An Elementary Proof of Franel Number Recurrence Relation

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Abstract

In this paper, we prove the recurrence relation for Franel number using an elementary method. Consequently, we generate another recurrence relation invloving the third powers of binomial coefficients.

Keywords: *binomial coefficients, recurrence relation, Franel number*

Introduction

The binomial coefficients are the positive integers that occur as coefficients in binomial expansion. For instance, the binomial coefficients of $(x+y)^4$ are $1, 4, 6, 4, 1$. A binomial coefficient is denoted by $\binom{n}{k}$, for $n \geq k \geq 0$, where n and k are nonnegative integers. Many identities involving binomial coefficients have been discovered. For instance, Boros and Moll [1, 14–15] showed that sums of the form $\sum_{k=0}^{n} \binom{n}{k} k^r$ are given by:

$$\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}, \tag{2.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1)2^{n-2}, \tag{2.2}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^3 = n(n+3)2^{n-3}, \tag{2.3}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^4 = n(n+1)(n^2+5n-2)2^{n-4}, \tag{2.4}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^5 = n^2(n^3+10n^2+15n-10)2^{n-5}, \tag{2.5}$$

and so on.

In 1894, Franel [2,] showed that if

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3,$$

then

$$(n+1)^2 f_{n+1} = (7n^2+7n+2)f_n + 8n^2 f_{n-1}. \tag{2.6}$$

Also, in 1895, Franel [3,] showed that if

$$P_n = \sum_{k=0}^{n} \binom{n}{k}^4,$$

$$(n+1)^3 P_{n+1} = 2(2n+1)(3n^2+3n+1)P_n + 4(4n-1)(4n+1)^2 P_{n-1}. \tag{2.7}$$

We should note that f_n is called the nth Franel number. They arise in first and second Strehl [4, 5,] identities which state that:

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}. \tag{2.8}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3. \tag{2.9}$$

The purpose of this paper is to prove (2.6) using an elementary method and as a result, we generate a recurrence relation for numbers of the form $\sum_{k=0}^{n}\binom{n}{k}^3k^2$. Both results are presented in section three. Also, we propose two problems involving the fourth powers of binomial coefficients in section four, which when proven true will be intrumental in generating recurrence relations for $\sum_{k=0}^{n}\binom{n}{k}^4k^2$ and $\sum_{k=0}^{n}\binom{n}{k}^4k^3$.

Main results

Theorem 1. *If* $f_n=\sum_{k=0}^{n}\binom{n}{k}^3$, *then*

$$(n+1)f_{n+1}=(7n^2+7n+2)f_n+8n^2f_{n-1}.$$

Proof. Let

$$S_{(n,i)}=\sum_{k=0}^{n}\binom{n}{k}^3k^i, \tag{3.1}$$

Since $\binom{n}{k}=\binom{n}{n-k}$ for $n\geq k\geq 0$, we see that (3.1) can also be written as

$$S_{(n,i)}=\sum_{k=0}^{n}\binom{n}{k}^3(n-k)^i. \tag{3.2}$$

Adding (3.1) and (3.2), we have

$$S_{(n,i)}=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}^3(k^i+(n-k)^i), \tag{3.3}$$

Setting $i=1$ in (3.3), we have

$$S_{(n,1)}=\frac{n}{2}\sum_{k=0}^{n}\binom{n}{k}^3,$$

$$S_{(n,1)}=\frac{n}{2}f_n, \tag{3.4}$$

$$\sum_{k=0}^{n} \binom{n}{k}^3 k = \frac{n}{2} f_n. \tag{3.5}$$

Setting $i = 3$ in (3.3), we have

$$S_{(n,3)} = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k}^3 (n^3 - 3n^2 k + 3nk^2),$$

$$S_{(n,3)} = \frac{1}{2}\big(n^3 f_n - 3n^2 S_{(n,1)} + 3n S_{(n,2)}\big),$$

$$S_{(n,2)} = \frac{1}{3n}\big(3n^2 S_{(n,1)} - n^3 f_n + 2S_{(n,3)}\big). \tag{3.6}$$

Setting $i = 3$ in (3.1), we have

$$S_{(n,3)} = \sum_{k=0}^{n} (\binom{n}{k} k)^3.$$

Since $\binom{n}{k}k = n\binom{n-1}{n-1}$ for $n \geq k > 0$, then

$$S_{(n,3)} = n^3 \sum_{k=0}^{n-1} \binom{n-1}{k-1}^3,$$

Shifting the index of k by 1, we have

$$S_{(n,3)} = n^3 \sum_{k=0}^{n-1} \binom{n-1}{k}^3. \tag{3.7}$$

Putting (3.4) and (3.7) in (3.6), we have

$$S_{(n,2)} = \frac{n^2}{6}\big(f_n + 4f_{n-1}\big), \tag{3.8}$$

$$\sum_{k=0}^{n} \binom{n}{k}^3 k^2 = \frac{n^2}{6}\big(f_n + 4f_{n-1}\big). \tag{3.9}$$

Setting $i = 4$ in (3.1), we have

$$S_{(n,4)} = \sum_{k=0}^{n} \binom{n}{k}^3 k^4,$$

$$S_{(n,4)} = \sum_{k=0}^{n} (\binom{n}{k} k)^3 k,$$

$$S_{(n,4)} = n^3 \sum_{k=0}^{n-1} \binom{n-1}{k-1}^3 k.$$

Shifting the index of k by 1, we have

$$S_{(n,4)} = n^3 \sum_{k=0}^{n-1} \binom{n-1}{k}^3 (k+1),$$

$$S_{(n,4)} = n^3(S_{(n-1,1)} + f_{n-1}). \tag{3.10}$$

Subtracting 1 from n in (3.4) and putting it in (3.10), we have

$$S_{(n,4)} = \frac{n^3(n+1)}{2} f_{n-1}. \tag{3.11}$$

Setting $i = 5$ in (3.3), we have

$$S_{(n,5)} = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k}^3 (n^5 - 5n^4 k + 10n^3 k^2 - 10n^2 k^3 + 5nk^4),$$

$$S_{(n,5)} = \frac{1}{2}\big(n^5 f_n - 5n^4 S_{n,1} + 10n^3 S_{(n,2)} - 10n^2 S_{(n,3)} + 5n S_{(n,4)}\big). \tag{3.12}$$

Putting (3.4), (3.7), (3.8) and (3.11) in (3.12), we see that

$$S_{(n,5)} = \frac{n^4}{12}\big(nf_n - 5(n-3)f_{n-1}\big),$$

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = \frac{n^4}{12}\big(nf_n - 5(n-3)f_{n-1}\big). \tag{3.13}$$

But

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = \sum_{k=0}^{n}(\binom{n}{k}k)^3 k^2,$$

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = n^3\sum_{k=0}^{n-1}\binom{n-1}{k-1}^3 k^2,$$

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = n^3\sum_{k=0}^{n-1}\binom{n-1}{k}^3 (k+1)^2,$$

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = n^3\sum_{k=0}^{n-1}\binom{n-1}{k}^3 (1+2k+k^2),$$

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = n^3(f_{n-1} + 2S_{(n-1,1)} + S_{(n-1,2)}). \tag{3.14}$$

Subtracting 1 from n in (3.4) and (3.8) and putting them in (3.14), we see that

$$\sum_{k=0}^{n}\binom{n}{k}^3 k^5 = \frac{n^3}{3}\big(\frac{n^2+4n+1}{2}f_{n-1} + 2(n-1)^2 f_{n-2}\big). \tag{3.15}$$

Adding 1 to n in (3.13), we have

$$\sum_{k=0}^{n+1} \binom{n+1}{k}^3 k^5 = \frac{(n+1)^4}{12}\big((n+1)f_{n+1} - 5(n-2)f_n\big). \tag{3.16}$$

Adding 1 to n in (3.15), we have

$$\sum_{k=0}^{n+1} \binom{n+1}{k}^3 k^5 = \frac{(n+1)^3}{3}\big(\frac{n^2+6n+6}{2}f_n + 2n^2 f_{n-1}\big). \tag{3.17}$$

Equating the right-hand sides of (3.16) and (3.17), we see that

$$(n+1)^2 f_{n+1} = (7n^2+7n+2)f_n + 8n^2 f_{n-1}. \tag{3.18}$$

We are done. $\square$

Theorem 2. *If* $S_{(n,2)} = \sum_{k=0}^{n} \binom{n}{k}^3 k^2$, *then*

$$n^2(3n-1)S_{(n+1,2)} = (21n^3+14n^2-5n-6)S_{(n,2)} + 8n(3n^2+5n+2)S_{(n-1,2)}.$$

Proof. Subtracting 1 from n in (3.8), we have

$$S_{(n-1,2)} = \frac{(n-1)^2}{6}\big(f_{n-1} + 4f_{n-2}\big), \tag{3.19}$$

Adding 1 to n in (3.8), we have

$$S_{(n+1,2)} = \frac{(n+1)^2}{6}\big(f_{n+1} + 4f_n\big), \tag{3.20}$$

Subtracting 1 from n in (3.18), we have

$$n^2 f_n = (7n^2-7n+2)f_{n-1} + 8(n-1)^2 f_{n-2}. \tag{3.21}$$

Making f_{n+1} the subject of the formula in (3.18) and putting it in (3.20), we have

$$f_{n-1} = \frac{6S_{(n+1,2)} - (11n^2+15n+6)f_n}{8n^2}. \tag{3.22}$$

Making f_{n-2} the subject of the formula in (3.21) and putting it in (3.19), we have

$$f_{n-1} = \frac{n^2 f_n - 12S_{(n-1,2)}}{n(5n-3)}. \tag{3.23}$$

Making f_{n-1} the subject of the formula in (3.8), we have

$$f_{n-1} = \frac{6S_{(n,2)} - n^2 f_n}{4n^2}. \tag{3.24}$$

Equating the right-hand sides of (3.22) and (3.23), we have

$$f_n = \frac{96nS_{(n-1,2)} + 6(5n-3)S_{(n+1,2)}}{8n^3 + (5n-3)(11n^2+15n+6)}. \tag{3.25}$$

Equating the right-hand sides of (3.23) and (3.24), we have

$$f_n = \frac{48nS_{(n-1,2)} + 6(5n-3)S_{(n,2)}}{4n^2 + n^2(5n-3)}. \tag{3.26}$$

Equating the right-hand sides of (3.25) and (3.26), we have

$$n^2(3n-1)S_{(n+1,2)} = (21n^3 + 14n^2 - 5n - 6)S_{(n,2)} + 8n(3n^2+5n+2)S_{(n-1,2)}. \tag{3.27}$$

We are done. □

Open Problems

If $P_n = \sum_{k=0}^{n} \binom{n}{k}^4$, show that

$$\sum_{k=0}^{n} \binom{n}{k}^4 k^2 = \frac{n}{5}\big(nP_n + (4n-1)P_{n-1}\big). \tag{3.28}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^4 k^3 = \frac{n^2}{20}\big(nP_n + 6(4n-1)P_{n-1}\big). \tag{3.29}$$

Conclusion

In this paper, we were able to prove the recurrence relation for Franel number using an elementary method. Consequently, we were able to able to generate a recurrence relation for $\sum_{k=0}^{n}\binom{n}{k}^3 k^2$ using (3.8) and (3.18). We should note that if (3.28) is true, then combining it with (2.7) will yield a recurrence relation for $\sum_{k=0}^{n}\binom{n}{k}^4 k^2$. Also, combining (3.29) and (2.7) will yield a recurrence relation for $\sum_{k=0}^{n}\binom{n}{k}^4 k^3$.

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