n.$$

We get

$$k_{4_n} = \pi/2 - k_{2_n}.$$

Substituting the above results into (3.12)

$$\begin{aligned}
u &= \sum_{n=1}^{s} \left(C_n\cos k_{2_n} - D_n\sin k_{4_n}\right)\sin\left(k_nx\right)\cos\left(k_nat\right) \\
&+ \sum_{n=1}^{s} \left(-C_n\sin k_{2_n} + D_n\cos k_{4_n}\right)\sin\left(k_nx\right)\sin\left(k_nat\right) \\
&= \sum_{n=1}^{s} \left(2C_n\cos k_{2_n}\sin\left(k_nx\right)\cos\left(k_nat\right) - 2C_n\sin k_{2_n}\sin\left(k_nx\right)\sin\left(k_nat\right)\right).
\end{aligned}$$

Namely

$$u = \sum_{n=1}^{s} 2C_n\left(\cos k_{2_n}\cos\left(k_nat\right) - \sin k_{2_n}\sin\left(k_nat\right)\right)\sin\left(k_nx\right). \tag{3.13}$$

Since C_n, k_n and k_{2_n} are all random parameters, set

$$k_n = \frac{n\pi}{l}, 2C_n\cos k_{2_n} = A_n, -2C_n\sin k_{2_n} = B_n. \tag{3.14}$$

Then (3.13) can be translated into (3.3). (3.2) was first discovered by d' Alembert, then Daniel Bernoulli found an infinite series solution

$$u = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right). \tag{3.15}$$

Both (3.2) and (3.15) have important application. Which one is the most basic solution and could replace the other has triggered a famed debate in the history of mathematics.[1] Many famous mathematicians have been involved in this lengthy and drastic debate, such as Euler, Daniel Bernoulli,d'Alembert, Lagrange, Laplace and so on. Even after discovering the Fourier series solution (3.3), the relationship between (3.2) and (3.3) was still unclear, now the problem is solved successfully. (3.3) and (3.15) can be get by using the complete series general solution (3.8), but (3.2) is not a complete general solution, so it could not be used to get (3.3) and (3.15). All important applications of (3.2), (3.3) and (3.15) may be obtained using (3.7) or (3.8).

4. Exact solutions of definite solution problems of some typical linear partial differential equations

Below we will use the general solutions get in the paper and our previous papers to obtain exact solutions of some typical definite solution problems.

Example 1. In $\mathbb{R}^3$, to obtain the exact solution of

$$u_t + u_x + u_y = \mathrm{e}^{t+x+y}, \tag{4.1}$$

in the condition of $u(0, x, y) = \phi(x, y)$, ϕ is a random known first differentiable function.

Solution. by Theorem 11* in our previous paper [22], the general solution of (4.1) is

$$u = f\left(\left(-c_2 - c_3\right)t + c_2 x + c_3 y, \left(-c_5 - c_6\right)t + c_5 x + c_6 y\right) + \frac{1}{3}\mathrm{e}^{t+x+y}. \tag{4.2}$$

So

$$u\left(0, x, y\right) = f\left(c_2 x + c_3 y, c_5 x + c_6 y\right) + \frac{1}{3}\mathrm{e}^{x+y} = \phi\left(x, y\right). \tag{4.3}$$

Set

$$c_2 x_2 + c_3 x_3 = \beta, c_5 x_2 + c_6 x_3 = \gamma.$$

We obtain

$$x = \frac{c_3\gamma - c_6\beta}{c_3 c_5 - c_2 c_6}, y = \frac{c_5\beta - c_2\gamma}{c_3 c_5 - c_2 c_6}.$$

Namely

$$\begin{aligned} f\left(c_2 x + c_3 y, c_5 x + c_6 y\right) &= f\left(\beta, \gamma\right) = \phi\left(x, y\right) - \frac{1}{3}\mathrm{e}^{x+y} \\ &= \phi\left(\frac{c_3\gamma - c_6\beta}{c_3 c_5 - c_2 c_6}, \frac{c_5\beta - c_2\gamma}{c_3 c_5 - c_2 c_6}\right) - \frac{1}{3}\mathrm{e}^{\frac{c_3\gamma - c_6\beta}{c_3 c_5 - c_2 c_6} + \frac{c_5\beta - c_2\gamma}{c_3 c_5 - c_2 c_6}}. \end{aligned}$$

Set

$$(-c_2 - c_3)t + c_2 x + c_3 y = \beta,$$
$$(-c_5 - c_6)t + c_5 x + c_6 y = \gamma.$$

Then

$$\frac{c_3\gamma - c_6\beta}{c_3 c_5 - c_2 c_6} = -t + x,$$
$$\frac{c_5\beta - c_2\gamma}{c_3 c_5 - c_2 c_6} = -t + y,$$

$$f\left(\left(-c_2 - c_3\right)t + c_2 x + c_3 y, \left(-c_5 - c_6\right)t + c_5 x + c_6 y\right) = \phi\left(-t + x, -t + y\right) - \frac{1}{3}\mathrm{e}^{-2t+x+y}.$$

By Eq. (4.2), the exact solution of the definite solution problem is

$$u\left(t,x,y\right)=\phi\left(-t+x,-t+y\right)-\frac{1}{3}\mathrm{e}^{-2t+x+y}+\frac{1}{3}\mathrm{e}^{t+x+y}. \tag{4.4}$$

By Example 1, we may directly obtain the exact solution of Eq. (4.1) in various initial value conditions. If the initial value condition is $u(0,x,y)=xy$, the exact solution is $u=t^2-ty-tx+xy-\frac{1}{3}\mathrm{e}^{-2t+x+y}+\frac{1}{3}\mathrm{e}^{t+x+y}$.

Example 2. In $\mathbb{R}^3$, to obtain the exact solution of

$$\mathrm{cos}y u_x+\left(\mathrm{e}^x-1\right)u_y-\mathrm{e}^x\mathrm{cos}y u_z=yz\mathrm{cos}y+xz\left(\mathrm{e}^x-1\right)-xy\mathrm{e}^x\mathrm{cos}y, \tag{4.5}$$

in the condition of $u(0,y,z)=\phi(y,z),\phi$ is a random known first differentiable function.

Solution. By Theorem 4* in our previous paper [22], the general solution of (4.5) is

$$u=f\left(e^x+z,x+\mathrm{sin}y+z\right)+xyz. \tag{4.6}$$

So

$$u\left(0,x,y\right)=f\left(1+z,\mathrm{sin}y+z\right)=\phi\left(y,z\right). \tag{4.7}$$

Set

$$1+z=\beta,\mathrm{sin}y+z=\gamma.$$

Then

$$z=\beta-1,y=\arcsin(\gamma-\beta+1).$$

And

$$f(\beta,\gamma)=\phi(\arcsin(\gamma-\beta+1),\beta-1),$$

$$f\left(e^x+z,x+\mathrm{sin}y+z\right)=\phi\left(\arcsin\left(x+\mathrm{sin}y-e^x+1\right),e^x+z-1\right).$$

By Eq. (4.6), the exact solution of the definite solution problem is

$$u=\phi\left(\arcsin\left(x+\mathrm{sin}y-e^x+1\right),e^x+z-1\right)+xyz. \tag{4.8}$$

Example 3. In $\mathbb{R}^3$, get the exact solution of

$$u_t+u_x+u_y+u=\mathrm{e}^{t+x+y}, \tag{4.9}$$

in the condition of $u(0,x,y)=\phi(x,y),\phi$ is a random known first differentiable function.

Solution. By Theorem 17* in our previous paper [22], the general solution of (4.9) is

$$u=\mathrm{e}^{-t}f\left(\left(-c_2-c_3\right)t+c_2x+c_3y,\left(-c_5-c_6\right)t+c_5x+c_6y\right)+\frac{1}{4}\mathrm{e}^{t+x+y}. \tag{4.10}$$

So

$$u\left(0,x,y\right)=f\left(c_2x+c_3y,c_5x+c_6y\right)+\frac{1}{4}\mathrm{e}^{x+y}=\phi\left(x,y\right). \tag{4.11}$$

Set

$$c_2x_2+c_3x_3=\beta,c_5x_2+c_6x_3=\gamma.$$

We obtain

$$x=\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},y=\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}.$$

Namely

$$f\left(c_2x+c_3y,c_5x+c_6y\right)=f\left(\beta,\gamma\right)=\phi\left(x,y\right)-\frac{1}{4}\mathrm{e}^{x+y}$$
$$=\phi\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)-\frac{1}{4}\mathrm{e}^{\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}+\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}}.$$

Set

$$(-c_2-c_3)t+c_2x+c_3y=\beta,$$
$$(-c_5-c_6)t+c_5x+c_6y=\gamma.$$

Then

$$\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}=-t+x,$$
$$\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}=-t+y,$$
$$f\left(\left(-c_2-c_3\right)t+c_2x+c_3y,\left(-c_5-c_6\right)t+c_5x+c_6y\right)=\phi\left(-t+x,-t+y\right)-\frac{1}{4}\mathrm{e}^{-2t+x+y}.$$

By Eq. (4.10), the exact solution of the definite solution problem is

$$u\left(t,x,y\right)=\mathrm{e}^{-t}\phi\left(-t+x,-t+y\right)-\frac{1}{4}\mathrm{e}^{-3t+x+y}+\frac{1}{4}\mathrm{e}^{t+x+y}. \tag{4.12}$$

By Example 3, we may directly obtain the exact solution of Eq. (4.9) in various initial value conditions. If the initial value condition is $u(0,x,y)=xy$, the exact solution is $u=\mathrm{e}^{-t}\left(t^2-ty-tx+xy\right)-\frac{1}{4}\mathrm{e}^{-3t+x+y}+\frac{1}{4}\mathrm{e}^{t+x+y}$.

Example 4. In $\mathbb{R}^2$, get the exact solution of

$$6u_{tt}-3u_{xx}+3u_{tx}-38u_t+19u_x=0, \tag{4.13}$$

in the conditions of $u(0,x)=\Phi(x)$ and $u_t(0,x)=\Psi(x)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. By Theorem 3, the general solution of (4.13) is

$$u=f\left(q\right)+\mathrm{e}^{2t+4x}g\left(p\right)=f\left(\frac{t}{2}+x\right)+\mathrm{e}^{2t+4x}g\left(-t+x\right). \tag{4.14}$$

So

$$u\left(0,x\right)=f\left(x\right)+\mathrm{e}^{4x}g\left(x\right)=f\left(q\right)+\mathrm{e}^{4x}g\left(p\right)=\Phi\left(x\right), \tag{4.15}$$
$$u_t\left(0,x\right)=\frac{1}{2}f_q^{'}+2\mathrm{e}^{4x}g-\mathrm{e}^{4x}g_p^{'}=\Psi\left(x\right). \tag{4.16}$$

When $t=0,p=q=x$, then

$$f_q^{'}+4\mathrm{e}^{4x}g+\mathrm{e}^{4x}g_p^{'}=\Phi'\left(x\right).$$

So

$$3\mathrm{e}^{4x}g_p^{'}=\Phi'\left(x\right)-2\Psi\left(x\right),$$
$$g_p^{'}=\frac{1}{3}\mathrm{e}^{-4x}\left(\Phi'\left(x\right)-2\Psi\left(x\right)\right),$$
$$g\left(p\right)=\frac{1}{3}\int_a^p\mathrm{e}^{-4x}\left(\Phi'\left(x\right)-2\Psi\left(x\right)\right)dp=\frac{1}{3}\int_a^p\mathrm{e}^{-4p}\left(\Phi'\left(p\right)-2\Psi\left(p\right)\right)dp,$$

$$f\left(q\right)=\Phi\left(x\right)-\mathrm{e}^{4x}g\left(p\right)=\Phi\left(q\right)-\frac{1}{3}\mathrm{e}^{4q}\int_a^q\mathrm{e}^{-4q}\left(\Phi'\left(q\right)-2\Psi\left(q\right)\right)dq.$$

Therefore

$$g\left(-t+x\right)=\frac{1}{3}\int_a^{-t+x}\mathrm{e}^{-4(-t+x)}\left(\Phi'\left(-t+x\right)-2\Psi\left(-t+x\right)\right)d\left(-t+x\right),$$

$$f\left(\frac{t}{2}+x\right)=\Phi\left(\frac{t}{2}+x\right)-\frac{1}{3}\mathrm{e}^{2t+4x}\int_a^{\frac{t}{2}+x}\mathrm{e}^{-2t-4x}\left(\Phi'\left(\frac{t}{2}+x\right)-2\Psi\left(\frac{t}{2}+x\right)\right)d\left(\frac{t}{2}+x\right).$$

On the basis of Eq. (4.14), the exact solution of the definite solution problem is

$$\begin{aligned}u=\Phi\left(\frac{t}{2}+x\right)&-\frac{1}{3}\mathrm{e}^{2t+4x}\int_a^{\frac{t}{2}+x}\mathrm{e}^{-4\left(\frac{t}{2}+x\right)}\left(\Phi'\left(\frac{t}{2}+x\right)-2\Psi\left(\frac{t}{2}+x\right)\right)d\left(\frac{t}{2}+x\right)\\+\frac{1}{3}\mathrm{e}^{2t+4x}&\int_a^{-t+x}\mathrm{e}^{-4(-t+x)}\left(\Phi'\left(-t+x\right)-2\Psi\left(-t+x\right)\right)d\left(-t+x\right).\end{aligned}\tag{4.17}$$

By Example 4, we may directly obtain the exact solution of Eq. (4.13) in various initial value conditions. If the initial value condition are $u(0,x)=x^2, u_t(0,x)=\sin x$, the exact solution is

$$\begin{aligned}u=&\left(\frac{t}{2}+x\right)^2-\frac{2}{51}\left(4\sin\left(\frac{t}{2}+x\right)+\cos\left(\frac{t}{2}+x\right)\right)+\frac{4\left(\frac{t}{2}+x\right)+1}{24}\\&+e^{6t}\left(\frac{2}{51}\left(4\sin(-t+x)+\cos(-t+x)\right)-\frac{4(-t+x)+1}{24}\right).\end{aligned}$$

Example 5. In $\mathbb{R}^2$, get the exact solution of

$$u_{tt}+u_{xx}-2u_{tx}+6u_t-6u_x=\left(x^2+t^2-2xt-4t+4x-7\right)\mathrm{e}^{tx+x},\tag{4.18}$$

in the conditions of $u(0,x)=\Phi(x)$ and $u_t(0,x)=\Psi(x)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. By Theorem 4, the general solution of (4.18) is

$$u=f\left(t+x\right)+\mathrm{e}^{3t+9x}g\left(t+x\right)+\mathrm{e}^{tx+x}.\tag{4.19}$$

Set $p=t+x$, then

$$u\left(0,x\right)=f\left(x\right)+\mathrm{e}^{9x}g\left(x\right)+\mathrm{e}^{x}=f+\mathrm{e}^{9x}g+\mathrm{e}^{x}=\Phi\left(x\right),\tag{4.20}$$

$$u_t\left(0,x\right)=f_p^{'}+3\mathrm{e}^{9x}g+\mathrm{e}^{9x}g_p^{'}+x\mathrm{e}^{x}=\Psi\left(x\right).\tag{4.21}$$

When $t=0, p=x$, so

$$\Phi'\left(x\right)=f_p^{'}+9\mathrm{e}^{9x}g+\mathrm{e}^{9x}g_p^{'}+\mathrm{e}^{x}.$$

And

$$\Phi'\left(x\right)-\Psi\left(x\right)=6\mathrm{e}^{9x}g+\mathrm{e}^{x}-x\mathrm{e}^{x}.$$

Then

$$g\left(x\right)=\frac{1}{6}\mathrm{e}^{-9x}\left(\Phi'\left(x\right)-\Psi\left(x\right)-\mathrm{e}^{x}+x\mathrm{e}^{x}\right),$$

$$f\left(x\right)=\Phi\left(x\right)-\mathrm{e}^{9x}g\left(x\right)-\mathrm{e}^{x}=\Phi\left(x\right)-\frac{1}{6}\left(\Phi'\left(x\right)-\Psi\left(x\right)-\mathrm{e}^{x}+x\mathrm{e}^{x}\right)-\mathrm{e}^{x}.$$

Therefore

$$f\left(t+x\right)=\Phi\left(t+x\right)-\frac{1}{6}\left(\Phi'\left(t+x\right)-\Psi\left(t+x\right)-\mathrm{e}^{t+x}+\left(t+x\right)\mathrm{e}^{t+x}\right)-\mathrm{e}^{t+x},$$

$$g\left(t+x\right)=\frac{1}{6}\mathrm{e}^{-9(t+x)}\left(\Phi'\left(t+x\right)-\Psi\left(t+x\right)-\mathrm{e}^{t+x}+\left(t+x\right)\mathrm{e}^{t+x}\right).$$

On the basis of Eq. (4.19), the exact solution of the definite solution problem is

$$\begin{aligned}u&=\Phi\left(t+x\right)-\frac{1}{6}\left(\Phi'\left(t+x\right)-\Psi\left(t+x\right)+\left(t+x\right)\mathrm{e}^{t+x}\right)-\frac{5}{6}\mathrm{e}^{t+x}\\&+\frac{1}{6}\mathrm{e}^{-6t}\left(\Phi'\left(t+x\right)-\Psi\left(t+x\right)-\mathrm{e}^{t+x}+\left(t+x\right)\mathrm{e}^{t+x}\right)+\mathrm{e}^{tx+x}.\end{aligned}\tag{4.22}$$

According to Example 5, we may directly get the exact solution of Eq. (4.18) in various initial value conditions. If the initial value condition are $u(0,x)=x\sin x, u_t(0,x)=\mathrm{e}^x\cos x$, the exact solution is

$$\begin{aligned}u&=-\frac{1}{6}\left(\sin(t+x)+(t+x)\cos(t+x)-\mathrm{e}^{t+x}\cos(t+x)+(t+x)\mathrm{e}^{t+x}\right)\\&+\frac{1}{6}\mathrm{e}^{-6t}\left(\sin(t+x)+(t+x)\cos(t+x)-\mathrm{e}^{t+x}\cos(t+x)-\mathrm{e}^{t+x}+(t+x)\mathrm{e}^{t+x}\right)\\&+(t+x)\sin(t+x)-\frac{5}{6}\mathrm{e}^{t+x}+\mathrm{e}^{tx+x}\end{aligned}$$

Example 6. In $\mathbb{R}^2$, get the exact solution of

$$u_{tt}-3u_{xx}-2u_{tx}=-2\mathrm{cos}tx+\left(3t^2-x^2+2tx\right)\mathrm{sin}tx,\tag{4.23}$$

in the conditions of $u(0,x)=\Phi(x)$ and $u_t(0,x)=\Psi(x)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. By Theorem 6, the general solution of (4.23) is

$$u=f(p)+g(q)+\mathrm{sin}tx=f(-t+x)+g(3t+x)+\mathrm{sin}tx.\tag{4.24}$$

So

$$u(0,x)=f(x)+g(x)=\Phi(x),\tag{4.25}$$

$$u_t\left(0,x\right)=-f_p^{'}\left(x\right)+3g_q^{'}\left(x\right)+x=\Psi\left(x\right).\tag{4.26}$$

When $t=0, p=q=x$, then

$$-f\left(x\right)+3g\left(x\right)+\frac{x^2}{2}=\int\Psi\left(x\right)dx.$$

So

$$g\left(x\right)=\frac{1}{4}\left(\Phi\left(x\right)+\int\Psi\left(x\right)dx-\frac{x^2}{2}\right),$$

$$f\left(x\right)=\Phi\left(x\right)-g\left(x\right)=\frac{3}{4}\Phi\left(x\right)-\frac{1}{4}\int\Psi\left(x\right)dx+\frac{x^2}{8}.$$

Therefore

$$f\left(-t+x\right)=\frac{3}{4}\Phi\left(-t+x\right)-\frac{1}{4}\int\Psi\left(-t+x\right)d\left(-t+x\right)+\frac{\left(-t+x\right)^2}{8},$$

$$g\left(3t+x\right)=\frac{1}{4}\Phi\left(3t+x\right)+\frac{1}{4}\int\Psi\left(3t+x\right)d\left(3t+x\right)-\frac{\left(3t+x\right)^2}{8}.$$

On the basis of Eq. (4.24), the exact solution of the definite solution problem is

$$\begin{aligned}u=&\frac{3}{4}\Phi\left(-t+x\right)-\frac{1}{4}\int\Psi\left(-t+x\right)d\left(-t+x\right)+\frac{\left(-t+x\right)^2}{8}+\frac{1}{4}\Phi\left(3t+x\right)\\&+\frac{1}{4}\int\Psi\left(3t+x\right)d\left(3t+x\right)-\frac{\left(3t+x\right)^2}{8}+\text{sin}tx.\end{aligned}\tag{4.27}$$

According to Example 6, we may directly get the exact solution of Eq. (4.23) in various initial value conditions. If the initial value condition are $u(0,x)=x\ln x, u_t(0,x)=\cos x$, the exact solution is

$$\begin{aligned}u=&\frac{3}{4}(-t+x)\ln(-t+x)-\frac{1}{4}\sin(-t+x)+\frac{(-t+x)^2}{8}\\&+\frac{1}{4}(3t+x)\ln(3t+x)+\frac{1}{4}\sin(3t+x)-\frac{(3t+x)^2}{8}+\sin tx\end{aligned}$$

Example 7. In $\mathbb{R}^2$, get the exact solution of

$$u_{tt}-a^2u_{xx}=A\left(x,t\right),\tag{4.28}$$

in the conditions of $u(0,x)=\varphi(x)$ and $u_t(0,x)=\psi(x)$. φ is a random known second differentiable function, ψ is a random known first differentiable function.

Solution. By Theorem 5, the general solution of (4.28) is

$$u=f\left(x+at\right)+g\left(x-at\right)-\frac{1}{4a^2}\iint A\left(\frac{p-q}{2a},\frac{p+q}{2}\right)dpdq,\tag{4.29}$$

where

$$p=x+at, q=x-at.$$

Set

$$B\left(t,x\right)=-\frac{1}{4a^2}\iint A\left(\frac{p-q}{2a},\frac{p+q}{2}\right)dpdq.\tag{4.30}$$

So

$$u(0,x)=f(x)+g(x)+B(0,x)=\varphi(x),\tag{4.31}$$

$$u_t(0,x)=af_p^{'}(x)-ag_q^{'}(x)+B_t(0,x)=\psi(x).\tag{4.32}$$

When $t=0, p=q=x$, then

$$f_p^{'}(x)+g_q^{'}(x)=\varphi^{'}(x)-B_x^{'}(0,x),$$

$$f_p^{'}\left(x\right)-g_q^{'}\left(x\right)=\frac{1}{a}\psi\left(x\right)-\frac{1}{a}B_t\left(0,x\right).$$

So

$$f_p^{'}\left(x\right)=\frac{1}{2}\varphi'\left(x\right)+\frac{1}{2a}\psi\left(x\right)-\frac{1}{2}B_x^{'}\left(0,x\right)-\frac{1}{2a}B_t\left(0,x\right),$$

$$f\left(x\right)=\frac{1}{2}\varphi\left(x\right)-\frac{1}{2}B\left(0,x\right)+\frac{1}{2a}\int_{x_0}^{x}\left(\psi\left(\xi\right)-B_t\left(0,\xi\right)\right)d\xi,$$

$$g(x)=\varphi(x)-B(0,x)-f(x)=\frac{1}{2}\varphi(x)-\frac{1}{2}B(0,x)+\frac{1}{2a}\int_{x_0}^{x}(\psi(\xi)-B_t(0,\xi))\,d\xi.$$

Therefore

$$f(x+at)=\frac{1}{2}\left(\varphi(x+at)-B(0,x+at)+\frac{1}{a}\int_{x_0}^{x+at}(\psi(\xi)-B_t(0,\xi))\,d\xi\right),$$

$$g(x-at)=\frac{1}{2}\left(\varphi(x-at)-B(0,x-at)-\frac{1}{a}\int_{x_0}^{x-at}(\psi(\xi)-B_t(0,\xi))\,d\xi\right).$$

By Eq. (4.29), the exact solution of the definite solution problem is

$$u=\frac{1}{2}\left(\varphi(x+at)+\varphi(x-at)-B(0,x+at)-B(0,x-at)+\frac{1}{a}\int_{x-at}^{x+at}(\psi(\xi)-B_t(0,\xi))\,d\xi\right)$$
$$+B(t,x). \tag{4.33}$$

Example 8. In $\mathbb{R}^2$, get the exact solution of

$$u_{xx}+u_{yy}-2u_{xy}+2u_x-2u_y+u=0, \tag{4.34}$$

in the conditions of $u(0,y)=\Phi(y)$ and $u_x(0,y)=\Psi(y)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. By Theorem 12, the general solution of (4.34) is

$$u=\mathrm{e}^{x+2y}(f(p)+qh(p))=\mathrm{e}^{x+2y}(f(x+y)+(2x+y)h(x+y)). \tag{4.35}$$

So

$$u(0,y)=\mathrm{e}^{2y}(f(y)+yh(y))=\mathrm{e}^{2y}f+y\mathrm{e}^{2y}h=\Phi(y), \tag{4.36}$$

$$u_x(0,y)=\mathrm{e}^{2y}f+y\mathrm{e}^{2y}h+2\mathrm{e}^{2y}h+\mathrm{e}^{2y}f_p'+y\mathrm{e}^{2y}h_p'=\Psi(y). \tag{4.37}$$

When $x=0, p=y$, then

$$2\mathrm{e}^{2y}f+\mathrm{e}^{2y}f_p'+\mathrm{e}^{2y}h+2y\mathrm{e}^{2y}h+y\mathrm{e}^{2y}h_p'=\Phi',$$

$$\mathrm{e}^{2y}f+y\mathrm{e}^{2y}h-\mathrm{e}^{2y}h=\Phi'-\Psi.$$

So

$$h(y)=\mathrm{e}^{-2y}\left(\Phi+\Psi-\Phi'\right),$$

$$f(y)=\mathrm{e}^{-2y}\left(\Phi-y\mathrm{e}^{2y}h\right)=\mathrm{e}^{-2y}\Phi-\mathrm{e}^{-2y}y\left(\Phi+\Psi-\Phi'\right).$$

Therefore

$$f(x+y)=\mathrm{e}^{-2(x+y)}\Phi(x+y)-\mathrm{e}^{-2(x+y)}(x+y)\left(\Phi(x+y)+\Psi(x+y)-\Phi'(x+y)\right),$$

$$h(x+y)=\mathrm{e}^{-2(x+y)}\left(\Phi(x+y)+\Psi(x+y)-\Phi'(x+y)\right).$$

By Eq. (4.35), the exact solution of the definite solution problem is

$$u=\mathrm{e}^{-x}\Phi(x+y)+x\mathrm{e}^{-x}\left(\Phi(x+y)+\Psi(x+y)-\Phi'(x+y)\right). \tag{4.38}$$

Example 9. In $\mathbb{R}^2$, get the exact solution of

$$2u_{xx} + 2u_{yy} - 5u_{xy} + u_x + u_y - u = \left(2x^2 + 2y^2 - 5xy - 5\right) \mathrm{e}^{x+y+xy}, \tag{4.39}$$

in the conditions of $u(0, y) = \Phi(y)$ and $u_x(0, y) = \Psi(y)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. By Theorem 11, the general solution of (4.39) is

$$u = \mathrm{e}^{x+y} \left(f\left(p\right) + h\left(q\right) + \mathrm{e}^{xy}\right) = \mathrm{e}^{x+y} \left(f\left(\frac{x}{2} + y\right) + h\left(2x + y\right) + \mathrm{e}^{xy}\right). \tag{4.40}$$

So

$$u\left(0, y\right) = \mathrm{e}^{y} \left(f\left(y\right) + h\left(y\right) + 1\right) = \mathrm{e}^{y} f + \mathrm{e}^{y} h + \mathrm{e}^{y} = \Phi\left(y\right), \tag{4.41}$$

$$u_x\left(0, y\right) = \mathrm{e}^{y} + \mathrm{e}^{y} f + \mathrm{e}^{y} h + \mathrm{e}^{y} y + \frac{1}{2}\mathrm{e}^{y} f_p^{'} + 2\mathrm{e}^{y} h_q^{'} = \Psi\left(y\right), \tag{4.42}$$

then

$$\mathrm{e}^{y} f + \mathrm{e}^{y} h = \Phi\left(y\right) - \mathrm{e}^{y},$$

$$f_p^{'} + 4h_q^{'} = 2\left(\mathrm{e}^{-y}\Psi\left(y\right) - \mathrm{e}^{-y}\Phi\left(y\right) - y\right).$$

When $x = 0, p = q = y$, then

$$f + 4h = 2\int_{y_0}^{y} \left(\mathrm{e}^{-\xi}\Psi\left(\xi\right) - \mathrm{e}^{-\xi}\Phi\left(\xi\right) - \xi\right) d\xi.$$

So

$$h\left(y\right) = \frac{2}{3}\int_{y_0}^{y} \left(\mathrm{e}^{-\xi}\Psi\left(\xi\right) - \mathrm{e}^{-\xi}\Phi\left(\xi\right) - \xi\right) d\xi - \frac{1}{3}\mathrm{e}^{-y}\Phi\left(y\right) - \frac{1}{3},$$

$$f\left(y\right) = \mathrm{e}^{-y}\Phi\left(y\right) - 1 - h = \frac{4}{3}\mathrm{e}^{-y}\Phi\left(y\right) - \frac{2}{3} - \frac{2}{3}\int_{y_0}^{y} \left(\mathrm{e}^{-\xi}\Psi\left(\xi\right) - \mathrm{e}^{-\xi}\Phi\left(\xi\right) - \xi\right) d\xi.$$

Therefore

$$h\left(2x + y\right) = \frac{2}{3}\int_{y_0}^{2x+y} \left(\mathrm{e}^{-\xi}\Psi\left(\xi\right) - \mathrm{e}^{-\xi}\Phi\left(\xi\right) - \xi\right) d\xi - \frac{1}{3}\mathrm{e}^{-2x-y}\Phi\left(2x + y\right) - \frac{1}{3},$$

$$f\left(\frac{x}{2} + y\right) = \frac{4}{3}\mathrm{e}^{-\frac{x}{2}-y}\Phi\left(\frac{x}{2} + y\right) - \frac{2}{3} - \frac{2}{3}\int_{y_0}^{\frac{x}{2}+y} \left(\mathrm{e}^{-\xi}\Psi\left(\xi\right) - \mathrm{e}^{-\xi}\Phi\left(\xi\right) - \xi\right) d\xi.$$

By Eq. (4.40), the exact solution of the definite solution problem is

$$u = \frac{2}{3}\mathrm{e}^{x+y}\int_{\frac{x}{2}+y}^{2x+y} \left(\mathrm{e}^{-\xi}\Psi\left(\xi\right) - \mathrm{e}^{-\xi}\Phi\left(\xi\right) - \xi\right) d\xi - \frac{1}{3}\mathrm{e}^{-x}\Phi\left(2x + y\right) + \frac{4}{3}\mathrm{e}^{\frac{x}{2}}\Phi\left(\frac{x}{2} + y\right) - \mathrm{e}^{x+y} + \mathrm{e}^{x+y+xy}. \tag{4.43}$$

Example 10. In $\mathbb{R}^3$, get the exact solution of

$$u_{xx} + u_{yy} + u_{zz} + 2u_{xy} + 2u_{xz} + 2u_{yz} = 0, \tag{4.44}$$

in the conditions of $u(0,y,z)=\Phi(y,z)$ and $u_x(0,y,z)=\Psi(y,z)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. For

$$u_{xx}+u_{yy}+u_{zz}+2u_{xy}+2u_{xz}+2u_{yz}=(D_x+D_y+D_z)^2u=0. \tag{4.45}$$

By Theorem 13* in our previous paper [22], the general solution of (4.45) is

$$u=f\left(p,q\right)+\left(\lambda_1x+\lambda_2y+\lambda_3z\right)g\left(p,q\right), \tag{4.46}$$

where f and g are arbitrary unary second differentiable functions; $\lambda_1,\lambda_2,\lambda_3$ are arbitrary parameters, and

$$p=c_1x+c_2y+c_3z, q=c_4x+c_5y+c_6z, r=c_7x+c_8y+c_9z, \tag{4.47}$$

$$-c_3c_5c_7+c_2c_6c_7+c_3c_4c_8-c_1c_6c_8-c_2c_4c_9+c_1c_5c_9\neq 0,$$
$$c_1=-c_2-c_3, c_4=-c_5-c_6, c_7\neq -c_8-c_9.$$

So

$$u\left(0,y,z\right)=\Phi\left(y,z\right)=f\left(c_2y+c_3z,c_5y+c_6z\right)+\left(\lambda_2y+\lambda_3z\right)g\left(c_2y+c_3z,c_5y+c_6z\right), \tag{4.48}$$

$$\begin{aligned}&u_x(0,y,z)=\Psi(y,z)\\&=\left(-c_2-c_3\right)f_p+\left(-c_5-c_6\right)f_q+\lambda_1g+\left(\lambda_2y+\lambda_3z\right)\left(\left(-c_2-c_3\right)g_p+\left(-c_5-c_6\right)g_q\right).\end{aligned} \tag{4.49}$$

When $x=0, p=c_2y+c_3z, q=c_5y+c_6z$, according to (4.48), we get

$$c_2f_p+c_5f_q+\lambda_2g+\left(\lambda_2y+\lambda_3z\right)\left(c_2g_p+c_5g_q\right)=\Phi_y, \tag{4.50}$$

$$c_3f_p+c_6f_q+\lambda_3g+\left(\lambda_2y+\lambda_3z\right)\left(c_3g_p+c_6g_q\right)=\Phi_z, \tag{4.51}$$

then

$$\left(\lambda_1+\lambda_2+\lambda_3\right)g=\Psi+\Phi_y+\Phi_z.$$

Namely

$$g\left(c_2y+c_3z,c_5y+c_6z\right)=\frac{\Psi+\Phi_y+\Phi_z}{\lambda_1+\lambda_2+\lambda_3}.$$

So

$$f\left(c_2y+c_3z,c_5y+c_6z\right)=\Phi-\frac{\left(\Psi+\Phi_y+\Phi_z\right)\left(\lambda_2y+\lambda_3z\right)}{\lambda_1+\lambda_2+\lambda_3}.$$

Set

$$c_2y+c_3z=\beta, c_5y+c_6z=\gamma,$$

then

$$y=\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}, z=\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}.$$

And

$$g(\beta,\gamma)=\frac{\Psi\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)+\Phi_y\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)+\Phi_z\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)}{\lambda_1+\lambda_2+\lambda_3},$$

$$f(\beta,\gamma)=\Phi\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)-\frac{\lambda_2\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}+\lambda_3\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}}{\lambda_1+\lambda_2+\lambda_3}\Psi\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)$$
$$-\frac{\lambda_2\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}+\lambda_3\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}}{\lambda_1+\lambda_2+\lambda_3}\Phi_y\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right)$$
$$-\frac{\lambda_2\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}+\lambda_3\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}}{\lambda_1+\lambda_2+\lambda_3}\Phi_z\left(\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6},\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}\right).$$

Set

$$(-c_2-c_3)x+c_2y+c_3z=\beta,(-c_5-c_6)x+c_5y+c_6z=\gamma.$$

So

$$\frac{c_3\gamma-c_6\beta}{c_3c_5-c_2c_6}=-x+y,\frac{c_5\beta-c_2\gamma}{c_3c_5-c_2c_6}=-x+z.$$

And

$$\begin{aligned}&g\left(\left(-c_2-c_3\right)x+c_2y+c_3z,\left(-c_5-c_6\right)x+c_5y+c_6z\right)\\&=\frac{\Psi\left(-x+y,-x+z\right)+\Phi_y\left(-x+y,-x+z\right)+\Phi_z\left(-x+y,-x+z\right)}{\lambda_1+\lambda_2+\lambda_3},\end{aligned}$$

$$\begin{aligned}&f\left(\left(-c_2-c_3\right)x+c_2y+c_3z,\left(-c_5-c_6\right)x+c_5y+c_6z\right)\\&=\Phi\left(-x+y,-x+z\right)+\frac{\left(\lambda_2+\lambda_3\right)x-\lambda_2y-\lambda_3z}{\lambda_1+\lambda_2+\lambda_3}\Psi\left(-x+y,-x+z\right)\\&+\frac{\left(\lambda_2+\lambda_3\right)x-\lambda_2y-\lambda_3z}{\lambda_1+\lambda_2+\lambda_3}\Phi_y\left(-x+y,-x+z\right)+\frac{\left(\lambda_2+\lambda_3\right)x-\lambda_2y-\lambda_3z}{\lambda_1+\lambda_2+\lambda_3}\Phi_z\left(-x+y,-x+z\right).\end{aligned}$$

By Eq. (4.46), the exact solution of the definite solution problem is

$$u=\Phi\left(-x+y,-x+z\right)+x\left(\Psi\left(-x+y,-x+z\right)+\Phi_y\left(-x+y,-x+z\right)+\Phi_z\left(-x+y,-x+z\right)\right). \tag{4.52}$$

According to Example 10, we may directly get the exact solution of Eq. (4.44) in various boundary value conditions. If the boundary value condition is $u(0,y,z)=yz, u_x(0,y,z)=y+z$, the exact solution is $u=-3x^2+xy+xz+yz$.

Example 11. In $\mathbb{R}^3$, get the exact solution of

$$u_{xx}+u_{yy}+2u_{zz}+2u_{xy}+3u_{yz}+3u_{zx}=0, \tag{4.53}$$

in the conditions of $u(0,y,z)=yz$ and $u_x(0,y,z)=y+z$.

Solution. According to Theorem 17, the general solution of (4.53) is

$$\begin{aligned}u=&f\left(\left(-c_2-c_3\right)x+c_2y+c_3z,\left(-c_5-c_6\right)x+c_5y+c_6z\right)\\&+g\left(\left(-c_2-2c_3\right)x+c_2y+c_3z,\left(-c_5-2c_6\right)x+c_5y+c_6z\right).\end{aligned} \tag{4.54}$$

Set $c_3=0$, then

$$\begin{aligned}u&=f(p,q)+g(p,r)\\&=f\left(-x+y,\left(-c_5-c_6\right)x+c_5y+c_6z\right)+g\left(-x+y,\left(-c_8-2c_9\right)x+c_8y+c_9z\right).\end{aligned} \tag{4.55}$$

And

$$u\left(0,y,z\right)=f\left(y,c_5y+c_6z\right)+g\left(y,c_8y+c_9z\right)=yz, \tag{4.56}$$

$$\begin{aligned} u_x(0,y,z) = &-f_p(y,c_5y+c_6z)-(c_5+c_6)f_q(y,c_5y+c_6z)-g_p(y,c_8y+c_9z) \\ &-(c_8+2c_9)g_r(y,c_8y+c_9z)=y+z. \end{aligned} \tag{4.57}$$

Namely

$$f+g=yz,$$

$$-f_p-(c_5+c_6)f_q-g_p-(c_8+2c_9)g_r=y+z.$$

When $x=0$,

$$p=y, q=c_5y+c_6z, r=c_8y+c_9z.$$

According to (4.56) and (4.57), we get

$$f_p+c_5f_q+g_p+c_8g_r=z,$$

$$c_6f_q+c_9g_r=y.$$

So

$$-c_9g_r=2y+2z.$$

And

$$\begin{aligned} g &= \frac{-2}{c_9}\int_{c_8y_0+c_9z_0}^{c_8y+c_9z}(y+z)\,dr=\frac{-2}{c_9}\int_{c_8y_0+c_9z_0}^{c_8y+c_9z}\left(p+\frac{r-c_8p}{c_9}\right)dr \\ &= \frac{-(c_8y+c_9z)^2}{c_9^2}-\frac{2(c_9-c_8)y(c_8y+c_9z)}{c_9^2}. \end{aligned}$$

Set

$$y=\alpha, c_8y+c_9z=\gamma,$$

then

$$g(\alpha,\gamma)=g(y,c_8y+c_9z)=\frac{-\gamma^2}{c_9^2}-\frac{2(c_9-c_8)\alpha\gamma}{c_9^2}.$$

Set

$$-x+y=\alpha, (-c_8-2c_9)x+c_8y+c_9z=\gamma.$$

So

$$\begin{aligned} &g(-x+y,(-c_8-2c_9)x+c_8y+c_9z) \\ &= \frac{-((-c_8-2c_9)x+c_8y+c_9z)^2}{c_9^2}-\frac{2(c_9-c_8)(-x+y)((-c_8-2c_9)x+c_8y+c_9z)}{c_9^2} \\ &= -8x^2+4xy+6xz-2yz-z^2+\frac{x^2c_8^2}{c_9^2}-\frac{2xyc_8^2}{c_9^2}+\frac{y^2c_8^2}{c_9^2}-\frac{2x^2c_8}{c_9}+\frac{4xyc_8}{c_9}-\frac{2y^2c_8}{c_9}. \end{aligned}$$

Set

$$y=\alpha, c_5y+c_6z=\beta,$$

then

$$z=\frac{\beta-c_5y}{c_6}=\frac{\beta-c_5\alpha}{c_6}.$$

According to (4.56), we get

$$\begin{aligned} f(\alpha,\beta) &= f(y,c_5y+c_6z)=yz-g=yz+\frac{(c_8y+c_9z)^2}{c_9^2}+\frac{2(c_9-c_8)y(c_8y+c_9z)}{c_9^2} \\ &= \frac{\beta^2}{c_6^2}-\frac{2\alpha\beta c_5}{c_6^2}+\frac{\alpha^2c_5^2}{c_6^2}+\frac{\alpha\beta}{c_6}-\frac{\alpha^2c_5}{c_6}+\frac{\alpha^2c_8^2}{c_9^2}+\frac{2\alpha\beta c_8}{c_6c_9}-\frac{2\alpha^2c_5c_8}{c_6c_9} \\ &\quad -\frac{2c_8\alpha\left(\alpha c_8+\frac{(\beta-\alpha c_5)c_9}{c_6}\right)}{c_9^2}+\frac{2\alpha\left(\alpha c_8+\frac{(\beta-\alpha c_5)c_9}{c_6}\right)}{c_9}. \end{aligned}$$

Set

$$(-x+y)=\alpha,((-c_5-c_6)x+c_5y+c_6z)=\beta.$$

So

$$\begin{aligned}&f\left(-x+y,\left(-c_5-c_6\right)x+c_5y+c_6z\right)\\&=4x^2-3xy-5xz+3yz+z^2-\tfrac{x^2c_8^2}{c_9^2}+\tfrac{2xyc_8^2}{c_9^2}-\tfrac{y^2c_8^2}{c_9^2}+\tfrac{2x^2c_8}{c_9}-\tfrac{4xyc_8}{c_9}+\tfrac{2y^2c_8}{c_9}.\end{aligned}$$

By Eq. (4.54), the exact solution of the definite solution problem is

$$u=-4x^2+yz+xy+xz. \tag{4.58}$$

Example 12. In $\mathbb{R}^3$, get the exact solution of

$$u_{tt}+u_{xx}-u_{tx}+u_{ty}-u_{xy}+2u_t-2u_x+2u_y=-y+x-t+2xy-2ty+2tx, \tag{4.59}$$

in the conditions of $u(0,x,y)=\Phi(x,y)$ and $u_t(0,x,y)=\Psi(x,y)$. Φ is a random known second differentiable function, Ψ is a random known first differentiable function.

Solution. By Theorem 18, the general solution of (4.59) is

$$\begin{aligned}u&=f\left(q,r\right)+\mathrm{e}^{-t+x+y}g\left(q,r\right)+txy\\&=f\left(t+x+2y,t+x+y\right)+\mathrm{e}^{-t+x+y}g\left(t+x+2y,t+x+y\right)+txy.\end{aligned} \tag{4.60}$$

So

$$u\left(0,x,y\right)=f\left(x+2y,x+y\right)+\mathrm{e}^{x+y}g\left(x+2y,x+y\right)=\Phi\left(x,y\right), \tag{4.61}$$

$$u_t\left(0,x,y\right)=f_q+f_r-\mathrm{e}^{x+y}g+\mathrm{e}^{x+y}g_q+\mathrm{e}^{x+y}g_r+xy=\Psi\left(x,y\right). \tag{4.62}$$

When $t=0,q=x+2y,r=x+y$,then

$$y=q-r,x=2r-q.$$

According to (4.61), we get

$$f_q+f_r+\mathrm{e}^{x+y}g+\mathrm{e}^{x+y}g_q+\mathrm{e}^{x+y}g_r=\Phi_x,$$

$$2f_q+f_r+\mathrm{e}^{x+y}g+2\mathrm{e}^{x+y}g_q+\mathrm{e}^{x+y}g_r=\Phi_y.$$

So

$$2\mathrm{e}^{x+y}g-xy=\Phi_x-\Psi,$$

$$g\left(q,r\right)=\frac{1}{2}\mathrm{e}^{-r}\left(\Phi_x\left(2r-q,q-r\right)-\Psi\left(2r-q,q-r\right)+\left(2r-q\right)\left(q-r\right)\right),$$

$$\begin{aligned}f\left(q,r\right)&=\Phi-\mathrm{e}^{x+y}g\\&=\Phi\left(2r-q,q-r\right)-\tfrac{1}{2}\left(\Phi_x\left(2r-q,q-r\right)-\Psi\left(2r-q,q-r\right)+\left(2r-q\right)\left(q-r\right)\right).\end{aligned}$$

Set

$$q=t+x+2y,r=t+x+y,$$

then

$$2r-q=t+x,q-r=y.$$

And

$$g\left(t+x+2y,t+x+y\right)=\frac{1}{2}\mathrm{e}^{-t-x-y}\left(\Phi_x\left(t+x,y\right)-\Psi\left(t+x,y\right)+ty+xy\right),$$

$$f\left(t+x+2y,t+x+y\right)=\Phi\left(t+x,y\right)-\frac{1}{2}\left(\Phi_x\left(t+x,y\right)-\Psi\left(t+x,y\right)+ty+xy\right).$$

By Eq. (4.60), the exact solution of the definite solution problem is

$$\begin{aligned}u&=\Phi\left(t+x,y\right)-\frac{1}{2}\left(\Phi_x\left(t+x,y\right)-\Psi\left(t+x,y\right)+ty+xy\right)\\&+\frac{1}{2}\mathrm{e}^{-2t}\left(\Phi_x\left(t+x,y\right)-\Psi\left(t+x,y\right)+ty+xy\right)+txy.\end{aligned}\tag{4.63}$$

Example 13. In $\mathbb{R}^2$, get the exact solution of

$$u_{ttt}+8u_{xxx}+6u_{ttx}+12u_{txx}=9\mathrm{e}^{t+x},\tag{4.64}$$

in the conditions of $u(0,x)=\Phi(x), u_t(0,x)=\Psi(x)$ and $u_{tt}(0,x)=\Omega(x)$. Φ is a random known third differentiable function, Ψ is a random known second differentiable function, Ω is a random known first differentiable function.

Solution. By Theorem 21, the general solution of (4.64) is

$$\begin{aligned}u&=f\left(q\right)+pg\left(q\right)+p^2h\left(q\right)+\frac{1}{3}\mathrm{e}^{t+x}\\&=f\left(-2t+x\right)+\left(t+x\right)g\left(-2t+x\right)+\left(t+x\right)^2h\left(-2t+x\right)+\frac{1}{3}\mathrm{e}^{t+x},\end{aligned}\tag{4.65}$$

$$p=t+x,q=-2t+x.$$

So

$$u\left(0,x\right)=f+xg+x^2h+\frac{1}{3}\mathrm{e}^{x}=\Phi\left(x\right),\tag{4.66}$$

$$u_t\left(0,x\right)=-2f_q+g-2xg_q+2xh-2x^2h_q+\frac{1}{3}\mathrm{e}^{x}=\Psi\left(x\right),\tag{4.67}$$

$$u_{tt}\left(0,x\right)=4f_{qq}-4g_q+4xg_{qq}+2h-8xh_q+4x^2h_{qq}+\frac{1}{3}\mathrm{e}^{x}=\Omega\left(x\right).\tag{4.68}$$

When $t=0,p=q=x$, according to (4.66), we have

$$f_q+g+xg_q+2xh+x^2h_q+\frac{1}{3}\mathrm{e}^{x}=\Phi'\left(x\right),$$

$$f_{qq}+2g_q+xg_{qq}+2h+4xh_q+x^2h_{qq}+\frac{1}{3}\mathrm{e}^{x}=\Phi''\left(x\right).$$

According to (4.67), we have

$$-2f_{qq}-g_q-2xg_{qq}+2h-2xh_q-2x^2h_{qq}+\frac{1}{3}\mathrm{e}^{x}=\Psi'\left(x\right),$$

then

$$3g+6xh+\mathrm{e}^{x}=\Psi\left(x\right)+2\Phi'\left(x\right),$$
$$3f_q+3xg_q+3x^2h_q=\Phi'\left(x\right)-\Psi\left(x\right),$$
$$12g_q+6h+24xh_q+\mathrm{e}^{x}=4\Phi''\left(x\right)-\Omega\left(x\right),$$
$$-6g_q+6h-12xh_q+\mathrm{e}^{x}=2\Psi'\left(x\right)+\Omega\left(x\right).$$

Therefore

$$18h+3\mathrm{e}^{x}=4\Psi'\left(x\right)+4\Phi''\left(x\right)+\Omega\left(x\right),$$

$$h\left(x\right)=\frac{2}{9}\Psi'\left(x\right)+\frac{2}{9}\Phi''\left(x\right)+\frac{1}{18}\Omega\left(x\right)-\frac{1}{6}\mathrm{e}^{x},$$

$$\begin{aligned}g\left(x\right)&=\frac{1}{3}\left(\Psi\left(x\right)+2\Phi'\left(x\right)-6xh-\mathrm{e}^{x}\right)\\&=\frac{1}{3}\Psi\left(x\right)+\frac{2}{3}\Phi'\left(x\right)-\frac{4x}{9}\Psi'\left(x\right)-\frac{4x}{9}\Phi''\left(x\right)-\frac{x}{9}\Omega\left(x\right)+\frac{x}{3}\mathrm{e}^{x}-\frac{1}{3}\mathrm{e}^{x},\end{aligned}$$

$$\begin{aligned}f\left(x\right)&=\Phi\left(x\right)-xg-x^2h-\frac{1}{3}\mathrm{e}^{x}\\&=\Phi\left(x\right)-\frac{x}{3}\Psi\left(x\right)-\frac{2x}{3}\Phi'\left(x\right)+\frac{2x^2}{9}\Psi'\left(x\right)+\frac{2x^2}{9}\Phi''\left(x\right)+\frac{x^2}{18}\Omega\left(x\right)-\frac{x^2}{6}\mathrm{e}^{x}+\frac{x}{3}\mathrm{e}^{x}-\frac{1}{3}\mathrm{e}^{x}.\end{aligned}$$

So

$$\begin{aligned}f\left(-2t+x\right)=&\Phi\left(-2t+x\right)-\frac{-2t+x}{3}\Psi\left(-2t+x\right)-\frac{2\left(-2t+x\right)}{3}\Phi'\left(-2t+x\right)\\&+\frac{2(-2t+x)^2}{9}\Psi'\left(-2t+x\right)+\frac{2(-2t+x)^2}{9}\Phi''\left(-2t+x\right)+\frac{(-2t+x)^2}{18}\Omega\left(-2t+x\right)\\&-\frac{(-2t+x)^2}{6}\mathrm{e}^{-2t+x}+\frac{-2t+x}{3}\mathrm{e}^{-2t+x}-\frac{1}{3}\mathrm{e}^{-2t+x},\end{aligned}$$

$$\begin{aligned}g\left(-2t+x\right)=&\frac{1}{3}\Psi\left(-2t+x\right)+\frac{2}{3}\Phi'\left(-2t+x\right)-\frac{4\left(-2t+x\right)}{9}\Psi'\left(-2t+x\right)\\&-\frac{4\left(-2t+x\right)}{9}\Phi''\left(-2t+x\right)-\frac{-2t+x}{9}\Omega\left(-2t+x\right)+\frac{-2t+x}{3}\mathrm{e}^{-2t+x}-\frac{1}{3}\mathrm{e}^{-2t+x},\end{aligned}$$

$$h\left(-2t+x\right)=\frac{2}{9}\Psi'\left(-2t+x\right)+\frac{2}{9}\Phi''\left(-2t+x\right)+\frac{1}{18}\Omega\left(-2t+x\right)-\frac{1}{6}\mathrm{e}^{-2t+x}.$$

By Eq. (4.65), the exact solution of the definite solution problem is

$$\begin{aligned}u=&\Phi\left(-2t+x\right)+t\Psi\left(-2t+x\right)+\frac{1}{2}t^2\Omega\left(-2t+x\right)+2t\Phi'\left(-2t+x\right)+2t^2\Psi'\left(-2t+x\right)\\&+2t^2\Phi''\left(-2t+x\right)+\frac{1}{3}\mathrm{e}^{t+x}-\frac{1}{3}\mathrm{e}^{-2t+x}-\mathrm{e}^{-2t+x}t-\frac{3}{2}\mathrm{e}^{-2t+x}t^2.\end{aligned}\tag{4.69}$$

Example 14. In $\mathbb{R}^2$, get the exact solution of

$$u_{ttt}-u_{xxx}-3u_{ttx}+3u_{txx}-6u_{tt}-6u_{xx}+12u_{tx}+8u_t-8u_x=-8t+8x+12,\tag{4.70}$$

in the conditions of $u(0,x)=\Phi(x), u_t(0,x)=\Psi(x)$ and $u_{tt}(0,x)=\Omega(x)$. Φ is a random known third differentiable function, Ψ is a random known second differentiable function, Ω is a random known first differentiable function.

Solution. By Theorem 23, the general solution of (4.70) is

$$u=f\left(q\right)+\mathrm{e}^{p}g\left(q\right)+\mathrm{e}^{2p}h\left(q\right)+xt=f\left(t+x\right)+\mathrm{e}^{t-x}g\left(t+x\right)+\mathrm{e}^{2t-2x}h\left(t+x\right)+xt,\tag{4.71}$$

$$p=t-x,q=t+x.$$

So

$$u\left(0,x\right)=f\left(x\right)+\mathrm{e}^{-x}g\left(x\right)+\mathrm{e}^{-2x}h\left(x\right)=\Phi\left(x\right),\tag{4.72}$$

$$u_t\left(0,x\right)=f_q+\mathrm{e}^{-x}g+\mathrm{e}^{-x}g_q+2\mathrm{e}^{-2x}h+\mathrm{e}^{-2x}h_q+x=\Psi\left(x\right),\tag{4.73}$$

$$u_{tt}\left(0,x\right)=f_{qq}+\mathrm{e}^{-x}g+2\mathrm{e}^{-x}g_q+\mathrm{e}^{-x}g_{qq}+4\mathrm{e}^{-2x}h+4\mathrm{e}^{-2x}h_q+\mathrm{e}^{-2x}h_{qq}=\Omega\left(x\right).\tag{4.74}$$

When $t=0,q=x$, according to (4.72), we have

$$f_q-\mathrm{e}^{-x}g+\mathrm{e}^{-x}g_q-2\mathrm{e}^{-2x}h+\mathrm{e}^{-2x}h_q=\Phi'\left(x\right),$$

$$f_{qq}+\mathrm{e}^{-x}g-2\mathrm{e}^{-x}g_q+\mathrm{e}^{-x}g_{qq}+4\mathrm{e}^{-2x}h-4\mathrm{e}^{-2x}h_q+\mathrm{e}^{-2x}h_{qq}=\Phi''(x).$$

According to (4.73), we get

$$f_{qq}-\mathrm{e}^{-x}g+\mathrm{e}^{-x}g_{qq}-4\mathrm{e}^{-2x}h+\mathrm{e}^{-2x}h_{qq}+1=\Psi'(x),$$

then

$$\begin{gathered}2\mathrm{e}^{-x}g+4\mathrm{e}^{-2x}h+x=\Psi(x)-\Phi'(x),\\ 2f_q+2\mathrm{e}^{-x}g_q+2\mathrm{e}^{-2x}h_q+x=\Psi(x)+\Phi'(x),\\ 2f_{qq}+2\mathrm{e}^{-x}g+2\mathrm{e}^{-x}g_{qq}+8\mathrm{e}^{-2x}h+2\mathrm{e}^{-2x}h_{qq}=\Omega(x)+\Phi''(x),\\ 4\mathrm{e}^{-x}g_q+8\mathrm{e}^{-2x}h_q=\Omega(x)-\Phi''(x),\\ 2\mathrm{e}^{-x}g+2\mathrm{e}^{-x}g_q+8\mathrm{e}^{-2x}h+4\mathrm{e}^{-2x}h_q-1=\Omega(x)-\Psi'(x).\end{gathered}$$

So

$$4\mathrm{e}^{-x}g+16\mathrm{e}^{-2x}h-2=2\Omega(x)-2\Psi'(x)-\Omega(x)+\Phi''(x)=\Omega(x)-2\Psi'(x)+\Phi''(x).$$

Therefore

$$\begin{aligned}&4\mathrm{e}^{-x}g+16\mathrm{e}^{-2x}h-2-4\mathrm{e}^{-x}g-8\mathrm{e}^{-2x}h-2x=8\mathrm{e}^{-2x}h-2-2x\\&=\Omega(x)-2\Psi'(x)+\Phi''(x)-2\Psi(x)+2\Phi'(x),\end{aligned}$$

then

$$h(x)=\frac{1}{8}\mathrm{e}^{2x}\big(\Omega(x)-2\Psi'(x)+\Phi''(x)-2\Psi(x)+2\Phi'(x)+2x+2\big),$$

$$\begin{aligned}g&=\tfrac{1}{2}\mathrm{e}^{x}\big(\Psi(x)-\Phi'(x)-4\mathrm{e}^{-2x}h-x\big)\\&=\tfrac{1}{2}\mathrm{e}^{x}\big(2\Psi(x)-2\Phi'(x)-\tfrac{1}{2}\Omega(x)+\Psi'(x)-\tfrac{1}{2}\Phi''(x)-2x-1\big),\end{aligned}$$

$$\begin{aligned}f(x)&=\Phi(x)-\mathrm{e}^{-x}g(x)-\mathrm{e}^{-2x}h(x)\\&=\tfrac{1}{8}\left(2+6x+8\Phi(x)-6\Psi(x)+\Omega(x)+6\Phi'(x)-2\Psi'(x)+\Phi''(x)\right).\end{aligned}$$

So

$$\begin{aligned}f(t+x)=&\frac{1}{8}(2+6(t+x)+8\Phi(t+x)-6\Psi(t+x)+\Omega(t+x)+6\Phi'(t+x)-2\Psi'(t+x)\\&+\Phi''(t+x)),\end{aligned}$$

$$\begin{aligned}&g(t+x)\\&=\frac{1}{2}\mathrm{e}^{t+x}\left(2\Psi(t+x)-2\Phi'(t+x)-\frac{1}{2}\Omega(t+x)+\Psi'(t+x)-\frac{1}{2}\Phi''(t+x)-2(t+x)-1\right),\\&h(t+x)\\&=\frac{1}{8}\mathrm{e}^{2(t+x)}\big(\Omega(t+x)-2\Psi'(t+x)+\Phi''(t+x)-2\Psi(t+x)+2\Phi'(t+x)+2(t+x)+2\big).\end{aligned}$$

By Eq. (4.71), the exact solution of the definite solution problem is

$$\begin{aligned}&u(t,x)\\&=\frac{1}{8}\big(2+6(t+x)+8\Phi(t+x)-6\Psi(t+x)+\Omega(t+x)+6\Phi'(t+x)-2\Psi'(t+x)+\Phi''(t+x)\big)\\&+\frac{1}{2}\mathrm{e}^{2t}\left(2\Psi(t+x)-2\Phi'(t+x)-\frac{1}{2}\Omega(t+x)+\Psi'(t+x)-\frac{1}{2}\Phi''(t+x)-2(t+x)-1\right)\\&+\frac{1}{8}\mathrm{e}^{4t}\big(\Omega(t+x)-2\Psi'(t+x)+\Phi''(t+x)-2\Psi(t+x)+2\Phi'(t+x)+2(t+x)+2\big)+xt.\end{aligned}\tag{4.75}$$

Example 15. In $\mathbb{R}^2$, get the exact solution of

$$u_{tttt} + 4u_{xxxx} - 6u_{tttx} + 13u_{ttxx} - 12u_{txxx} = 0, \tag{4.76}$$

in the conditions of $u(0,x) = x^2 + \sin x$, $u_t(0,x) = 2x + 2\cos x + e^x - xe^x$, $u_{tt}(0,x) = 2 - 4\sin x - 3xe^x$ and $u_{ttt}(0,x) = -8\cos x - 6e^x - 7xe^x$.

Solution. By Theorem 25, the general solution of (4.76) is

$$\begin{aligned} u &= f(p) + g(q) + qh(p) + pw(q) \\ &= f(t+x) + g(2t+x) + (2t+x)h(t+x) + (t+x)w(2t+x), \end{aligned} \tag{4.77}$$

$$p = t + x, q = 2t + x.$$

So

$$u(0,x) = f(x) + g(x) + xh(x) + xw(x) = x^2 + \sin x, \tag{4.78}$$

$$u_t(0,x) = f_p(x) + 2g_q(x) + 2h(x) + xh_p(x) + w(x) + 2xw_q(x) = 2x + 2\cos x + e^x - xe^x, \tag{4.79}$$

$$u_{tt}(0,x) = f_{pp}(x) + 4g_{qq}(x) + 4h_p(x) + xh_{pp}(x) + 4w_q(x) + 4xw_{qq}(x) = 2 - 4\sin x - 3xe^x, \tag{4.80}$$

$$\begin{aligned} u_{ttt}(0,x) &= f_{ppp}(x) + 8g_{qqq}(x) + 6h_{pp}(x) + xh_{ppp}(x) + 12w_{qq}(x) + 8xw_{qqq}(x) \\ &= -8\cos x - 6e^x - 7xe^x. \end{aligned} \tag{4.77}$$

When $t = 0, p = q = x$, according to (4.78), we have

$$f_p + g_q + h + xh_p + w + xw_q = 2x + \cos x,$$

$$f_{pp} + g_{qq} + 2h_p + xh_{pp} + 2w_q + xw_{qq} = 2 - \sin x,$$

$$f_{ppp} + g_{qqq} + 3h_{pp} + xh_{ppp} + 3w_{qq} + xw_{qqq} = -\cos x.$$

By (4.79), we have

$$f_{pp} + 2g_{qq} + 3h_p + xh_{pp} + 3w_q + 2xw_{qq} = 2 - 2\sin x - xe^x,$$

$$f_{ppp} + 2g_{qqq} + 4h_{pp} + xh_{ppp} + 5w_{qq} + 2xw_{qqq} = -2\cos x - e^x - xe^x.$$

According to (4.80), we obtain

$$f_{ppp} + 4g_{qqq} + 5h_{pp} + xh_{ppp} + 8w_{qq} + 4xw_{qqq} = -4\cos x - 3e^x - 3xe^x.$$

Then

$$g_q + h + xw_q = 2x + 2\cos x + e^x - xe^x - 2x - \cos x = \cos x + e^x - xe^x,$$

$$f_p + xh_p + w = 4x + 2\cos x - 2x - 2\cos x - e^x + xe^x = 2x - e^x + xe^x,$$

$$f_{pp} + 2h_p + xh_{pp} + 2w_q = 4 - 4\sin x - 2xe^x - 2 + 4\sin x + 3xe^x = 2 + xe^x,$$

$$f_{pp} + h_p + xh_{pp} + w_q = 4 - 2\sin x - 2 + 2\sin x + xe^x = 2 + xe^x.$$

So

$$h_p + w_q = 0.$$

Namely

$$h + w = C_1.$$

For

$$7g_{qqq} + 3h_{pp} + 9w_{qq} + 7xw_{qqq} = -8\cos x - 6\mathrm{e}^x - 7x\mathrm{e}^x + \cos x = -7\cos x - 6\mathrm{e}^x - 7x\mathrm{e}^x,$$

$$\begin{aligned} 2g_{qqq} + h_{pp} + 3w_{qq} + 2xw_{qqq} &= -4\cos x - 3\mathrm{e}^x - 3x\mathrm{e}^x + 2\cos x + \mathrm{e}^x + x\mathrm{e}^x \\ &= -2\cos x - 2\mathrm{e}^x - 2x\mathrm{e}^x. \end{aligned}$$

So

$$h_{pp} + 3w_{qq} = -14\cos x - 14\mathrm{e}^x - 14x\mathrm{e}^x + 14\cos x + 12\mathrm{e}^x + 14x\mathrm{e}^x = -2\mathrm{e}^x.$$

That is

$$h + 3w = -2\mathrm{e}^x + C_2x + C_3.$$

Then

$$2w(x) = -2\mathrm{e}^x + C_2x + C_3 - C_1 = -2\mathrm{e}^x + C_2x + C_4,$$

$$w_q = -\mathrm{e}^x + \frac{C_2}{2},$$

$$h(x) = C_1 - w = C_1 + \mathrm{e}^x - \frac{C_2}{2}x - \frac{C_4}{2} = \mathrm{e}^x - \frac{C_2}{2}x + C_5.$$

So

$$g_q = \cos x + \mathrm{e}^x - x\mathrm{e}^x - h - xw_q = \cos x - C_5.$$

Namely

$$g(x) = \sin x - C_5x + C_6.$$

Therefore

$$f(x) = x^2 + \sin x - g - xh - xw = x^2 - \frac{C_4}{2}x - C_6.$$

Then

$$f(t+x) = (t+x)^2 - \frac{C_4}{2}(t+x) - C_6,$$

$$g(2t+x) = \sin(2t+x) - C_5(2t+x) + C_6,$$

$$h(t+x) = \mathrm{e}^{t+x} - \frac{C_2}{2}(t+x) + C_5,$$

$$w(2t+x) = -\mathrm{e}^{2t+x} + \frac{C_2}{2}(2t+x) + \frac{C_4}{2}.$$

By Eq. (4.77), the exact solution of the definite solution problem is

$$u(t,x) = (t+x)^2 + \sin(2t+x) + (2t+x)\mathrm{e}^{t+x} - (t+x)\mathrm{e}^{2t+x}. \tag{4.82}$$

5. Z_4 Transformation

We present Z_4 transformation as follows.

Z_4 Transformation. *In the domain D, $(D \subseteq \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$, setting $u = \Psi(f_1, f_2, \ldots f_s), (s \geq 1), \Psi$ is a random known function, $f_i = f_i\left(y_{i_1}, \ldots y_{i_{k_i}}\right)$ and $y_{i_j} = y_{i_j}(x_1, \ldots x_n)$ are all undetermined functions, $y_{i_1}, \ldots y_{i_{k_i}}$ are independent of each other, $\Psi, f_i, y_{i_k} \in C^m(D), (i = 1, 2, \ldots s), (1 \leq k \leq k_i \leq n)$, then substitute $u = \Psi(f_1, f_2, \ldots f_s)$ and its partial derivatives into $F = 0$*

1. In case of working out f_i and y_{i_k}, then $u = \Psi(f_1, f_2, \ldots f_s)$ is the solution of $F = 0$,

2. In case of dividing out $f_{l_1}, f_{l_2}, \ldots f_{l_p}$ and their partial derivative, also working out $f_{l_{p+1}}, f_{l_{p+2}}, \ldots f_{l_s}$ and $y_{i_k}, (1 \leq p \leq s, l_i \in \{1, 2, \ldots s\})$, $u = \Psi(f_1, f_2, \ldots f_s)$ is the solution of $F = 0$, and $f_{l_1}, f_{l_2}, \ldots f_{l_p}$ are arbitrary mth-differentiable functions,
3. In case of getting $k = 0$, but in fact $k \neq 0$, $u = \Psi(f_1, f_2, \ldots f_s)$ is not the solution of $F = 0$.

For solve some PDEs we could be required to set Ψ undetermined or set $f_1, f_2, \ldots f_p$ known and so on. $(1 \leq p \leq s)$. The forms of these laws are similar to Z_4 transformation, we will not propose here.

In $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} + a_{k+1} u = A\left(x_1, x_2, \ldots x_n\right), (2 \leq k \leq n), \tag{5.1}$$

where a_i are random known constants and $A(x_1, x_2, \ldots x_n)$ is any known function, $(1 \leq i \leq k+1)$, we can use Z_4 transformation to get a general solution of Eq. (5.1).

By Z_4 transformation, set $u\left(x_1, \ldots x_n\right) = f\left(v_1, v_2, \ldots v_k, x_{k+1}, x_{k+2}, \ldots x_n\right) + g\left(x_1, \ldots x_k\right) h\left(y_1, y_2, \ldots y_k, x_{k-}\right.$ $v_i = l_{i1} x_1 + l_{i2} x_2 + \ldots + l_{ik} x_k, y_i = c_{i1} x_1 + c_{i2} x_2 + \ldots + c_{ik} x_k$ and

$$\frac{\partial\left(v_1, v_2, \ldots v_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0. \tag{5.2}$$

So

$$\begin{aligned}
& a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_n} + a_{k+1} u \\
& = a_1 \sum_{i=1}^{k} l_{i1} f_{v_i} + a_2 \sum_{i=1}^{k} l_{i2} f_{v_i} + \ldots + a_k \sum_{i=1}^{k} l_{ik} f_{v_i} + a_{k+1} f + a_1 g_{x_1} h + a_1 g \sum_{i=1}^{k} c_{i1} h_{y_i} \\
& + a_2 g_{x_2} h + a_2 g \sum_{i=1}^{k} c_{i2} h_{y_i} + \ldots + a_k g_{x_k} h + a_k g \sum_{i=1}^{k} c_{ik} h_{y_i} + a_{k+1} g h \\
& = \left(a_1 l_{11} + a_2 l_{12} + \ldots + a_k l_{1k}\right) f_{v_1} + \left(a_1 l_{21} + a_2 l_{22} + \ldots + a_k l_{2k}\right) f_{v_2} + \ldots \\
& + \left(a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}\right) f_{v_k} + a_{k+1} f + \left(a_1 c_{11} + a_2 c_{12} + \ldots + a_k c_{1k}\right) g h_{y_1} \\
& + \left(a_1 c_{21} + a_2 c_{22} + \ldots + a_k c_{2k}\right) g h_{y_2} + \ldots + \left(a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}\right) g h_{y_k} \\
& + \left(a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g\right) h = A\left(v_1, v_2, \ldots v_k, x_{k+1}, x_{k+2}, \ldots x_n\right).
\end{aligned}$$

Set

$$l_{i1} = \frac{-a_2 l_{i2} - a_3 l_{i3} - \ldots - a_k l_{ik}}{a_1}, (1 \leq i \leq k-1), \tag{5.3}$$

$$c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \ldots - a_k c_{ik}}{a_1}, (1 \leq i \leq k-1), \tag{5.4}$$

$$a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = 0. \tag{5.5}$$

And set $g(x_1, \ldots x_k) = g(x_i), (i = 1, 2, \ldots k)$, Then

$$a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = a_i g_{x_i} + a_{k+1} g = 0 \Rightarrow g\left(x_i\right) = \lambda_i e^{\frac{-a_{k+1} x_i}{a_i}}, \tag{5.6}$$

where λ_i are random constants. Namely $g\left(x_1, \ldots x_k\right) = \sum_{i=1}^{k} \lambda_i e^{\frac{-a_{k+1} x_i}{a_i}}$ is a particular solution of $a_1 g_{x_1} + a_2 g_{x_2} + \ldots + a_k g_{x_k} + a_{k+1} g = 0$, Thus

$$\begin{aligned}
& a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_n} + a_{k+1} u \\
& = \left(a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}\right) f_{v_k} + a_{k+1} f + \left(a_1 c_{k1} + a_2 c_{k2} + \ldots + a_k c_{kk}\right) g h_{y_k} \\
& = A\left(v_1, v_2, \ldots v_k, x_{k+1}, x_{k+2}, \ldots x_n\right).
\end{aligned}$$

Set

$$h_{y_k} = 0,$$

Namely

$$h = \Phi\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right),$$

where Φ is a random first differentiable function. So

$$\left(a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}\right) f_{v_k} + a_{k+1} f = A\left(v_1, v_2, \ldots v_k, x_{k+1}, x_{k+2}, \ldots x_n\right). \tag{5.7}$$

The solution of (5.7) is

$$f = \mathrm{e}^{\frac{-a_{k+1} v_k}{a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}}} \left(C + \int \mathrm{e}^{\frac{a_{k+1} v_k}{a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}}} \frac{A\left(v_1, v_2, \ldots v_k, x_{k+1}, x_{k+2}, \ldots x_n\right)}{a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}} dv_k\right).$$

Set

$$B = \left(a_1 l_{k1} + a_2 l_{k2} + \ldots + a_k l_{kk}\right)^{-1}. \tag{5.8}$$

So the general solution of Eq. (5.1) is

$$\begin{aligned} u = gh + f = {} & \Phi\left(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n\right) \sum_{i=1}^{k} \lambda_i \mathrm{e}^{\frac{-a_{k+1} x_i}{a_i}} \\ & + \mathrm{e}^{\frac{-a_{k+1} v_k}{B}} \left(C + B \int \mathrm{e}^{\frac{a_{k+1} v_k}{B}} A\left(v_1, v_2, \ldots v_k, x_{k+1}, x_{k+2}, \ldots x_n\right) dv_k\right). \end{aligned} \tag{5.9}$$

In our subsequent papers, we will use the Z_4 transformation to research more partial differential equations.

6. Conclusions

This paper uses our previously presented Z transformations to get general solutions of a large number of second-order, third-order and fourth-order linear PDEs very concisely, and proves the effectiveness of using Z transformations to get general solutions of linear PDEs. By general solutions, we get exact solutions for many typical definite solution problems of first-, second-, third- and fourth-order linear PDEs, reflecting the important value of general solutions.

Using the series general solution of the one-dimensional homogeneous wave equation, we successfully obtained the Fourier series solution, which satisfactorily resolve a famed debate in the history of mathematics about the relationship between general solution and series solution.

This paper proposes the Z_4 transformation for the first time and puts forward an application case. In subsequent papers, we will continue to research various PDEs using Z transformations.

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