# A New Home for Bivectors

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**Abstract**

The impetus for the work is this quote:

> *"...as shown by Gel'fand's approach, we can only abstract a unique manifold if our algebra is commutative." (Hiley and Callaghan, 2010)*

Geometric algebra is non-commutative. Components of different grades can be staged on different manifolds. As operations on those elements proceed, they will effect the promotion and/or demotion of components to higher and/or lower grades, and thus to different manifolds. This paper includes imagery that visually displays bivector addition and rotation on a sphere.

David Hestenes interpreted the vector product or rotor in two-dimensions:

> "*as a directed arc of fixed length that can be rotated at will on the unit circle, just as we interpret a vector* $\boldsymbol{a}$ *as a directed line segment that can be translated at will without changing its length or direction…" (Hestenes, 2003)*

Rotors can be used to develop addition and multiplication of bivectors on a sphere. For those rotational dynamics, rotors of length $\pi/2$ are the basis elements. The geometric algebra of bivectors – Hamilton's "pure quaternions" – is thus shown to transparently reside on a spherical manifold.

*Keywords:* Bivectors/Visualization/Rotors/Spherical Manifold/Quaternions/Non-Commutative Algebra

## Introduction

Hiley's quest for an algebra of process led him to develop rules that quickly take shape in one form as a quaternion algebra (Hiley, 2012). In his seminal textbook (Macdonald, 2011), Alan Macdonald redefines the pure quaternions, which exclude the scalar part, for use as the bivectors of geometric algebra. Those sit comfortably on a spherical manifold as we will see. In three dimensions, that manifold hosts visual metaphors for both bivector multiplication, and bivector addition, representations which transparently subsume the algebraic formulation of those operations. The algebra is a symbolic analog for transformations in 3D space, highlighting the power of the geometric algebra developed by David Hestenes (Hestenes, 1986).

In the case formulated by Hiley which led him to the quaternions, the quest for the process in question implicates rotational dynamics on a spherical manifold in three dimensions. That process further equipped with the dynamics of time for those operations, will encompass swirling movements both expansive and contractive throughout the spherical domain. The trajectory of those movements will be a geometrical mirror for the dynamical process in question.

# An algebra that embeds into multiple geometries

In a follow-up paper (Hiley and Callaghan, 2010), Hiley and Callaghan expand on the reasoning for pursuing an algebra for that structure process:

> *Most of the work in (Hiley, 2012) was to establish how it is possible to produce an algebraic description of this structure process. Having demonstrated how this was possible, we went on to show that this algebra had enfolded in it a series of what we called 'shadow manifolds'. We deliberately choose the plural 'manifolds' because* ***we have a non-commutative algebra and as shown by Gel'fand's approach, we can only abstract a unique manifold if our algebra is commutative****. [emphasis added]*
>
> *Notice that our approach stands the conventional approach on its head, as it were, because we start with the algebra and then abstract the geometry. We do not start with a a priori given manifold and then build an algebra on that.*

For a non-commutative geometric algebra, the premise is that algebraic components of different grades can be staged on different geometric manifolds. As operations on those elements proceed, they will effect the promotion and/or demotion of components to higher and/or lower grades, and thus to different manifolds. The model I've used in my writing is of a baseball given both linear and rotational motion by the pitcher – think of a curveball headed towards the plate here. The batter can "operate" on the baseball, providing the force necessary to transform, eliminate, or modify those graded elements.

As one example, the rotational dynamics of the ball can transmute into purely linear motion. What had been a regime of spin dynamics on a spherical manifold – the baseball – has changed grade and can properly be modeled with vectors in 3D euclidean space. Or the batter might undercut the ball, exchanging one set of rotational dynamics for another – a within-grade transformation of the dynamics on a spherical manifold.

In his Oersted Medal Lecture (Hestenes, 2003), Hestenes introduced the key to bivectors in two dimensions, one that translates to three dimensions as well. He established a definition for rotors that parallels the definition for vectors in the plane. In his words:

> *...we should interpret* $\boldsymbol{U_\theta}$ [1] *as a directed arc of fixed length that can be rotated at will on the unit circle, just as we interpret a vector* $\boldsymbol{a}$ *as a directed line segment that can be translated at will without changing its length or direction.*

Corresponding to those two-dimensional bivectors, the basis components of a three-dimensional bivector algebra can be staged and rotated on a spherical manifold, each translated without changing its length or direction. Hamilton's pure quaternions are revealed to be homologous to the coordinate-free rendering of bivectors on that manifold.

The notation below is used, defined by the following relationship of bivectors to Hamilton's pure quaternions $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ [2]:

$$\boldsymbol{i}_1 = \boldsymbol{e}_2 \boldsymbol{e}_3 = \boldsymbol{j}$$
$$\boldsymbol{i}_2 = \boldsymbol{e}_3 \boldsymbol{e}_1 = \boldsymbol{k}$$
$$\boldsymbol{i}_3 = \boldsymbol{e}_1 \boldsymbol{e}_2 = \boldsymbol{i}$$

1 $\boldsymbol{U_\theta}$ is the bivector product of two vectors.

2 The basis bivectors in Macdonald (Macdonald, 2011) are negatives of these.

## Bivector addition on a spherical manifold

The $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are orthonormal basis vectors for axes $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ in three dimensions (Figure 1).

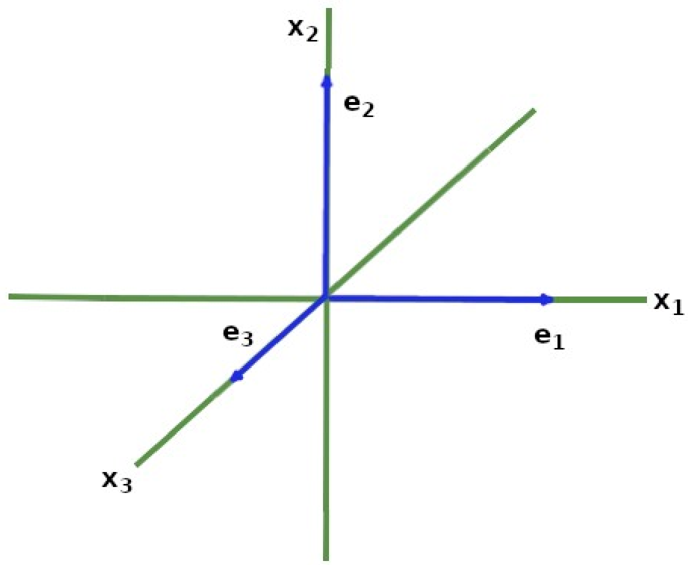


*Figure 1: Standard basis for Euclidean space.*

Bivector $\boldsymbol{e}_1 \boldsymbol{e}_2$ can be derived by multiplying corresponding unit vectors (Figure 1).While $\pi/2$ takes on the role of the unit element on the sphere, for explanatory purposes the diagram posits it as twice that length for the sake of displaying bivector addition. As mentioned previously, the bivectors are rotational elements corresponding to rotors in two dimensions. That is to say, they can be considered as existing anywhere on the great circle they inhabit. The images explaining bivector addition present a visual extension of that unitary motion.

Near the end of one of his online tutorials (*Geometric Algebra 3*, 2015), Macdonald works through an example using geometric algebra to derive a result that, as will be shown, can be transparently visualized on the sphere. Carefully explaining how rotations compose in the algebra, his example starts with the composition of rotations:

$$\exp(-\boldsymbol{i}_2 \theta_2/2)\exp(-\boldsymbol{i}_1 \theta_1/2)\boldsymbol{u}\exp(\boldsymbol{i}_1 \theta_1/2)\exp(\boldsymbol{i}_2 \theta_2/2)$$

from which he derives the equivalent normalized unit bivector in the 3D geometric algebra $G^3$ :

$$\exp(\boldsymbol{i}\,\theta/2)=\exp\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3}{\sqrt{3}}\frac{\pi}{3}\right)$$

The two-dimensional plane where $\boldsymbol{\theta}$ operates is defined by the bracketed pseudoscalar $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3$ . A crucial point is that this equation specifies how the pseudoscalar defining the two-dimensional hosting plane can be described using the bivector bases in the three-dimensional space where it is located. Such is the transparency of geometric algebra.

The normal to that plane in the dual space can be obtained through right multiplication by $\boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1$ , the reversion of the three dimensional pseudoscalar $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ so that:

$$(\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3)\boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1=\boldsymbol{e}_3+\boldsymbol{e}_2-\boldsymbol{e}_1$$

With this elegant algebraic formulation for the plane and the normal to it, we can proceed to develop its visualization on the sphere.

As in Figure 1 we proceed by developing a bivector on the plane defined by the $x_2$ and $x_3$ axes in the same fashion, extending it to a half-circle. (Figure 2).

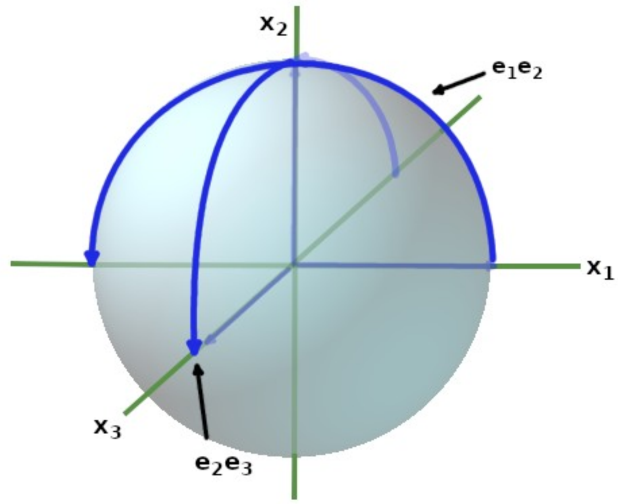


*Figure 2: The two rotors $e_1 e_2$ and $e_2 e_3$ on the sphere*

The logical route to the addition of bivectors is to assume their position on the sphere mediates the resulting bivector. The operation should, therefore, result in a bivector that is halfway between them (Figure 3).

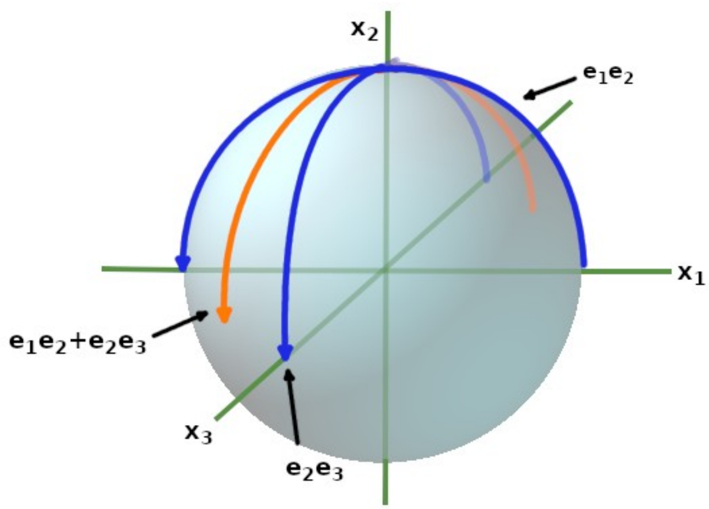


*Figure 3: The addition of two bivectors halfway between them*

Note that the bivector sum $e_1 e_2 + e_2 e_3$ is displaced by $\pm\pi/4$ from its parents, which informs the next calculation. The bivector in the third plane defined by $x_1$ and $x_3$ can next be displayed on the sphere as we proceed (Figure 4). Note that this addition is commutative.

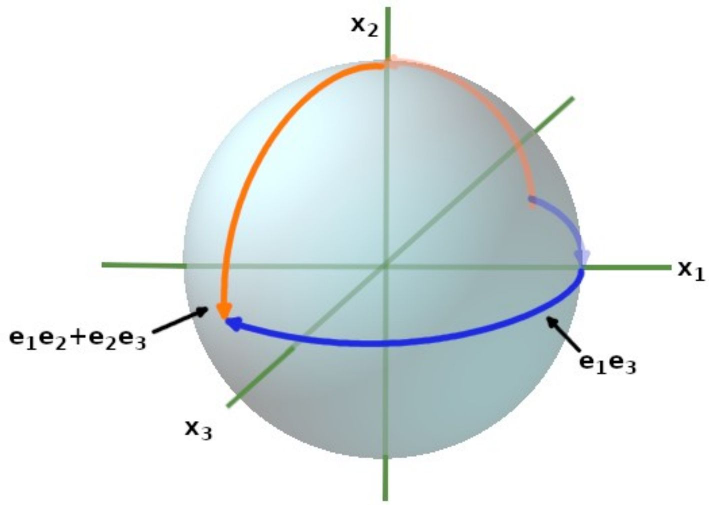


*Figure 4: Bivector* $\boldsymbol{e}_1\boldsymbol{e}_3$ *overlaid on the sphere*

We now have a framework in place to complete the summation. Adding $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3$ to $\boldsymbol{e}_1\boldsymbol{e}_3$ splits the difference once again, with the bivector positioned halfway between the two. The bivector sum, itself a new bivector, is angled between them. The plane the bivector rotates on is shown in the image to offer perspective. That plane runs through the center of the sphere, as do the different planes which host all the bivectors on the sphere.

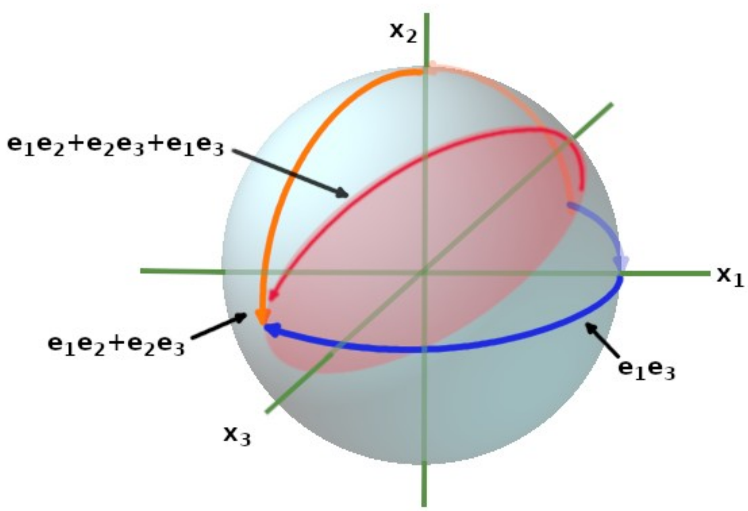


*Figure 5: The bivector sum* $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3$

The normal $\boldsymbol{e}_1+\boldsymbol{e}_3-\boldsymbol{e}_2$ to the plane of the bivector sum, projects downward into the $\boldsymbol{x}_1, -\boldsymbol{x}_2, \boldsymbol{x}_3$ portion of the sphere (Figure 6).

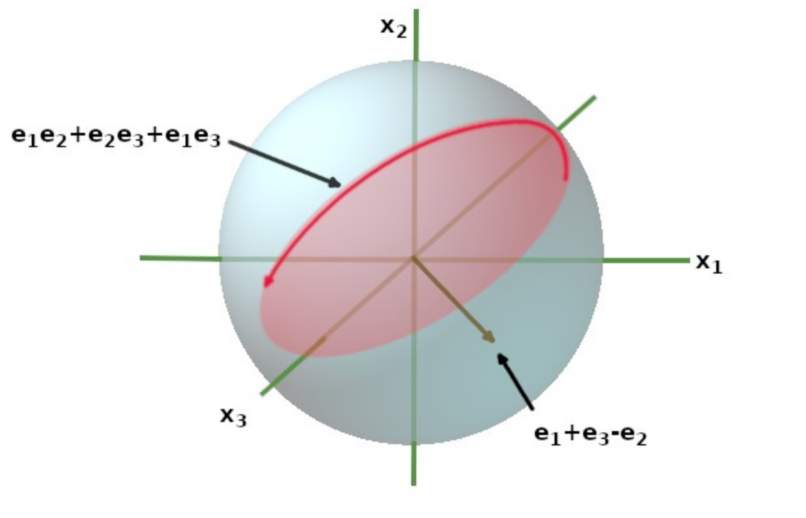


*Figure 6: Bivector sum, its plane and normal to it:* $\boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_2$

This mirrors the algebraic calculation with bivector addition residing comfortably on a sphere in this example. This provides the impetus for mobilizing spherical geometry to derive properly weighted bivectors emerging from calculations.

## Bivector multiplication on the sphere

Multiplication of bivectors can be properly understood as the rotation of one bivector into a different axial plane when acted upon by another. The result is that anti-commutative multiplication of basis bivectors has a representation on a sphere in three dimensions. Once again, the starting point is a set of orthonormal basis vectors for axes $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ in three-dimensions (Figure 7).

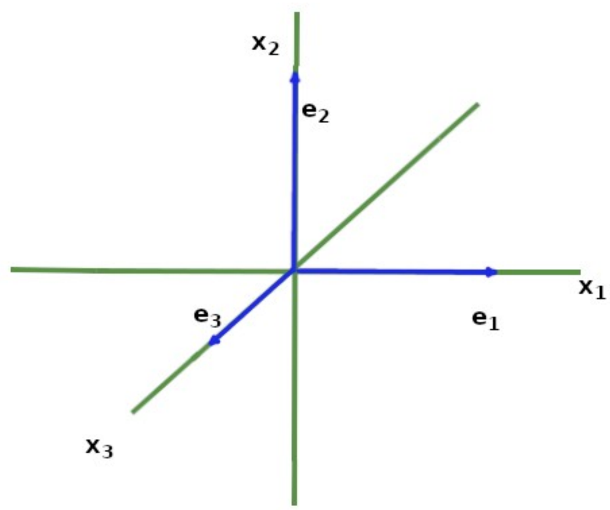


*Figure 7: Standard basis in Euclidean space.*

This model for bivector multiplication on a sphere was derived from earlier work researching the imagery necessary to visualize the operation. Movements initially portrayed in Euclidean space were eventually seen to migrate naturally to a sphere. Careful thought led to the realization that the paradigm for bivectors which had been developed to portray rotors in two-dimensions was also meaningful for a proper understanding of bivectors operations in three dimensions. Each of the three dimensional planes anchoring a spherical manifold can host a canonical unit rotor on a sphere of radius $1$ which is $\pi/2$ in length. With that understanding, we proceed to bivector multiplication.

The unit bivector $\boldsymbol{i}_1 = \boldsymbol{e}_2 \boldsymbol{e}_3$ is shown in (Figure 8).

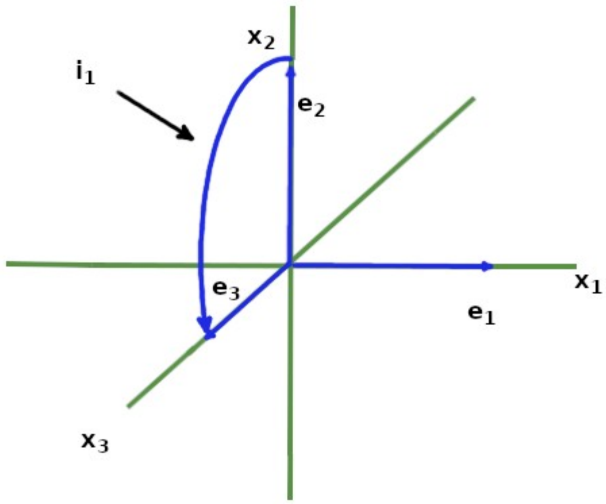


*Figure 8: Unit bivector* $\boldsymbol{i}_1$ *from the vector multiplication* $\boldsymbol{e}_2 \boldsymbol{e}_3$ .

The unit bivector $\boldsymbol{i}_2 = \boldsymbol{e}_3 \boldsymbol{e}_1$ , can be synthesized in the same fashion (Figure 9).

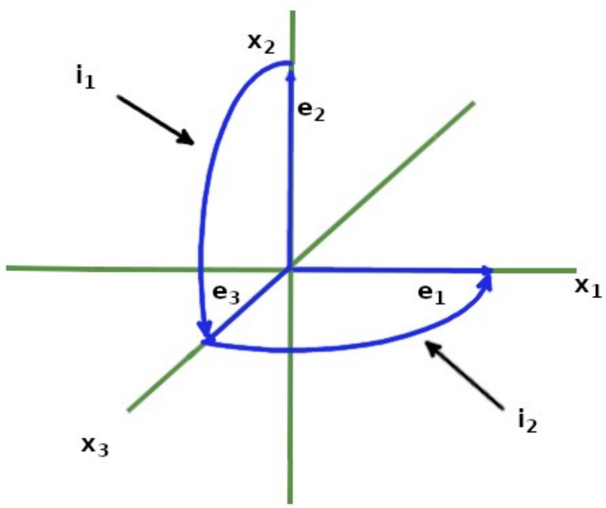


*Figure 9: Unit bivectors* $\boldsymbol{i}_1$ *and* $\boldsymbol{i}_2$ .

Bivectors $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ are next shown on a unit sphere with the multiplication $\boldsymbol{i}_1\boldsymbol{i}_2$ generating $\boldsymbol{i}_3$. All the bivectors have the properties of rotors which Hestenes defined in two dimensions: they have unit length and their placement on a great circle is immaterial (Figure 10).

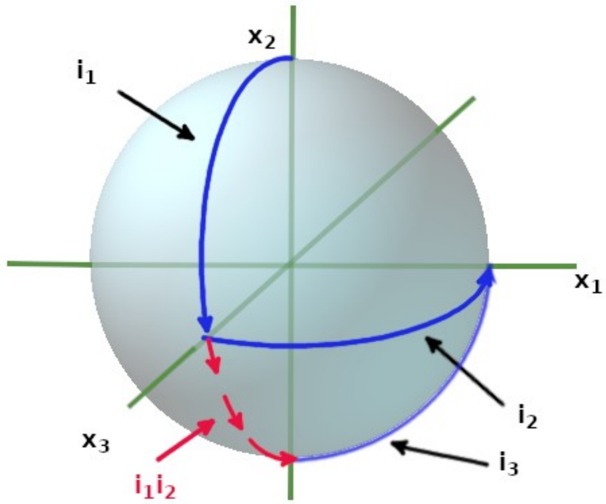


*Figure 10: Bivector multiplication with* $\boldsymbol{i}_1\boldsymbol{i}_2=\boldsymbol{i}_3$ .

The anti-commutative property of bivectors then flows naturally from the definition of unit rotors on the sphere. The resulting rotation reverses direction when the order changes. That is in line with the algebraic formulation since $\boldsymbol{i}_2\boldsymbol{i}_1=-\boldsymbol{i}_3$ in $\boldsymbol{G}^3$ (Figure 11).

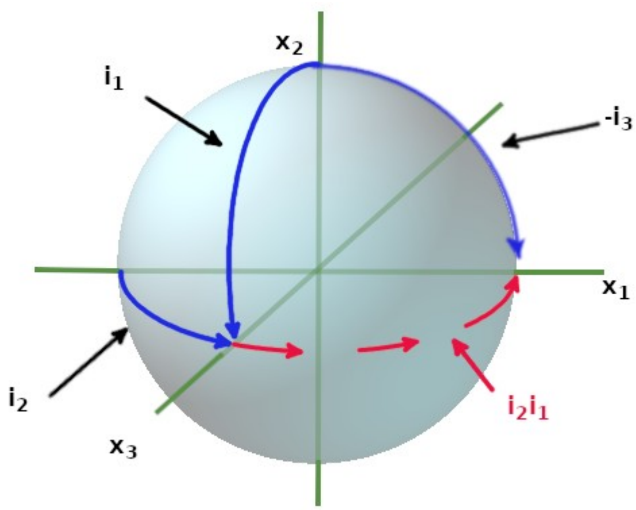


*Figure 11: Bivector multiplication with* $\boldsymbol{i}_2\boldsymbol{i}_1=-\boldsymbol{i}_3$ .

The symmetry of the sphere insures the proof of anti-commutativity for each of the other unit bivector multiplications. An appropriate bias will inform the trajectory of all bivectors regardless of their length and placement on the sphere once impacted by other bivectors – a swirling orbital path the result.

## The ecology of multi-dimensional space

The profoundly visionary work of Hermann Grassmann(Grassmann, 1995) which he pursued for decades and which eventually led Clifford to his epiphany(Clifford, 1878), reveals itself to transcend Euclidean space, providing the conceptual tools to move smoothly between multiple manifolds. Singular points on a real line, rotations on circles, angular Euclidean movements, twisting spherical spirals ... these are the two- and three-dimensional abstract dynamisms available to geometric algebra. They only hint at the extra-dimensional extensions that will find their proper home in this all-encompassing mathematical framework.

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