Generalized (σ,τ)-Derivations on Associative Rings Satisfying Certain Identities

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Abstract

The main purpose of this paper is to study a number of results concerning the generalized (σ, τ)-derivation D associated with the derivation d of the semiprime ring and prime ring R such that D and d are zero power valued on R, where the mappings σ and τ act as automorphism mappings.

Precisely, this article divided into two sections, in the first section, we emphasize on generalized (σ, τ)-derivation D associated with the derivation d of the semiprime ring and prime ring R while in the second section, we study the effect of the compositions of generalized (σ, τ)-derivations of the semiprime ring and prime ring R such that D is period $(n - 1)$ on R, for some positive integer n.

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1 Introduction

One of the natural questions of Ring Theory is to determine conditions implying commutativity of the ring. During the last two decades, the commutativity of associative rings with derivations have become one of the focus point of several authors and a significant work has been done in this direction. The concept of derivations and automorphisms of affiliated rings are a particular milestone in the advancement of classical Galois Theory and Theory of Invariants. Commutative ring theory is fundamental role role in analysis, algebraic geometry and algebra.

The study of generalized derivations of partially ordered sets has its roots in the study of the Krull dimension of rings and modules, where the concept of Krull dimension of commutative rings was originally developed by E. Noether and W. Krull in the 1920s. In fact, there are some applications of (σ,τ)-derivations which can help to develop an approach to deformation of Lie algebras, and which have various applications in modelling quantum phenomena and in the analysis of complex systems. The map has been extensively investigated in pure algebra. Recently, it has been treated for Banach algebra theory [1].

There are several results in the existing literature that deal with centralizing and commuting mappings on rings. Basically, the study of derivation was initiated during the 1950s and 1960s. The study of centralizing mappings was first undertaken by E. C. Posner [2], who stated that the existence of a non-zero centralizing derivation on a prime ring forces the ring to be commutative (referred to as Posner's Second Theorem). In an attempt to generalize the above result, J. Vukman [3] confirmed that if R is a 2-torsion free prime ring and $d: R \to R$ is a non-zero derivation such that the map $x \to [d(x), x]$ is commuting on R, then R is commutative.

Atteya [4], proved that if R is a 2-torsion free semiprime ring and U is a non-zero ideal of R admits a derivation d satisfying the condition $[d(x^2), d(y^2)] - [x, y] \in Z(R)$ for all $x, y \in U$ then R contains a non-zero central ideal. M. Ashraf, A. Khan and C. Haetinger [5] showed that under certain conditions on a prime ring R, every Jordan (σ,τ)-higher derivation of R is a (σ,τ)-higher derivation of R. B. Dhara and A. Pattanayak [6] proved that if R is a semiprime ring, U a non-zero ideal of R, and σ and τ are two epimorphisms of R, then an additive mapping $D: R \to R$ is a generalized (σ,τ)-derivation of R if there exists a (σ,τ)-derivation $d: R \to R$ such that $D(xy) = D(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. If $\tau(U)d(U) \neq 0$, then R contains a non-zero central ideal of R if the condition $D[x, y] = \pm(xoy)_{\sigma,\tau}$ holds.

Additionally, the results determined by Ajda Fošner in [7] concentrated on the assumption that U was a separated set of an M-bimodule contained in the algebra generated by all idempotents in A, and let α, β be endomorphisms of A such that $\alpha(U) = U$, $\beta(U) = U$. Then, every local generalized (α,β)-derivation (local (α,β)-derivation, resp.) from an algebra A into an A-bimodule M is a generalized (α,β)-derivation ((α,β)-derivation, resp.). Conversely, Marubayashi *et al* [8] stated

numerous results connecting derivations,
(σ,τ)-derivations and generalized derivations to the generalized (σ,τ)-derivation of R. More precisely, the authors studied the commutativity of a prime ring R admitting a generalized (σ,τ)-derivation F, satisfying certain conditions such as $[F(x),x]_{\sigma,\tau}=0$ for all x in an appropriate subset of R, where σ,τ are automorphisms of R. Basically, H. E.Bell and W. S. Matindale III [9] assert that R is a prime ring and U is a nontrivial left ideal of R. If R admits a nonidentity endomorphism d which is one-to-one on U and centralizing on U, then R is commutative.

Throughout the this paper, R will denote an associative ring with center $Z(R)$. Let $x,y,z\in R$. We write the notation $[y,x]$ for the commutator $yx-xy$ and $x\circ y$ for anticommutator $xy+yx$ also make use of the identities $[xy,z]=[x,z]y+x[y,z]$ and $[x,yz]=[x,y]z+y[x,z]$. Recall that R is semiprime if $aRa=0$ implies $a=0$ and R is prime if $aRb=0$ implies $a=0$ or $b=0$. Every prime ring is semiprime ring, but the converse is not true always. R is said to be commutative if $xy=yx$ for all $x,y\in R$. An analogous notion is that of anticommutativity of rings. A ring R is said to be anticommutative if $xy=-yx$ for all $x,y\in R$. A ring R is said to be n-torsion free if for $x\in R$, $nx=0$ implies $x=0$. A map $d\colon R\to R$ is said to be n-commuting on R if $[d(x),x^n]=0$ holds for all $x\in R$.

An additive map $d\colon R\to R$ is called a derivation if the Leibniz's rule $d(xy)=d(x)y+xd(y)$ holds for all $x,y\in R$. Also, an additive mapping $D\colon R\to R$ is called a generalized derivation if there exists an additive mapping d on R such that $D(xy)=D(x)y+xd(y)$ for all $x,y\in R$.

Further this as a motivation we define an additive mapping $d\colon R\to R$ is called a (σ,τ)-derivation if there exists automorphisms $\sigma,\tau\colon R\to R$ such that $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$ for all $x,y\in R$. Also, $D\colon R\to R$ is called a generalized (σ,τ)-derivation if there exists automorphisms $\sigma,\tau\colon R\to R$ and d is a (σ,τ)-derivation such that $D(xy)=D(x)\sigma(y)+\tau(x)d(y)$ for all $x,y\in R$.

If $S\subseteq R$, then a mapping $d\colon R\to R$ preserves S if $d(S)\subseteq S$. A mapping $d\colon R\to R$ is zero-power valued on S if d preserves S and if for each $x\in S$, there exists a positive integer $n(x)>1$ such that $d^{n(x)}(x)=0$. A mapping $d\colon R\to R$ is strong commutativity-preserving (SCP) on S if $[x,y]=[d(x),d(y)]$ for all $x,y\in S$.

Furthermore, a mapping $d: R \to R$ is called period 2 on R if $d^2(x) = x$ for all $x \in R$. We shall use, without explicitly mentioning, the following basic identities:

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y,$$

$$[x, yz]_{\sigma,\tau} = \sigma(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$$

$$(xo(yz))_{\sigma,\tau} = (xoy)_{\sigma,\tau}\sigma(z) - \tau(y)(xoz)_{\sigma,\tau} = \tau(y)(xoz)_{\sigma,\tau} + (xoy)_{\sigma,\tau}\sigma(z).$$

where $[x, y]_{(\sigma,\tau)}$ for the commutator $x\sigma(y)-\tau(y)x$ and $(x \circ y)_{(\sigma,\tau)}$ for anti-commutator $x\sigma(y) + \tau(y)x$.

In the present paper, we establish a number of results concerning the generalized (σ, τ) -derivation D associated with the derivation d of the semiprime ring and prime ring R, in addition to presenting the general formula for the composition of a generalized (σ, τ) -derivation D, and some example applications of such.
We assume the composition $\sigma \circ D = D \circ \sigma, \tau \circ D = D \circ \tau, \sigma \circ d = d \circ \sigma$ and $\tau \circ d = d \circ \tau$ of R. Also, we used the well-known fact about the center of semiprime rings:
The center of semiprime ring contains no non-zero nilpotent elements.
We begin with the following known results, on which our derivation subsequently depends:

Lemma 1.1. *([10, Proposition 8.5.3, Page 330]) Let R be a ring. Then every intersection of prime ideals is semiprime. Conversely every semiprime ideal is an intersection of prime ideals.*

Lemma 1.2. *([11, Lemma 2.1]) Let R be a semiprime ring, U a non-zero two-sided ideal of R and $a \in R$ such that $axa = 0$ for all $x \in U$, then $a = 0$.*

Lemma 1.3. *([12, Lemma 2.4]) Let R be a semiprime ring and $a \in R$. Then $[a, [a, x]] = 0$ holds for all $x \in R$ if and only if $a^2, 2a \in Z(R)$.*

Lemma 1.4. *([13, Lemma 2]) Let R be a prime ring. If $a, b, c \in R$ such that $axb = cxa$ for all $x \in R$, then either $a = 0$ or $c = b$.*

Lemma 1.5 (14, Lemma 1.1). *Let R be a semiprime ring. If $a, b \in R$ such that $axb = 0$ for all $x \in R$ then $ab = ba = 0$.*

2 On Generalized (σ, τ)-Derivation of Semiprime Rings

In this section, we emphasize on a number of results concerning the generalized (σ, τ) -derivation D associated with the derivation d of the semiprime ring and prime ring R has the property of torsion free restricted, where the mappings σ and τ act as automorphisms mappings.

Theorem 2.1. *Let R be a 2 and 3-torsion free semiprime ring and σ, τ be automorphism mappings of R. If D is a generalized (σ, τ)-derivation which is zero power valued index 2 on R then $d = 0$, where R satisfies the relation $aRb \subset Z(R), a, b \in R$ and $\sigma^2 = \sigma$.*

Proof. From our hypothesis, we have D is zero power valued on R. Then there exists an integer $n(r) > 1$ such that $D^{n(r)}(r) = 0$ for all $r \in R$. Since D is zero power valued index 2 on R, we deduce that $D^2(r) = 0$ for all $r \in R$. Replacing r with rs for all $r, s \in R$, we find that
$D(D(rs)) = D(D(r)\sigma(s) + \tau(r)d(s)) = 0$. We rewrite the above relation as
$D(D(rs)) = D(D(r)\sigma(s)) + D(\tau(r)d(s)) = 0$. Simple calculation, we see that
$D^2(r)\sigma^2(s) + \tau(D(r))d(\sigma(s)) + D(\tau(r))\sigma(d(s)) + \tau^2(r)d^2(s) = 0$.
Since $\sigma \circ d = d \circ \sigma$ and $\tau \circ D = D \circ \tau$ of R, we obtain

$$D^2(r)\sigma(s) + D(\tau(r))d(\sigma(s)) + D(\tau(r))d(\sigma(s)) + \tau(r)d^2(s) = 0. \tag{1}$$

Due to the fact that D is zero power valued on R, we conclude that

$$D(\tau(r))d(\sigma(s)) + D(\tau(r))d(\sigma(s)) + \tau(r)d^2(s) = 0.$$

Since σ and τ are automorphisms of R. In this case $\sigma, \tau : R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ, τ are automorphisms of R, we use $\sigma(s) = q, \tau(r) = p$ in the above relation, we find that

$$2D(p)d(q) + pd^2(q) = 0. \tag{2}$$

In (2), we substitute q by tq, $t \in R$, we obtain

$$2D(p)(d(t)\sigma(q) + \tau(t)d(q)) + pd(d(t)\sigma(q) + \tau(t)d(q)) = 0.$$

Moreover, the left side of this relation imply

$$\begin{aligned} &2D(p)d(t)\sigma(q) + 2D(p)\tau(t)d(q)) + pd^2(t)\sigma^2(q) \\ &+ p\tau(d(t))d(\sigma(q)) + pd(\tau(t))\sigma(d(q)) + p\tau^2(t)d^2(q) = 0. \end{aligned} \tag{3}$$

In agreement with (2), the first term of (3) becomes $-pd^2(t)\sigma(q)$, which cancel with the item $pd^2(t)\sigma^2(q)$ after applying the condition $\sigma^2(q) = \sigma(q)$.
Due to σ and τ are automorphisms of R, then (3) becomes

$$2D(p)\tau(t)d(q) + p\tau(d(t))d(q) + pd(\tau(t))\sigma(d(q)) + p\tau^2(t)d^2(q) = 0. \tag{4}$$

Applying that $\sigma \circ d = d \circ \sigma$ and $\tau \circ D = D \circ \tau$ of R and σ and τ are automorphisms of R. In this case $\sigma, \tau : R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ and τ are automorphisms of R, we use $\sigma(t) = y, \sigma(q) = e, \tau(t) = x, \tau(x) = w$ in (4), we find that
$2D(p)xd(q) + pd(x)d(q) + pd(x)d(e) + pwd^2(q) = 0$. Replacing x by t, we obtain $2D(p)td(q) + pd(t)d(q) + pd(t)d(e) + pwd^2(q) = 0$. Now, replacing p by x, q by y, w by t and e by y, we deduce

$$2D(x)td(y) + 2xd(t)d(y) + xtd^2(y) = 0. \tag{5}$$

Using (2) in relation (5), we find that

$$2D(x)td(y) + 2xd(t)d(y) - 2xD(t)d(y) = 0. \tag{6}$$

Replacing x by $D(x)$ and applying $D^2(R) = 0$, we see that

$$2D(x)d(t)d(y) - 2D(x)D(t)d(y) = 0. \tag{7}$$

According to (2), we rewrite (7) as follows
$-xd^2(t)d(y) + D(x)td^2(y) = 0$.
Replacing x with $D(x)$ and using the fact $D^2(R) = 0$, we conclude that

$$D(x)d^2(t)d(y) = 0. \tag{8}$$

In (8) replacing y with ys, $s \in R$ and applying the result imply $D(x)d^2(t)yd(s) = 0$. Replacing y with $yD(x)$ and s with $d(t)$ and using the semiprimeness of R, we obtain

$$D(x)d^2(t) = 0. \tag{9}$$

In (2) we set $y = d(y)$, we show that $2D(x)d^2(y) + xd^3(y) = 0$.
Applying (9) and using the semiprimeness of R, we find that $d^3(y) = 0$.
In this result replacing y by xy, we obtain $3d^2(x)d(y)+3d(x)d^2(y) = 0$, for all $x, y \in R$.
Substitution x by $d(x)$ and employing R is 3-torsion free, we conclude that $d^3(x)d(y)+ d^2(x)d^2(y) = 0$, for all $x, y \in R$.
Due to the result $d^3(y) = 0$ for all $x \in R$. The first term becomes zero, that means the above relation reduces to
$d^2(x)d^2(y) = 0$.
Right-multiplying by $rd^2(x)$, $r \in R$ and left-multiplying by $d^2(y)r$, $r \in R$, with applying the center of semiprime ring contains no non-zero nilpotent elements and the fact that $aRb \subset Z(R), a, b \in R$, we obtain
$d^2(y)Rd^2(x) = 0$. Based on the semiprimeness of R, we conclude that $d^2(x) = 0$, for all $x \in R$. Replacing x by xy and using the result $d^2(x) = 0$, we arrive to

$2d(x)d(y) = 0$, for all $x, y \in R$.
Using the fact R is 2-torsion free and left-multiplying by $d(y)R$ and right-multiplying by $Rd(x)$ with applying the center of semiprime ring contains no non-zero nilpotent elements and $aRb \in Z(R)$. It follows that $d(y)Rd(x) = 0$ yields $d = 0$.
Using the same argument as in the last part of the proof, we obtain the required result. □

The following example shows the condition $aRb \subset Z(R)$ for the results necessary *i.e.*, we can not exclude it from the hypothesis.

Example 1. *Let* $R = \{\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \setminus x, y \in \mathbb{F}\}$ *be a ring over a field* $\mathbb{F}$ *such that* x *and* y *are nilpotent index 2 also* y *is an annihilator element. Define the mappings* $g, h: R \to R$ *as follows:*

$$g(t) = g(\begin{pmatrix} 0 & 0 \\ n & m \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix} \text{ and } h(s) = h(\begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}$$

for all $t, s \in R, n, m, p, q \in \mathbb{F}$.

Obviously, $guh = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}$, *where* $u \in R$.
Thus, we find that

$$[u, guh] = [\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 \\ y^2nq & 0 \end{pmatrix}.$$

Due to $y^2 = 0$ *for all* $y \in \mathbb{F}$ *this matrix reduces to* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Also, with applying the relation $x^2 = 0$, *we conclude that*

$$[g(t), guh] = [\begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}] = 0 \text{ and } [h(s), guh] = [\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ nyq & 0 \end{pmatrix}]$$

$= 0$. *Hence, we obtain* $guh \subset Z(R)$. *Let us keeping the definition of* g *and* h *with* $u \in R$. *We now suppose*

$R^* = \{\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}, u \in R\}$, *where* R^* *is a ring has no divisors of zero.*
Let $d: R^* \to R^*$ *be an additive mapping define as*

$d(s) = \begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h & g \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}$ *for all* $s \in R^*$. *Clearly,* d *is a derivation of* R^*.

Suppose $\sigma, \tau: R^* \to R^*$ *be a mappings defined by*

$$\sigma(r_1) = \sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} \text{ and } \tau(r_2) = \tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$$

for all $r_1, r_2 \in R^*$.

Moreover, we check whether d *is* (σ,τ)*-derivation on* R^*. *Hence, we assume* $d(r_1r_2) = d(r_1)\sigma(r_2) + \tau(r_1)d(r_2)$, *for all* $r_1, r_2 \in R^*$.
We consider $r_1 = \begin{pmatrix} h & g \\ u & 0 \end{pmatrix}$ *and* $r_2 = \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$. *The right-side*

$$= d\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix} + \tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -gzuw & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & -hz \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -gzuw & -gz) \\ uw & 0 \end{pmatrix},$$

since $zuw \subset Z(R)$, *then*

$$= \begin{pmatrix} -zuwg & -hz \\ uw & 0 \end{pmatrix}, \text{ where } z = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \text{ and } w = \begin{pmatrix} 0 & 0 \\ e & c \end{pmatrix} =$$

$\begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}$ *for all* $a, b, e, c \in \mathbb{Z}, z, w \in R^{\star}$ *yields* $zuwg = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $hz = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then, this matrix reduces to $\begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}$.

While the left-side

$$d(r_1 r_2) = d(\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) = d(\begin{pmatrix} hw + ug & hz \\ uw & uz \end{pmatrix}) = \begin{pmatrix} 0 & -hz \\ uw & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}.$$

Thus, d *is* (σ, τ)*-derivation of* R^*.

We now investigate on generalized (σ, τ)*-derivation of* R^**. Let* D *be additive mapping on* R^* *defined*

$$D(t) = D(\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}) = \begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix}.$$ *Then, we check*

$$D(r_1 r_1) = D(r_1)\sigma(r_2) + \tau(r_1)d(r_2), \text{ for all } r_1, r_1 \in R^*.$$

Take $r_1 = \begin{pmatrix} h & g \\ u & 0 \end{pmatrix}$ *and* $r_2 = \begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$,
where $\tau(r_1) = \tau\begin{pmatrix} h & 0 \\ u & 0 \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\sigma(r_2) = \sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix} = \begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix}$ *for all* $r_1, r_2 \in R^*$.

The left-side give us

$$D(r_1 r_2) = D(\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) = D\begin{pmatrix} hw + gu & hz \\ uw & uz \end{pmatrix} = \begin{pmatrix} hz & 0 \\ uw & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}.$$

Furthermore, the right-side provide

$$D(r_1)\sigma(r_2) + \tau(r_1)d(r_2) = D\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix} + \tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix}d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}$$

Applying the definitions of the mappings, we find that

$$= \begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} w & o \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} gw & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & hz \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix}.$$

Thus, D is generalized (σ,τ)-derivation of R^*. We have enough information to determine whether $D^2 = 0$. Then

$$D^2(r_1r_1) = D^2(r_1)\sigma^2(r_2)+\tau(D(r_1))d(\sigma(r_2))+D(\tau(r_1))\sigma(d(r_2))+\tau^2(r_1)d^2(r_2).....(*).$$

$$= D(D\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})\sigma(\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(D\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + D(\tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})$$

$$\sigma(d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(\tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}).$$

$$= D\begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix}\sigma\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + \tau\begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix}d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + D\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix} + \tau\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}d\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}.$$

Thus, we conclude that

$$= \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0(zuw) & 0 \end{pmatrix} + \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & hz \\ 0 & 0 \end{pmatrix}.$$

Due to the action of the entries of u and w, then $uw = 0$. This result modifies this matrix to become zero i.e. $D^2 = 0$.

Let us move to show that $d^2 = 0$ *too.*

$$d^2(r_1r_1) = d^2(r_1)\sigma^2(r_2) + \tau(d(r_1))d(\sigma(r_2)) + d(\tau(r_1))\sigma(d(r_2)) + \tau^2(r_1)d^2(r_2)$$

$$= d(d\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})\sigma(\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(d\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(\sigma\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + d(\tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})$$

$$\sigma(d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}) + \tau(\tau\begin{pmatrix} h & g \\ u & 0 \end{pmatrix})d(d\begin{pmatrix} w & z \\ u & 0 \end{pmatrix}).$$

$$= d\begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}\sigma\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + \tau\begin{pmatrix} 0 & -g \\ u & 0 \end{pmatrix}d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} + d\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\sigma\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$$

$$+\tau\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}d\begin{pmatrix} 0 & -z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0(zuw) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ zuw & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ u & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & g \\ u & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & z \\ u & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 0 \\ uw & 0 \end{pmatrix} + \begin{pmatrix} 0 & hz \\ 0 & 0 \end{pmatrix}.$$

Since $uw = 0$ *and* $hz = 0$*, this matrix becomes zero* i.e $d^2 = 0$.

Substituting the values of D^2 *and* d^2 *in Relation* $*$*, we find that*

$\tau(D(r_1))d(\sigma(r_2)) + D(\tau(r_1))\sigma(d(r_2)) = 0.$

Due to σ *commute with* d *this relation modify to* $(\tau(D(r_1)) + D(\tau(r_1)))d(\sigma(r_2)) = 0$ *for all* $r_1, r_2 \in R^*$*. Then*

$(\tau\begin{pmatrix} g & 0 \\ u & 0 \end{pmatrix} + D\begin{pmatrix} -h & 0 \\ 0 & 0 \end{pmatrix})d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = 0.$ *Moreover,*

$$(\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = 0.$$

Basically, R^* has no divisors of zero. Hence, we arrive to either

$\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} = 0$ *yields contradiction or* $d\begin{pmatrix} w & 0 \\ zuw & 0 \end{pmatrix} = 0$*. Thus* $d = 0$.

Theorem 2.2. *Let R be a 2-torsion free semiprime ring, σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ, τ)-derivation D such that $[D(x), x]_{\sigma,\tau} = 0$ for all $x \in R$. If d is central period 2 on R then D is zero power valued on R.*

Proof. Let us introduce a mapping $\gamma: R \times R \to R$ by the relation
$\gamma(r_1, r_2) = [D(r_1), r_2]_{\sigma,\tau} + [D(r_2), r_1]_{\sigma,\tau}$ for all $r_1, r_2 \in R$.
It is symmetric and additive in both arguments. Notice that for all $r_1, r_2, z \in R$,
$\gamma(r_1 r_2, z) = [D(r_1 r_2), z]_{\sigma,\tau} + [D(z), r_1 r_2]_{\sigma,\tau}$.
Using the definition of (σ, τ)-generalized derivation, we expand the right-hand side as

$$\begin{aligned} \gamma(r_1 r_2, z) =& D(r_1)[\sigma(r_2), z]_{\sigma,\tau} + [D(r_1), z]_{\sigma,\tau}\sigma(r_2) + \tau(r_1)[d(r_2), z]_{\sigma,\tau} \\ &+ [\tau(r_1), z]_{\sigma,\tau} d(r_2) + r_1[D(z), r_2]_{\sigma,\tau} + [D(z), r_1]_{\sigma,\tau} r_2. \end{aligned} \tag{10}$$

Applying that $\sigma \circ d = d \circ \sigma$ and $\tau \circ D = D \circ \tau$ of R and σ and τ are automorphisms of R. In this case $\sigma, \tau: R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ, τ are automorphisms of R, we use $\sigma(r_2) = y, \tau(r_1) = x$ in the above relation, we find that

$$\begin{aligned} \gamma(xy, z) =& D(x)[y, z]_{\sigma,\tau} + [D(x), z]_{\sigma,\tau} y + x[d(y), z]_{\sigma,\tau} + [x, z]_{\sigma,\tau} d(y) \\ &+ x[D(z), y]_{\sigma,\tau} + [D(z), x]_{\sigma,\tau} y. \end{aligned}$$

Replacing y with xy in the main relation, we find that $\gamma(x, xy) = [D(x), xy]_{\sigma,\tau} + [D(xy), x]_{\sigma,\tau} = 0$.

Further, we conclude that $[D(x), xy]_{\sigma,\tau} + [D(x)\sigma(y) + \tau(x)d(y), x]_{\sigma,\tau} = 0$. Expanding the left-hand side, we obtain

$$x[D(x), y]_{\sigma,\tau} + [D(x), x]_{\sigma,\tau}y + D(x)[\sigma(y), x]_{\sigma,\tau}$$
$$+ [D(x), x]_{\sigma,\tau}\sigma(y) + \tau(x)[d(y), x]_{\sigma,\tau} + [\tau(x), x]_{\sigma,\tau}d(y) = 0. \tag{11}$$

Applying that $\sigma \circ d = d \circ \sigma$ and $\tau \circ D = D \circ \tau$ of R and σ and τ are automorphisms of R. In this case $\sigma, \tau: R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ, τ are automorphisms of R, we use $\sigma(y) = t, \tau(x) = s$ in the (11) becomes $x[D(x), y]_{\sigma,\tau}+[D(x), x]_{\sigma,\tau}y+D(x)[t, x]_{\sigma,\tau}+[D(x), x]_{\sigma,\tau}t+s[d(y), x]_{\sigma,\tau}+[s, x]_{\sigma,\tau}d(y) = 0$ for all $t, s \in R$.

Replacing y by $d(t)$, s by $d(s)$ and t by $d(t)$ with employing d acts as central mapping, we find that

$[D(x), x]_{\sigma,\tau}d(t) + [D(x), x]_{\sigma,\tau}d(t) = 0$ for all $t, s \in R$. Since R is 2-torsion free, we deduce

$$[D(x), x]_{\sigma,\tau}d(t) = 0. \tag{12}$$

Writing this relation as $([D(x)\sigma(x) - \tau(x)D(x))d(t) = 0$. According to σ, τ are automorphisms of R, we use $\sigma(x) = y$ and $\tau(x) = t$ in this relation, we conclude that $(D(x)y - tD(x))d(t) = 0$.

Now replacing y by $-y$ and combine with this relation, we find that

$2tD(x)d(t) = 0$. Applying the fact that R is 2-torsion free, we conclude that

$tD(x)d(t) = 0$. Substituting this result in the relation $(D(x)y - tD(x))d(t) = 0$. It modifies to $D(x)yd(t) = 0$. Replacing t by $d(t)$ and using d is a period 2, we see that

$D(x)yt = 0$. In this relation, replacing x by $D(x)$ and t by $D^2(x)$ with using the semiprimeness of R, we arrive to

$D^2(x) = 0$ for all $x \in R$. This completes the proof. □

By the same manner of prime ring, we can obtain the same result without the condition period 2 of d when d is non-zero.

Proposition 2.3. *Let R be a 2-torsion free prime ring, σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ,τ)-derivation D such that $[D(x),x]_{\sigma,\tau}=0$ for all $x\in R$. If d is non-zero central on R then D is zero power valued index 2 on R.*

Proposition 2.4. *Let σ and τ be two ring automorphisms of R. Suppose and there exists a generalized (σ,τ)-derivation D such that $D(x)[x,y]_{\sigma,\tau}=0$. Then*

(i) if R is semiprime and U is a maximal ideal of R then either $D(R)$ is commuting on R or $[t,r]_{\sigma,\tau}=0$ for all $t,r\in R$,

(ii) if R is a 2-torsion free prime ring then either D has zero power valued index 2 on R or $D(R)=0$,

(iii) if D is a period 2 on prime ring R then $[D(x),x]_{\sigma,\tau}=0$..

Proof. (i) In the main relation $D(x)[x,y]_{\sigma,\tau}=0$ for all $x,y\in R$, replacing y by yt, $t\in R$. That gives

$$D(x)\sigma(y)[x,t]_{\sigma,\tau}+D(x)[x,t]_{\sigma,\tau}\sigma(t)=0. \tag{13}$$

Obviously, the second term vanishes in view of the main relation. This leads to $D(x)y[x,t]_{\sigma,\tau}=0$ for all $x,y,t\in R$. By reason of R is a semiprime ring, so in this relation we replace y with xR and t by $D(x)$, we arrive to

$$D(x)R[x,t]_{\sigma,\tau}=0. \tag{14}$$

Due to R is semiprime, we consider the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha=\{0\}$.

In agreement with Lemma 2, we show the intersection $\{P_\alpha\}$ of prime ideals of R is a semiprime ideal. Based on U is a maximal ideal of R there are no other ideals contained between U and R. Hence, we find that $\cap P_\alpha\subseteq U$.

If P is a typical member of $\cap P_\alpha$ and $x\in U$, it follows that $[w,x]\in P$ or $D(x)\in P$.

Construct two additive subgroup $T_1=\{x\in U\mid [x,t]_{\sigma,\tau}\in P\}$ and $T_2=\{x\in U\mid D(x)\in P\}$, where any ideal of a ring R is subgroup of the additive group of R. Then

$T_1 \cup T_2 = U$.

Since a group can not be a union of two its proper subgroups, either $T_1 = U$ or $T_2 = U$, that is, either $[x,t]_{\sigma,\tau} \in P$ or $D(x) \in P$. Thus, both cases yield $[x,t]_{\sigma,\tau} \in \cap P_\alpha$ or $D(x) \in \cap P_\alpha$. In other words,$[x,t]_{\sigma,\tau} \in \cap P_\alpha \subseteq U$ or $D(x) \in \cap P_\alpha \subseteq U$. In what follows, we obtain either $[x,t]_{\sigma,\tau} \in U$ for all $x \in U, w \in R$ or $D(x) \in U$ for all $x \in U$.

We divide the proof into two cases.

Case 1: If $[x,t]_{\sigma,\tau} \in U$ for all $x \in U, w \in R$ then $[x,t]_{\sigma,\tau} = 0$ for all $x \in U, t \in R$. Replacing x with $xr, r \in R$, we find that $x[t,r]_{\sigma,\tau} + [t,x]_{\sigma,\tau} r = 0$ for all $x \in U, t, r \in R$. Applying the relation $[t,x]_{\sigma,\tau} = 0$ to this result, we conclude that $x[t,r]_{\sigma,\tau} = 0$ for all $x \in U, w, r \in R$. We write this relation as follows $U[t,r]_{\sigma,\tau} = (0)$. According to Lemma 3, we obtain $[t,r]_{\sigma,\tau} = 0$ for all $t, r \in R$.

Case 2: If $D(x) \in U$ for all $x \in R$ then $D(x) = 0$ for all $x \in R$, we arrive to D is commuting on R.

(ii) Suppose R is prime, we have the relation

$$D(x)y[x,t]_{\sigma,\tau} = 0, \quad x, y, t \in R.$$

Substituting of y by R, we see that $D(x)R[x,t]_{\sigma,\tau} = 0$. Replacing x by $D(x)$, we find that $D^2(x)R[D(x),t]_{\sigma,\tau} = 0$. Since R is prime, we come to the following results: either $D^2(x) = 0$ that is mean has zero power valued index 2 on R or $[D(x),t]_{\sigma,\tau} = 0$. Given that σ and τ are automorphisms of R. In this case $\sigma, \tau : R \to R$ are 1-1 and onto. $(\sigma(R) = R; \tau(R) = R)$: In particular, since σ, τ are automorphisms of R, we use $\sigma(t) = w, \tau(t) = y$, we conclude that $D(x)w = yD(x)$ for all $w, y \in R$. Putting $y = -y$ and combine with the previous result, we deduce $2D(x)w = 0$ for all $w, x \in R$. Applying R is a 2-torsion free yields $D(R) = 0$.

(iii) Using the same technique of Branch(ii), we arrive to $D^2(x)R[D(x),t]_{\sigma,\tau} = 0$. In ducat to D is period 2 on R this relation modifies to $xR[D(x),x]_{\sigma,\tau} = 0$. Since R is prime ring then $[D(x),x]_{\sigma,\tau} = 0$. Hence we get the required result. □

Theorem 2.5. *Let R be a 2-torsion free semiprime ring and σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ,τ)-derivation D*

such that d has zero power valued index 2 on R and $D(xy) = D(yx)$ for all $x, y \in R$. Then $d([x,y]_{\sigma,\tau}) = 0$ for all $x, y \in R$.

Proof. Suppose $c \in R$ is a constant, *i.e.*, an element such that $D(c) = 0$, and let c be an arbitrary element of R. According to our hypothesis, we have $D(rw) = D(wr)$ for all $r, w \in R$. Replacing r with c and w with z, we arrive to $D(cz) = D(zc)$ for all $z \in R$.
Then

$$D(c)\sigma(z) + \tau(c)d(z) = D(z)\sigma(c) + \tau(z)d(c). \tag{15}$$

Applying the fact that $D(c) = 0$ to (15), we find that

$$\tau(c)d(z) = \tau(z)d(c). \tag{16}$$

For all $p, q \in R$, the commutator $[p,q]_{\sigma,\tau}$ is a constant. Hence from (16), we obtain

$$\tau([p,q]_{\sigma,\tau})d(z) = \tau(z)d([p,q]_{\sigma,\tau}), \quad \text{for all } p, q, z \in R.$$

Since τ is automorphism of R. In this case $\tau : R \to R$ is 1-1 and onto. $\tau(R) = R$): In particular, since τ is automorphism of R, we use $\tau([p,q]) = [x,y], \tau(z) = t$, this equation becomes

$$[x,y]_{\sigma,\tau}d(t) = td([x,y]_{\sigma,\tau}). \tag{17}$$

Replacing t by $d(t)$ and using d has zero power valued index 2 on R, we see that $d(t)d([x,y]_{\sigma,\tau}) = 0$. Wring $[x,y]_{\sigma,\tau}$ for t, we find that
$d([x,y]^2_{\sigma,\tau}) = 0$.
In agreement with Lemma 4, we obtain
$2[d([x,y]_{\sigma,\tau}), r]_{\sigma,\tau} = 0$ for all $x, y, r \in R$.
Since R is 2-torsion free, this relation modifies to $[d([x,y]_{\sigma,\tau}), r]_{\sigma,\tau} = 0$ for all $x, y, r \in R$. Furthermore,
$d([x,y]_{\sigma,\tau}) \in Z(R)$ for all $x, y \in R$.
Indicate to the center of semiprime ring contains no non-zero nilpotent element, the

relation $d([x,y]^2_{\sigma,\tau}) = 0$ yields $d([x,y]_{\sigma,\tau}) = 0$. □

Theorem 2.6. *Let R be a prime ring, σ and τ be two automorphisms of R. Suppose that there exists a generalized (σ,τ)-derivation D such that d is period 2 of R commute with D and $[D(r_1), D(r_2)]_{\sigma,\tau} = 0$ for all $r_1, r_2 \in R$. Then either $[D(r_1), r_1]_{\sigma,\tau} = 0$ or $d(R) = 0$.*

Proof. Replacing r_2 with r_1r_2 in the main relation $[D(r_1), D(r_2)]_{\sigma,\tau} = 0$, we obtain $[D(r_1), D(r_1)\sigma(r_2) + \tau(r_1)d(r_2)]_{\sigma,\tau} = 0$ for all $r_1, r_2 \in R$. Moreover, we find that

$$D(r_1)[D(r_1), \sigma(r_2)]_{\sigma,\tau} + \tau(r_1)[D(r_1), d(r_2)]_{\sigma,\tau} + [D(r_1), \tau(r_1)]_{\sigma,\tau}d(r_2) = 0. \quad (18)$$

In (18), we substitute r_2 with $D(z)$, $z \in R$. Due to σ and τ are automorphisms of R. Applying the same previous technique which used in the proof of Theorem 1 and thanks to $[D(x), D(z)]_{\sigma,\tau} = 0$, we find that
$x[D(x), d(D(z))]_{\sigma,\tau} = -[D(x), x]d(D(z))$, for all $x, z \in R$. Putting $b = [D(x), d(D(z))]_{\sigma,\tau}$ and $a = -[D(x), x]d(D(z))$ yields $xb = -a$. Left-multiplying by a and right-multiplying by xa, we have $ax(bxa) = -a^2xa$. According to Lemma 5, we conclude that either $a = [D(x), x]_{\sigma,\tau}d(D(z))$ is equal to zero for all $x, z \in R$ or $(bxa) = -a^2$. Now we focus on the term $[D(x), x]_{\sigma,\tau}d(D(z)) = 0$. Due to d and D commute with each other, we have $[D(x), x]_{\sigma,\tau}D(d(z)) = 0$ for all $x, z \in R$. Replacing z by $d(z)$ and using d is period 2 of R, we find that
$[D(x), x]_{\sigma,\tau}D(z) = 0$ for all $x, z \in R$. Replacing z by yz, we deduce
$[D(x), x]_{\sigma,\tau}yd(z) = 0$. Applying the primeness of R,we complete the proof.

□

3 The Compositions of Generalized (σ,τ)-Derivations with Their Applications

In [15], Ajda and Mehsin derived a Leibniz's formula for the compositions of generalized (σ,τ)-derivations and some results based on it.

Definition 3.1. Let D be a generalized (σ,τ)-derivation of a ring R, σ and τ be automorphisms of R such that σ and τ commute with D and d. Then we define the compositions of D as

$D^n(xy) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y))$ for all $x, y \in R$, where n and r are a positive integers (we adopt the convention $D^0 = d^0 = id$).

Theorem 3.1. *Let R be a 2-torsion free prime ring, σ and τ be two automorphisms of R. For some positive integer n, suppose that D is a non zero generalized (σ,τ)-derivation satisfying $D^n(x) \in Z(R)$ for all $x \in R$ and has period $n-1$ of R. Then $[x,y]_{\sigma,\tau} \in Z(R)$ for all $x, y \in R$.*

Proof. From the hypothesis, we have $[D^n(x), r] = 0$ for all $x, r \in R$. Basically we have the relation $[D^n(x), x^n]_{\sigma,\tau} = 0$ for all $x \in R$. Replacing x with yx, we obtain $[D^n(xy), (xy)^n]_{\sigma,\tau} = 0$ for all $x, y \in R$. Now applying the previous definition to this relation, we find that

$$\left[\sum_{r=0}^{n} \binom{n}{r} D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n\right]_{\sigma,\tau} = 0,$$

for all $x, y \in R$. Then

$$\begin{aligned}&\left[\binom{n}{0} D^n(\sigma^n(x))d^0(\tau^0(y)) + \binom{n}{1} D^{n-1}(\sigma^{n-1}(x))d(\tau(y))\right.\\ &+\binom{n}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y)) + \cdots + \binom{n}{n} D^{n-n}(\sigma^{n-n}(x))\\ &d^n(\tau^n(y)), (xy)^n]_{\sigma,\tau} = 0.\end{aligned}$$

From the this relation, we obtain

$$\begin{aligned}&[D^n(\sigma^n(x))y + nD^{n-1}(\sigma^{n-1}(x))d(\tau(y)) + \frac{n(n-1)!}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y))\\ &+ \cdots + xd^n(\tau^n(x)), (xy)^n]_{\sigma,\tau} = 0.\end{aligned}$$

We rewrite this relation as a sum of two commutators

$$\begin{aligned}&\left[nD^{n-1}(\sigma^{n-1}(x))d(\tau(y)) + \frac{n(n-1)!}{2} D^{n-2}(\sigma^{n-2}(x))d^2(\tau^2(y))\right.\\ &+ \cdots + xd^n(\tau^n(x)), (xy)^n]_{\sigma,\tau} + \left[D^n(\sigma^n(x))y, (xy)^n\right]_{\sigma,\tau} = 0.\end{aligned}$$

Furthermore,

$$\sum_{r=1}^{n}\binom{n}{r}[D^{n-r}(\sigma^{n-r}(x))d^{r}(\tau^{r}(y)),(xy)^{n}]_{\sigma,\tau}+[D^{n}(\sigma^{n}(x))y,(xy)^{n}]_{\sigma,\tau}=0. \qquad (19)$$

Multiplying (19) by $t \in R$ on the left and right, we see that

$$\sum_{r=1}^{n}\binom{n}{r}t[D^{n-r}(\sigma^{n-r}(x))d^{r}(\tau^{r}(y)),(xy)^{n}]_{\sigma,\tau}t+t[D^{n}(\sigma^{n}(x))y,(xy)^{n}]_{\sigma,\tau}t=0.$$

This relation has the form

$$at+tb=0, \qquad (20)$$

where we set $a=\sum_{r=1}^{n}\binom{n}{r}t[D^{n-r}(\sigma^{n-r}(x))d^{r}(\tau^{r}(y)),(xy)^{n}]_{\sigma,\tau}$ and
$b=[D^{n}(\sigma^{n}(x))y,(xy)^{n}]_{\sigma,\tau}t$.
Multiplying (20) by $s \in R$ on the left, we arrive to

$$sat+stb=0. \qquad (21)$$

Replacing t by st yields in (20) gives $ast+stb=0$ for all $s,t \in R$. Subtracting this result from (21), we obtain

$$[s,a]_{\sigma,\tau}t=0. \qquad (22)$$

We replace t by st in the definition of a, we find that

$$a=\sum_{r=1}^{n}\binom{n}{r}st[D^{n-r}(\sigma^{n-r}(x))d^{r}(\tau^{r}(y)),(xy)^{n}]_{\sigma,\tau},$$

Hence, (22) give us

$$\sum_{r=1}^{n}\binom{n}{r}[s,st[D^{n-r}(\sigma^{n-r}(x))d^{r}(\tau^{r}(y)),(xy)^{n}]_{\sigma,\tau},(xy)^{n}]_{\sigma,\tau}]_{\sigma,\tau}t=0.$$

Replacing s with t and setting
$h=\sum_{r=1}^{n}\binom{n}{r}[D^{n-r}(\sigma^{n-r}(x))d^{r}(\tau^{r}(y)),(xy)^{n}]_{\sigma,\tau}$, this relation becomes $t^{2}[t,h]_{\sigma,\tau}t=0$.
Multiplying by $t[t,h]_{\sigma,\tau}$ on the left, we obtain $(t[t,h]_{\sigma,\tau}t)^{2}=0$.

Applying Lemma 4, we obtain $2(t[t,h]_{\sigma,\tau}t) \in Z(R)$ which implies $2[(t[t,h]_{\sigma,\tau}t), r] = 0$ for all $r \in R$. Using the fact R is 2-torsion free. Obviously, $(t[t,h]_{\sigma,\tau}t) \in Z(R)$. According to the fact that the center of semiprime ring contains no non-zero nilpotent elements, we arrive to

$t[t,h]_{\sigma,\tau}t = 0$. Right-multiplying by $[t,h]_{\sigma,\tau}$, we see that $(t[t,h]_{\sigma,\tau})^2 = 0$. Repeating the same technique as before to this result, we find that

$$t[t,h]_{\sigma,\tau} = 0. \tag{23}$$

Multiplying (23) by $[s,r]_{\sigma,\tau}$ on the left, we conclude that $[s,r]_{\sigma,\tau}t[t,h]_{\sigma,\tau}$ $= 0$ for all $s, r \in R$. Using Lemma 6 with replacing s by t and r by h, we find that $[t,h]^2_{\sigma,\tau} = 0$ for all $t \in R$.

Based on Lemma 4 and the fact that the center of semiprime ring contains no non-zero nilpotent elements with R is 2-torsion free, we obtain $[t,h]_{\sigma,\tau} = 0$ for all $t \in R$.

Clearly, we find that $h \in Z(R)$ yields

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), (xy)^n]_{\sigma,\tau} \in Z(R). \tag{24}$$

From (24), we obtain

$$\sum_{r=1}^{n} \binom{n}{r} [D^{n-r}(\sigma^{n-r}(x))d^r(\tau^r(y)), r]_{\sigma,\tau} \in Z(R).$$

Again, in the same manner we find from (19) that

$$[D^n(\sigma^n(x))y, r]_{\sigma,\tau} \in Z(R).$$

Moreover, applying $D^n(\sigma^n(x)) \in Z(R)$ to this commutator, we find that $D^n(\sigma^n(x))[y,r]_{\sigma,\tau} \in Z(R)$.

Then $[D^n(\sigma^n(x))[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$ for all $x, y, r, s \in R$. Since $D^n(\sigma^n(x)) \in Z(R)$, this relation becomes $D^{n-1}(D(\sigma^n(x)))[[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$ for all $x, y, r, s \in R$. Due to D is period $n-1$ of R, we conclude that

$D(\sigma^n(x)))[[y,r]_{\sigma,\tau}, s]_{\sigma,\tau} = 0$. According to D is a nonzero generalized (σ,τ)-derivation

and using the primeness of R, we find that
$[[y,r]_{\sigma,\tau},s]_{\sigma,\tau}=0$. Clearly, this option imply that $[y,r]_{\sigma,\tau}\in Z(R)$. $\square$

We close our paper with the following theorem.

Theorem 3.2. *Let R be a semiprime ring, σ and τ be two automorphisms of R. Suppose D is zero power valued index 2 on R. Then $\prod_{i=0}^{n+1} d^i=0$, for some positive integer n.*

Proof. From the hypothesis, we have $D^2(R)=0$. For all $x,y\in R$, we obtain $D(D(x)\sigma(y)+\tau(x)d(y))=0$. Moreover,

$$D(x)^2\sigma^2(y)+\tau(D(x))d(\sigma(y))+D(\tau(x))\sigma(d(y))+\tau^2(x)d^2(y)=0.$$

Due to the assumption D is zero power valued index 2 on R, this relation reduces to

$$\tau(D(x))d(\sigma(y))+D(\tau(x))\sigma(d(y))+\tau^2(x)d^2(y)=0. \tag{25}$$

Since σ and τ commute with D and d then (25) becomes $D(\tau(x))d(\sigma(y))+D(\tau(x))d(\sigma(y))+\tau^2(x)d^2(y)=0$. Obviously, we find that

$$2D(\tau(x))d(\sigma(y))+\tau^2(x)d^2(y)=0. \tag{26}$$

Replacing x by $D(x)$ and using the facts $D^2=0$ and σ and τ are automorphisms of R. From (26), we find that $D(x)d^2(y)=0$. Replacing x by $xr, r\in R$ in this relation and using the fact σ and τ are automorphisms of R, we conclude that $D(x)rd^2(y)+xd(r)d^2(y)=0$. Replacing r by $d^2(y)$ and using $D(x)d^2(y)=0$, we deduce $xd(d^2(y))d^2(y)=0$.
Left-multiplying by $d(d^2(y))d^2(y)$ and using the semiprimeness of R, we arrive to $d(d^2(y))d^2(y)=0$.
Right-multiplying by $d(y)y$ and left-multiplying by $(d^{n+1}(y)d^n(y)d^{n-1}(y)\cdots)$, we conclude that $\prod_{i=0}^{n+1} d^i(R)=0$. By reason of R is a semiprime. This is the required result. $\square$

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