Octonions with associative property using Geometric Algebra

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Abstract

In this paper, we will use geometric algebra to derive sets of octonions that have associative property (unlike the original ones that cannot keep it). The price to pay is that is not possible to keep all the elements of the diagonal with -1 but at least one of them has to be +1. Also, the anticommutative property is affected.

Here, two sets of octonions that fulfill the associative property:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	-1	b_4	$-b_6$	$-b_2$	b_7	b_3	$-b_5$
b_2	$-b_4$	-1	b_5	b_1	$-b_3$	b_7	$-b_6$
b_3	b_6	$-b_5$	-1	b_7	b_2	$-b_1$	$-b_4$
b_4	b_2	$-b_1$	b_7	-1	b_6	$-b_5$	$-b_3$
b_5	b_7	b_3	$-b_2$	$-b_6$	-1	b_4	$-b_1$
b_6	$-b_3$	b_7	b_1	b_5	$-b_4$	-1	$-b_2$
b_7	$-b_5$	$-b_6$	$-b_4$	$-b_3$	$-b_1$	$-b_2$	1

Fig.4

And

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
c_1	-1	c_4	c_7	$-c_2$	c_6	$-c_5$	$-c_3$
c_2	$-c_4$	-1	$-c_5$	c_1	c_3	$-c_7$	c_6
c_3	c_7	c_5	-1	c_6	$-c_2$	-c4	$-c_1$
c_4	c_2	$-c_1$	$-c_6$	-1	c_7	c_3	$-c_5$
c_5	$-c_6$	$-c_3$	c_2	c_7	-1	c_1	$-c_4$
c_6	c_5	$-c_7$	c4	$-c_3$	$-c_1$	-1	c_2
c_7	$-c_3$	c_6	$-c_1$	$-c_5$	$-c_4$	c_2	1

Fig.6

Keywords

Octonions, associative property, Geometric Algebra

1. Introduction

In this paper, we will use geometric algebra to derive sets of octonions that have associative property (unlike the original ones that cannot keep it). First, we will comment about octonions. Then we will have a little insight in geometric algebra. Finally, we will use the geometric algebra to derive octonions that have associative property.

2. Octonions

The octonions as they are defined, follow the next table of multiplication [1][2]:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Fig.1

The product is anticommutative so the order matters. The product is considering first the octonion in the left column followed by the element in the top row.

This means:

$$e_1e_2 = e_4$$

But:

$$e_2e_1 = -e_4$$

One of the properties of the octonions is that the associative property is also not fulfilled. As an example:

$$(e_1e_2)e_3 = e_4e_3 = -e_6$$

But:

$$e_1(e_2e_3) = e_1e_5 = e_6$$

In the following chapters, we will create octonions that keep the associative property. But there is a price to pay.

You can see in Fig. 1 that the diagonal is filled always with -1. If you want to create octonions with associative property, you need to have at least one +1 in this diagonal. It is not possible to keep all the -1 in the diagonal. Also, somehow the anticommutative property will be affected. We will see now.

3. Geometric Algebra

I will never get tired of recommending this masterpiece by D. Hestenes [3]. There, you can find an insight of what geometric algebra is and all the things that this mathematic framework is capable of. You can find also a lot of documentation regarding geometric algebra in the literature as [4].

To create the octonions with associative property, we will use a basis composed by vectors in Geometric Algebra.

Unlike some of my previous papers [5][6] where I used non-Euclidean metrics, so non orthonormal bases were used, here we will use an orthonormal basis. I will comment the minimum you need to know about an orthonormal basis in geometric algebra, so you can follow the steps in this paper.

We consider an orthonormal basis in three dimensions in geometric algebra. This means we have the following vectors:

$$\{a_1, a_2, a_3\}$$

We will consider a negative signature for the three vectors (normally called $Cl_{0,3}$ Algebra). The Cl stands for Clifford (Geometric) Algebra. The 0,3 stands for 0 basis vectors with positive signature and 3 vectors with negative signature.

This means (as the basis is orthonormal with negative signature):

$$a_1a_1 = {a_1}^2 = -1 \qquad (1)$$
$$a_2a_2 = {a_2}^2 = -1 \qquad (2)$$
$$a_3a_3 = {a_3}^2 = -1 \qquad (3)$$

The product we are using between vectors is called geometric product [3][4], but you do not know its meaning to understand the paper. You just have to follow the properties we will be commenting regarding these vectors.

The next property you need to know s the anticommutative property for these basis vectors:

$$a_1a_2 = -a_2a_1 \quad (4)$$
$$a_2a_3 = -a_3a_2 \quad (5)$$
$$a_3a_1 = -a_1a_3 \quad (6)$$

The geometric product follows the associative and distribute properties:

$$(a_1a_2)a_3 = a_1(a_2a_3)$$

$$a_1(a_2 + a_3) = a_1a_2 + a_1a_3$$

So, summing up, the geometric product for these basis vectors is anticommutative (changes the sign when reversing the order), associative and distributive.

So, how do we operate when we have a product of a lot of different vectors? Let's see the following example. We want to multiply:

$$a_1a_2a_1$$

How do we do it? We want always to have the basis vectors that are the same (in this case a_1) together so we can apply the equations to (1)(2)(3) to simplify the products of the vectors (convert them to -1). And to get there, we use the equations (4)(5)(6) to swap terms.

So, starting in:

$$a_1a_2a_1$$

First, we swap the first a_1 and a_2 using (4):

$$a_1a_2a_1 = (-a_2a_1)a_1 = -a_2a_1a_1$$

Now, we have the two a_1 together so we can use (1):

$$-a_2a_1a_1 = -a_2{a_1}^2 = -a_2(-1) = a_2$$

Let's see another example to settle the process:

$$a_3a_1a_2a_3a_2$$

We need to put the two a_2 together and also the two a_3 together to simplify them. So, first we swap the last a_3a_2 using (5):

$$a_3a_1a_2a_3a_2 = a_3a_1a_2(-a_2a_3) = -a_3a_1a_2a_2a_3$$

Now, we can simplify the two a_2 using (2):

$$-a_3a_1a_2a_2a_3 = -a_3a_1{a_2}^2a_3 = -a_3a_1(-1)a_3 = a_3a_1a_3$$

Now, using (6) we swap a_3 and a_1 (as the geometric product has associative property, it does not matter if you swap the first two terms or the last to terms, check it!). We will swap the first two terms (the result is the same as swapping the last two ones):

$$a_3a_1a_3 = (-a_1a_3)a_3 = -a_1a_3a_3$$

Using (3):

$$-a_1a_3a_3 = -a_1{a_3}^2 = -a_1(-1) = a_1$$

So, summing up, you swap terms until you simplify all the repeated ones. When all the terms are different in the product you cannot simplify more. This means you can finish with a scalar (+1 or -1 if all the elements simplify) or with a_1, a_2, a_3 or a combination (a product) of them (with all the elements different). If any term is repeated, it means you can still simplify more, there are some swapping or simplifications pending.

Ok, so what any of this has to do with octonions? Ok, let's go to the next chapter.

4. Creating a table of octonions using geometric algebra

So, what we will do is to create a table like the following. Once, we have the results, we will change the nomenclature to have an octonion-like table. Let's follow, the process and you will see.

	a_1	a_2	a_3	a_1a_2	a_2a_3	a_3a_1	$a_1a_2a_3$
a_1							
a_2							
a_3							
a_1a_2							
a_2a_3							
a_3a_1							
$a_1a_2a_3$							

Fig.2

So, if we want to fulfill the cell in the first row and first column, we use equation (1)

$$a_1a_1 = {a_1}^2 = -1 \qquad (1)$$

For the cell in the first row and second column, it is directly:

$$a_1a_2$$

In the same first row third column, we would have:

$$a_1a_3$$

But we want the nomenclature to correspond always with the principal row and column (the grey ones including the original elements), so we apply (6) to get:

$$a_1a_3 = \ -a_3a_1$$

So, we have an element that has the same name as the sixth element in the principal (the grey) row but in negative.

If we go for a diagonal one as 5th row, 5th column (using equations (1) to (6):

$$a_2a_3a_2a_3 = -a_3a_2a_2a_3 = -a_3(-1)a_3 = a_3a_3 = -1$$

Let's go for a trickier one. For example, 4th row and 6th column (using equations from (1) to (6)):

$$a_1a_2a_3a_1 = -a_1a_2a_1a_3 = a_1a_1a_2a_3 = (-1)a_2a_3 = -a_2a_3$$

Or, another example 6th row, 7th column:

$$a_3a_1a_1a_2a_3 = a_3(-1)a_2a_3 = -a_3a_2a_3 = a_2a_3a_3 = a_2(-1) = -a_2$$

Ok, last one, because this is special. 7th row and 7th column:

$$a_1a_2a_3a_1a_2a_3 = -a_2a_1a_3a_1a_2a_3 = a_2a_3a_1a_1a_2a_3 = -a_2a_3a_2a_3 = a_3a_2a_2a_3$$
$$= -a_3a_3 = -(-1) = 1$$

We see that we have an element in the diagonal that it is not -1 as it happens with the original octonions, but we have a +1 in one diagonal element. This is one of the prices to pay to be able to have associative symmetry.

If we fulfill the table completely. we have:

	a_1	a_2	a_3	a_1a_2	a_2a_3	a_3a_1	$a_1a_2a_3$
a_1	-1	a_1a_2	$-a_3a_1$	$-a_2$	$a_1a_2a_3$	a_3	$-a_2a_3$
a_2	$-a_1a_2$	-1	a_2a_3	a_1	$-a_3$	$a_1a_2a_3$	$-a_3a_1$
a_3	a_3a_1	$-a_2a_3$	-1	$a_1a_2a_3$	a_2	$-a_1$	$-a_1a_2$
a_1a_2	a_2	$-a_1$	$a_1a_2a_3$	-1	a_3a_1	$-a_2a_3$	$-a_3$
a_2a_3	$a_1a_2a_3$	a_3	$-a_2$	$-a_3a_1$	-1	a_1a_2	$-a_1$
a_3a_1	$-a_3$	$a_1a_2a_3$	a_1	a_2a_3	$-a_1a_2$	-1	$-a_2$
$a_1a_2a_3$	$-a_2a_3$	$-a_3a_1$	$-a_1a_2$	$-a_3$	$-a_1$	$-a_2$	1

Fig.3

Now, we change the nomenclature to a more typical octonion one. We define the following octonions:

$$b_1 = a_1$$
$$b_2 = a_2$$
$$b_3 = a_3$$
$$b_4 = a_1a_2$$
$$b_5 = a_2a_3$$
$$b_6 = a_3a_1$$
$$b_7 = a_1a_2a_3$$

So, we change the nomenclature of the table of Fig.3 to Fig. 4:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	-1	b_4	$-b_6$	$-b_2$	b_7	b_3	$-b_5$
b_2	$-b_4$	-1	b_5	b_1	$-b_3$	b_7	$-b_6$
b_3	b_6	$-b_5$	-1	b_7	b_2	$-b_1$	$-b_4$
b_4	b_2	$-b_1$	b_7	-1	b_6	$-b_5$	$-b_3$
b_5	b_7	b_3	$-b_2$	$-b_6$	-1	b_4	$-b_1$
b_6	$-b_3$	b_7	b_1	b_5	$-b_4$	-1	$-b_2$
b_7	$-b_5$	$-b_6$	$-b_4$	$-b_3$	$-b_1$	$-b_2$	1

Fig.4

You can check that these octonions follow the associative property. You can check examples:

$$b_4(b_2b_3) = b_4b_5 = b_6$$
$$(b_4b_2)b_3 = (-b_1)b_3 = -b_1b_3 = -(-b_6) = b_6$$

Another one:

$$b_5(b_7b_1) = b_5(-b_5) = -b_5b_5 = -(-1) = 1$$
$$(b_5b_7)b_1 = (-b_1)b_1 = -(-1) = 1$$

You see that the associative property is always fulfilled. This comes from the original geometric algebra basis vectors that we also associative. So, the property is inherited.

The distributed property is always assured as the original basis vectors had it also.

The issue comes with the anticommutative property, it is not followed. In fact, the element b_7 which square is +1 instead of -1 is an issue.

The commutative-anticommutative property now is as the following.

If we have (for i≠j):

$$b_ib_j = b_k$$

And all i, j and k are not 7 (none of the elements in the product or the result itself is b_7) then:

$$b_ib_j = -b_jb_i$$

But if i, j or k are equal to 7 (one of the elements in the product or in the result is b_7) then:

$$b_ib_j = b_jb_i$$

This is the second price to have associative property. Remember that the first one was the square of b_7 being +1 instead of -1. And the second one, is that we cannot define this set of octonions as commutative or anticommutative. This depends on if the element b_7 is involved in the product or not (as product element or as result).

5. Creating the most similar table of octonions to the original one, but having associative property

If instead of Fig. 3 we use the following table with this order. We have changed the order of some rows/columns. Also, the name of the column a_3a_1 we have renamed as a_1a_3:

	a_1	a_2	a_2a_3	a_1a_2	a_3	a_1a_3	$a_1a_2a_3$
a_1	-1	a_1a_2	$a_1a_2a_3$	$-a_2$	a_1a_3	$-a_3$	$-a_2a_3$
a_2	$-a_1a_2$	-1	$-a_3$	a_1	a_2a_3	$-a_1a_2a_3$	a_1a_3
a_2a_3	$a_1a_2a_3$	a_3	-1	a_1a_3	$-a_2$	$-a_1a_2$	$-a_1$
a_1a_2	a_2	$-a_1$	$-a_1a_3$	-1	$a_1a_2a_3$	a_2a_3	$-a_3$
a_3	$-a_1a_3$	$-a_2a_3$	a_2	$a_1a_2a_3$	-1	a_1	$-a_1a_2$
a_1a_3	a_3	$-a_1a_2a_3$	a_1a_2	$-a_2a_3$	$-a_1$	-1	a_2
$a_1a_2a_3$	$-a_2a_3$	a_1a_3	$-a_1$	$-a_3$	$-a_1a_2$	a_2	1

Fig.5

We arrive to the following table of octonions in Fig.6:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
c_1	-1	c_4	c_7	$-c_2$	c_6	$-c_5$	$-c_3$
c_2	$-c_4$	-1	$-c_5$	c_1	c_3	$-c_7$	c_6
c_3	c_7	c_5	-1	c_6	$-c_2$	-c4	$-c_1$
c_4	c_2	$-c_1$	$-c_6$	-1	c_7	c_3	$-c_5$
c_5	$-c_6$	$-c_3$	c_2	c_7	-1	c_1	$-c_4$
c_6	c_5	$-c_7$	c4	$-c_3$	$-c_1$	-1	c_2
c_7	$-c_3$	c_6	$-c_1$	$-c_5$	$-c_4$	c_2	1

Fig.6

You can see that it is very similar to the original octonions table (Fig.1) but just some sign changes in the orange cells. This is the most similar I have got into. The difference between Fig.1 and Fig. 6 is that Fig.1 (original octonions) are not associative. Fig 6 is associative but not strictly commutative or anticommutative (see end of chapter 4 regarding this point).

6. Other possible tables changing the signature of the geometric algebra space

As I have commented we have used $Cl_{0,3}$ (0 basis vectors with positive signature and 3 with negative).

If instead of that, we create a space where equations similar to (1)-(3) now are different:

$$a_1 a_1 = {a_1}^2 = +1$$
$$a_2 a_2 = {a_2}^2 = +1$$
$$a_3 a_3 = {a_3}^2 = -1$$

This will be a $Cl_{2,1}$ for example. All the subsequent equations (4)-(6) and alike should be recalculated and will yield different results. We would create a complete different set of tables of octonions using these vectors. The diagonal will have more than one +1 for example.

As commented, it is impossible to have associative property not paying the price of having at least one +1 in the diagonal.

9. Conclusions

Using geometric algebra, we have been able to create set of octonions that keep the associative property. The price to pay is to have at least one +1 in the diagonal (not all -1). Also, the commutative-anticommutative property depends on the elements of the product (see end of chapter 4).

One set of octonions with associative property is:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	-1	b_4	$-b_6$	$-b_2$	b_7	b_3	$-b_5$
b_2	$-b_4$	-1	b_5	b_1	$-b_3$	b_7	$-b_6$
b_3	b_6	$-b_5$	-1	b_7	b_2	$-b_1$	$-b_4$
b_4	b_2	$-b_1$	b_7	-1	b_6	$-b_5$	$-b_3$
b_5	b_7	b_3	$-b_2$	$-b_6$	-1	b_4	$-b_1$
b_6	$-b_3$	b_7	b_1	b_5	$-b_4$	-1	$-b_2$
b_7	$-b_5$	$-b_6$	$-b_4$	$-b_3$	$-b_1$	$-b_2$	1

Fig.4

Another example, very similar to the original octonions table (but with the changes marked in orange) is:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
c_1	-1	c_4	c_7	$-c_2$	c_6	$-c_5$	$-c_3$
c_2	$-c_4$	-1	$-c_5$	c_1	c_3	$-c_7$	c_6
c_3	c_7	c_5	-1	c_6	$-c_2$	-c4	$-c_1$
c_4	c_2	$-c_1$	$-c_6$	-1	c_7	c_3	$-c_5$
c_5	$-c_6$	$-c_3$	c_2	c_7	-1	c_1	$-c_4$
c_6	c_5	$-c_7$	c4	$-c_3$	$-c_1$	-1	c_2
c_7	$-c_3$	c_6	$-c_1$	$-c_5$	$-c_4$	c_2	1

Fig.6

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16. Acknowledgements

AAAAÁBCCCDEEIIILLLLLMMMOOOPSTU

To my family and friends.

If you consider this helpful, do not hesitate to drop your BTC here:

bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f

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