# Limits of sequences on topological groups

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**Abstract**

In this paper, I present a generalization of the key notions of limits of sequences and Cauchy sequences from analysis and metric spaces to more general topological groups. In the second part of this paper, the process of constructing the real numbers from Cauchy sequences of rationals is generalized, allowing us to construct new groups from non-complete topological groups.

## Introduction

The notion of a topological group dates back to 1873–1874. The idea of topological group originated with Sophus Lie, who was concerned with a very important special case of such groups, now known as Lie groups. Topological groups and Lie groups form a important sub-field of mathematics.

In the theory of analysis and metric spaces, sequences play a key role, however, the generalization of sequences to topological spaces is usually achieved using filters and nets. This is because a normal sequences of points in a topological space fail to capture the general topological structure of the space. Nonetheless, its study still has the potential to give group theorists/topologists some tools to help generalize or prove some other results. In this paper we explore some basic constructions and results related to sequences on topological groups. Thought I am not a professional mathematician, these are the first results I found while trying to generalize these central concepts of analysis and metric spaces to the realm of topological groups.

## 1 Preliminaries on topological groups

In this section we review some basic definitions and results of topological groups.

**Definition 1.1 (Topological Group):** A topological group $G$ is a topological space that is also a group such that the group operation:

$$\begin{aligned} \cdot : G \times G &\to G \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

and the inversion map:

$$\begin{aligned} ^{-1} : G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

are continuous functions. Note that here we are considering $G \times G$ as a topological space with the product topology.
Now let's introduce a proposition that will be very useful later in section 2:

**Proposition 1.2:** Let $G$ be a topological group:

1. Let $a, b \in G$ and let $W$ be an open neighbourhood of $a \cdot b$. Then, there are open neighbourhoods $V$ and $U$ of $a$ and $b$ respectively such that $V \cdot U \subseteq W$, where $V \cdot U := \{v \cdot u : v \in V, u \in U\}$.
2. Let $a \in G$ and let $W$ be an open neighbourhood of $a^{-1}$. Then, there is an open neighbourhood $U$ of $a$ such that $U^{-1} \subseteq W$, where $U^{-1} := \left\{u^{-1} : u \in U\right\}$

**Proof:**

1) Let $a, b \in G$ and let $W$ be an open neighbourhood of $ab$. Let $M = \cdot^{-1}(W)$, where $\cdot^{-1}(W)$ denotes the pre-image of $W$ with respect to the group operation, i.e, $\cdot^{-1}(W) := \{(x, y) \in G \times G : x \cdot y \in W\}$ .

Because $M$ is the pre-image of the open set $W$ and because the function $\cdot$ is continuous, $M$ is open in $G \times G$. Because $a \cdot b = ab$, then $(a, b) \in M$. Now let $\mathcal{B} = \{B_i\}_{i \in I}$ be a basis for the topology on $G$, then $\{B_i \times B_j : i, j \in I\}$ is a basis for the product topology in $G \times G$ [1]. Because $M$ is open and $(a, b) \in M$ then

$$(a, b) \in B_i \times B_j \subseteq M$$

for some $i, j \in I$. Note that $B_i$ and $B_j$ are open neighbourhoods of $a$ and $b$ respectively. Now we just need to show that $B_i \cdot B_j \subseteq W$. Let $x \in B_i$ and $y \in B_j$, then $(x, y) \in B_i \times B_j \subseteq M$. By the definition of $M$ we conclude that $x \cdot y \in W$, and thus $B_i \cdot B_j \subseteq W$

2 ) For the sake of simplicity, we shall denote $a^{-1}$ as $\text{inv}(a)$. Let $a \in G$ and let $W$ be an open neighbourhood of $\text{inv}(a)$. Let $U = \text{inv}^{-1}(W)$. Because $W$ is open and inv is continuous, $U$ is an open subset of $G$ and $a \in U$. Now, let $x \in U$. Because $U$ is the pre-image of $W$ with respect to inv, $\text{inv}(x) \in W$. So we can conclude that $\text{inv}(U) \subseteq W$. ■

# 2 Limits of sequences on topological groups

In this section, the basic notions about limits of sequences on topological groups are explicitly stated and some results about them are proven. Among those results, we give special attention to the generalization of Cauchy sequences and complete spaces to the area of topological groups. Later, on section 3, we will use Cauchy sequences to "extend" some topological groups (this notion will be clearly defined and explained then).

Let's start by defining limit of a sequence in a topological space:

**Definition 2.1 (Limit):** Let $G$ be a topological group and let $(x_n)_n$ be a sequence of points in $G$. We say that $(x_n)_n$ converges to $a \in G$ if, for any open neighbourhood $V$ of $a$, we have that:

$$\exists k \in \mathbb{N} : n \geq k \implies x_n \in V$$

If $(x_n)_n$ converges to $a$, we write: $\lim_n x_n = a$, $\lim x_n = a$ or simply $x_n \to a$.

This definition of limit is purely a topological concept, and it's compatible with the already existing definition of limit of a sequence when studying metric spaces, and in particular, $\mathbb{R}$.

Note that in general, the limit of a sequence $(a_n)_n$ may not be unique, for example, let $G$ be a topological group with the trivial topology, this is $\tau = \{\emptyset, G\}$. Then every sequence $(a_n)_n$ on $G$ is convergent and it converges to every point on $G$. To see this, let $g \in G$ be any element. Then there is only one open neighbourhood of $g$, and that neighbourhood is the entire group $G$. Now, note that $n \geq 1 \implies a_n \in G$, so indeed $\lim a_n = g$.

On the next proposition we will see a sufficient condition what will make every limit on $G$ unique:

**Proposition 2.2:** Let $G$ be an Hausdorff topological group. Then, if a sequence $(x_n)_n$ of points in $G$ converges, its limit is unique.

**Proof:** Suppose that $\lim_n x_n = a$ and $\lim_n x_n = b$ with $a \neq b$. Because $G$ is an Hausdorff space, there are open neighbourhoods $A$ and $B$ of $a$ and $b$ respectively such that $A \cap B = \emptyset$. Because $\lim_n x_n = a$ , there is a natural number $k_1$ such that:

$$n \geq k_1 \implies x_n \in A$$

and because $\lim_n x_n = b$ , it exists a natural number $k_2$ such that:

$$n \geq k_2 \implies x_n \in B$$

Let $k = \max\{k_1, k_2\}$ then, if $n \geq k$ we have that $x_n \in A \ \wedge \ x_n \in B \iff x_n \in A \cap B = \emptyset$ which is impossible. Thus it's impossible for $a$ to be different from $b$, meaning that the limit is unique. ■

**Note:** From now on, we will always assume (unless stated otherwise) that any every topological group we are working with is an Hausdorff space as we only want to keep working in spaces where the limit of a sequence is unique.

Now let's see how the notion of limit (which is purely a topological concept) interacts with the group structure of $G$:

**Proposition 2.3:** Let $(a_n)_n$ and $(b_n)_n$ be two convergent sequences on a topological group $G$, then:

1. The sequence $(a_n \cdot b_n)_n$ converges and $\lim(a_n \cdot b_n) = \lim(a_n) \cdot \lim(b_n)$
2. The sequence $(a_n^{-1})_n$ converges and $\lim(a_n^{-1}) = \lim(a_n)^{-1}$

**Proof:**

1) Let $\lim a_n = a$, $\lim b_n = b$ and $W$ be an open neighbourhood of $a \cdot b$. According to proposition 1.2, there are open neighbourhoods $V$ and $U$ of $a$ and $b$, respectively, such that: $V \cdot U \subseteq W$. Using the fact that $(a_n)_n$ converges to $a \in V$ and $(b_n)_n$ to $b \in U$, there are constants $k_1$ and $k_2$ such that: $n \geq k_1 \implies a_n \in V$ and $n \geq k_2 \implies b_n \in U$. Let $k = \max\{k_1, k_2\}$, then

$$n \geq k \implies a_n \in V \ \wedge \ b_n \in U$$

This means that $n \geq k \implies a_n \cdot b_n \in V \cdot U \subseteq W$. Thus we conclude that $\lim(a_n \cdot b_n) = a \cdot b = \lim(a_n) \cdot \lim(b_n)$.

2) Let $\lim a_n = a$ and let $W$ be an open neighbourhood of $a^{-1}$. According to proposition 1.2, there is an open neighbourhood $V$ of $a$ such that $V^{-1} \subseteq W$. Using the fact that $(a_n)_n$ converges to $a \in V$, there is a constant $k$ such that: $n \geq k \implies a_n \in V$, i.e:

$$n \geq k \implies a_n^{-1} \in V^{-1} \subseteq W$$

Thus we conclude that $\lim(a_n^{-1}) = a^{-1} = \lim(a_n)^{-1}$. ■

Throughout the rest of the paper proposition 2.3 will be mentioned in a lot of proofs, so keep it in mind.

The next proposition is also interesting in itself, but its true purpose is to aid us on a proof in the next section.

**Proposition 2.4:** Let $f : G \to H$ be an continuous surjective map, and let $(a_n)_n$ be a convergent sequence on $G$. Then, $(f(a_n))_n$ converges on $H$ and $\lim f(a_n) = f(\lim a_n)$.

**Proof:** Let $a \in G$ be the limit of the sequence $(a_n)_n$ in $G$ and $V$ an open neighbourhood of $f(a)$ in $H$. Then $f^{-1}(V)$ is an open neighbourhood of $a$ in $G$. This means that $\exists N \in \mathbb{N} : n \geq N \implies a_n \in f^{-1}(V) \implies f(a_n) \in f(f^{-1}(V)) \implies f(a_n) \in V$ [2]. So, $f(a_n)$ converges to $f(a) = f(\lim a_n)$ in $H$. ■

Now we are ready to generalize and introduce the notion of Cauchy sequences on topological groups:

**Definition 2.5:** Let $(a_n)_n$ be a sequence of points in a topological group $G$. We say that $(a_n)_n$ is a Cauchy sequence if, for every open neighbourhood $V$ of $e$ (here $e$ denotes the identity element of $G$), there is a constant $k \in \mathbb{N}$ such that:

$$n, m \geq k \implies a_n \cdot a_m^{-1} \in V \ \vee a_n^{-1} \cdot a_m \in V$$

Note that if the topological group is abelian, then the condition above just becomes:

$$n, m \geq k \implies a_n \cdot a_m^{-1} \in V$$

This notion of Cauchy sequence is compatible with the one we had when working with $\mathbb{R}^n$ and $\mathbb{Q}$.

Now, we shall see (exactly like what happens in every metric space), that every converging sequence is a Cauchy sequence as well.

**Proposition 2.6:** Let $(a_n)_n$ be a convergent sequence in a topological group $G$. Then $(a_n)_n$ is a Cauchy sequence.

**Proof:** Let $\lim a_n = a \in G$ and let $V$ be an open neighbourhood of $e$. According to proposition 2.3, the sequence $(a_n^{-1})$ converges to $a^{-1}$ and because $a \cdot a^{-1} = e$, there are open neighbourhoods $W_1$ and $W_2$ of $a$ and $a^{-1}$ such that $W_1 \cdot W_2 \subseteq V$ (proposition 1.2).

Now, let $N_1, N_2 \in \mathbb{N}$ be such that:

$$n \geq N_1 \implies a_n \in W_1$$

$$n \geq N_2 \implies a_n^{-1} \in W_2$$

and let $N = \max\{N_1, N_2\}$. Then:

$$m \geq n \geq N \implies a_n \in W_1 \ \wedge a_m^{-1} \in W_2 \iff a_n \cdot a_m^{-1} \in W_1 \cdot W_2 \subseteq V \ \blacksquare$$

With this in mind, the most natural thing to do next is to study the convergence of Cauchy sequences on a general Hausdorff topological group $G$. As we will see, sometimes, a sequence being Cauchy does not imply that it converges. For example, consider the following sequence on $(\mathbb{Q}, +)$ : 3, 3.1, 3.14, 3.141, 3.1415, ... This sequence (whose terms form the decimal expansion of $\pi$) is a Cauchy sequence but obviously it does not converge on $\mathbb{Q}$. This notion will play a key role in the definition of the Cauchy extension of a topological group in the next section. But first, we will prove the following proposition, which is very similar in structure to proposition 2.4.

**Proposition 2.7:** Let $G$ and $H$ be topological groups and $f : G \to H$ be a continuous surjective homomorphism between them.

If $(a_n)_n$ is a Cauchy sequence on $G$, then $(f(a_n))_n$ is also a Cauchy sequence on $H$.

**Proof:** Let's assume that for every open neighbourhood $V$ of $e_G$, we have that $\exists k \in \mathbb{N} :$ $n, m \geq k \implies a_n \cdot a_m^{-1} \in V$ (in the case where $a_n^{-1} \cdot a_m \in V$ the proof is analogous). Let $L$ be an open neighbourhood of $e_H$ in $H$. Because $f$ is continuous, $f^{-1}(L)$ is an open neighbourhood of $e_G$ in $G$, which means that: $\exists k \in \mathbb{N} : n, m \geq k \implies a_n \cdot a_m^{-1} \in f^{-1}(L)$. But this implies that $f(a_n \cdot a_m^{-1}) \in f(f^{-1}(L)) \iff f(a_n) \cdot f(a_m)^{-1} \in L$ [2]. So, for any open neighbourhood $L$ of $e_H$, there is some $k \in \mathbb{N}$ such that:

$$m, n \geq k \implies f(a_n) \cdot f(a_m)^{-1} \in L$$

and therefore $(f(a_n))_n$ is a Cauchy sequence on $H$. $\blacksquare$

Now we are ready to define what a complete topological group is using Cauchy sequences:

**Definition 2.8:** Let $G$ be a topological group. We say that $G$ is complete if every Cauchy sequence converges.

Now, on the next propositions, we will see how this notion of a complete topological group interacts with the group and topological structure of $G$.

**Proposition 2.9:** Let $G$ be a topological group and let $F \leq G$ be a closed subgroup with the usual subspace topology induced by $G$, then:

1. If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence on $G$ such that, for all $n$, $a_n \in F$, then its limit is also a point of $F$.

2. if $G$ is complete, then $F$ is also complete.

**Proof:**

1. Let $a$ be the limit of $(a_n)_{n\in\mathbb{N}}$ and suppose that $a \in G \setminus F$. Because $F$ is closed, $G \setminus F$ is an open neighbourhood of $a$. Thus, according to the definition of convergence:

$$\exists N \in \mathbb{N} : n \geq N \implies a_n \in G \setminus F \iff a_n \notin F$$

which is a contradiction, so $a \in F$.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of points in $F$. Let's start by proving that $(a_n)_n$ is also a Cauchy sequence on $G$. Let's assume that for every open neighbourhood $V$ of $e$ in $F$, we have that $\exists k \in \mathbb{N} : n, m \geq k \implies a_n \cdot a_m^{-1} \in V$ (in the case where $a_n^{-1} \cdot a_m \in V$ the proof is analogous). Let $W$ be any open neighbourhood of $e$ in $G$. Then $W \cap F$ is an open neighbourhood of $e$ in $F$ and thus:

$$\exists k \in \mathbb{N} : n, m \geq k \implies a_n \cdot a_m^{-1} \in W \cap F \implies a_n \cdot a_m^{-1} \in W$$

This leads to $(a_n)_n$ being Cauchy on $G$. As $G$ is complete, this sequence converges and, according to the first item in this proposition, its limit is in $F$. Thus, the sequence converges in $F$, meaning that every Cauchy sequence in $F$ converges. ■

Now we will see a way of constructing new complete topological groups by considering the cartesian product of complete topological groups. We'll start by proving a proposition that will be very useful in doing so.

**Proposition 2.10:** Let $\{G_i\}_{i=1}^k$ be a collection of topological groups and let $(a_n^i)_n$ be a sequence on $G_i$. If each $(a_n^i)_n$ converges on $G_i$, then sequence $(a_n^1, ..., a_n^k)_n$ in $G_1 \times ... \times G_k$ converges and, in that case, we have that $\lim(a_n^1, ..., a_n^k) = (\lim a_n^1, ..., \lim a_n^k)$.

**Proof:** Let's start by assuming that each $(a_n^i)_n$ converges on $G_i$ and let its limit be $a_i$. Let $b_n = (a_n^1, ..., a_n^k)$. Our goal is to show that $\lim b_n = (a_1, ..., a_k)$.

Let $W$ be an open neighbourhood of $(a_1, ..., a_k)$ and let $\mathcal{B}^i = \left\{B_j^i\right\}_{j\in J_i}$ be a basis for the topology on $G_i$. Then, $\mathcal{M} := \left\{B^1 \times ... \times B^k : B^i \in \mathcal{B}^i\right\}$ is a basis for the product topology on $G_1 \times ... \times G_k$ [1]. Because $W$ is open, there is some $L \in \mathcal{M}$ such that:

$$(a_1, ..., a_k) \in L \subseteq W$$

$L$ can be written as $B_{j_1}^1 \times ... \times B_{j_k}^k$ for some $j_i \in J_i$. Note that $B_{j_i}^i$ is an open neighbourhood of $a_i$, and so, for all $i$:

$$\exists N_i \in \mathbb{N} : n \geq N_i \implies a_n^i \in B_{j_i}^i$$

Let $N = \max_i \{N_i\}$. Then,

$$n \geq N \implies a_n^i \in B_{j_i}^i, \ \forall i \in \{1, ..., k\}$$

,and $a_n^i \in B_{j_i}^i, \ \forall i \in \{1, ..., k\} \iff (a_n^1, ..., a_n^k) \in B_{j_1}^1 \times ... \times B_{j_k}^k \iff b_n \in L$. So,

$$n \geq N \implies b_n \in L \subseteq W$$

and thus $\lim b_n = (a_1, ..., a_k) = (\lim a_n^i, ..., \lim a_n^k)$. ■

**Proposition 2.11:** Let $G_1, ..., G_k$ be topological groups and $G = \prod_{j=1}^k G_j$. Then $G$ is complete if and only if $G_i$ is complete, for all $i = 1, ..., k$.

**Proof:**

1. We shall begin by proving that if $G_i$ is complete, for all $i = 1, ..., k$, then $G$ is complete:

Let $(b_n)_n$ be a Cauchy sequence on $G$. We can write each term of the sequence as: $b_n = \left(a_n^1, ..., a_n^k\right)$, where $(a_n^i)_n$ is a sequence on $G_i$, for any $i \in \{1, ..., k\}$.

Let's start by proving that for any $i \in \{1, ..., k\}$, the sequence $(a_n^i)_n$ is a Cauchy sequence on $G_i$:

Let $i \in \{1, ..., k\}$ and let $V_i$ be any open neighbourhood of $e_i$ in $G_i$, where $e_i$ is the identity element of $G_i$ . Then the set $\prod_{j=1}^k V_j \subseteq G$ is also open, and thus an open neighbourhood of $\bar{e} := (e_1, ..., e_k)$, which is the identity element on $G$. Because $(b_n)_n$ is a Cauchy sequence:

$$\exists N \in \mathbb{N} : n, m \geq N \implies b_n \cdot b_m^{-1} \in \prod_{j=1}^k V_j$$

But note that $b_n \cdot b_m^{-1} = \left(a_n^1 \cdot \left(a_m^1\right)^{-1}, ..., a_n^i \cdot \left(a_m^i\right)^{-1}, ..., a_n^k \cdot \left(a_m^k\right)^{-1}\right) \in \prod_{j=1}^k V_j \implies a_n^i \cdot \left(a_m^i\right)^{-1} \in V_i$. Thus we conclude that $\exists N \in \mathbb{N} : n, m \geq N \implies a_n^i \cdot \left(a_n^i\right)^{-1} \in V_i$, for all $i \in \{1, ..., k\}$, meaning that the sequence $(a_n^i)_n$ is a Cauchy sequence on $G_i$.

Because $G_i$ is complete, for any $i \in \{1, ..., k\}$, the sequence $(a_n^i)_n$ converges on $G_i$. Let $\lim a_n^i = a_i \in G_i$. According to proposition 2.10, $b_n$ converges and its limit is $(\lim a_n^1, ..., \lim a_n^k) = (a_1, ..., a_k)$.

2. Now let's prove that if $G$ is complete, then $G_i$ is complete, for all $i = 1, ..., k$:

For this, I will assume that $i = 1$ just for the sake of simplicity. However, the proof for an arbitrary $i$ is done in the exact same way.

Let $(a_n)_n$ be an Cauchy sequence on $G_1$ and consider the sequence $(a_n, e_2, ..., e_k)_n$ on $G$. We will start by showing that $(a_n, e_2, ..., e_k)_n$ is a Cauchy sequence on $G$: Let $V$ be an open neighbourhood of $\bar{e} = (e_1, ..., e_k)$. Because $V$ is open, using the notation we used to prove the previous item, there exists a set $L \in \mathcal{M}$ such that:

$$\bar{e} \in L \subseteq V$$

and $L$ can be written as $B^1_{j_1} \times ... \times B^k_{j_k}$ for some $j_i \in J_i$. Note now that each $B^i_{j_i}$ is an open neighbourhood of $e_i$ and in particular $B^1_{j_1}$ is an open neighbourhood of $e_1$ in $G_1$. Because $(a_n)_n$ is a Cauchy sequence on $G_1$ we conclude that there exists an $N \in \mathbb{N}$ such that:

$$n, m \geq N \implies a_n \cdot a_m^{-1} \in B^1_{j_1}$$

And so we can conclude that:

$$n, m \geq N \implies (a_n, e_2, ..., e_k) \cdot (a_m, e_2, ..., e_k)^{-1} \in B^1_{j_1} \times ... \times B^k_{j_k} \subseteq V$$

So $(a_n, e_2, ..., e_k)$ is indeed a Cauchy sequence, and this allows us to conclude that, because $G$ is complete, that $\lim(a_n, e_2, ..., e_k)$ exists.

Let $(a_n, e_2, ..., e_k) = (a_1, ..., a_k)$ and $K$ be an open neighbourhood of $a_1$. Then, the set

$$K \times \prod_{j=2}^{k} G_j$$

is an open neighbourhood of $(a_1, ..., a_k)$. Therefore

$$\exists N \in \mathbb{N} : n \geq N \implies (a_n, e_2, ..., e_k) \in K \times \prod_{j=2}^{k} G_j \implies a_n \in K$$

And thus, $\lim a_n = a_1$, meaning that $G_1$ is complete. ■

The next lemma is purely in the area of topology. We will use it in the proof of proposition 2.13.

**Lemma 2.12:** Let $X$ be a finite Hausdorff topological space. Then $X$ is a discrete space.

**Proof:** Let $X = \{x_1, ..., x_n\}$. Let's start by proving that each $\{x_i\}$ is closed. Let $x_i$ be fixed and let $x_j \in X$ be any other element of $X$. Because $X$ is Hausdorff, there are open neighbourhoods of $x_i$ and $x_j$, $V_i$ and $V_j$ such that $V_i \cap V_j = \emptyset$. In particular, $x_i \notin V_j$ for all $j \neq i$. Now we only have to prove that $X \setminus \{x_i\}$ is open. We can do this be verifying that:

$$X \setminus \{x_i\} = \bigcup_{j \neq i} V_j$$

Let $x \in X \setminus \{x_i\}$,then $x = x_j$ for some $j$ and thus $x \in V_j \subseteq \bigcup_{j \neq i} V_j$. Now let $x \in \bigcup_{j \neq i} V_j$, then $x \in V_j$ for some $j$. Because $x_i \notin V_j$ then $x \neq x_i$ and thus $x \in X \setminus \{x_i\}$.

Because each $V_j$ is open, then $X \setminus \{x_i\} = \bigcup_{j \neq i} V_j$ is an open set, and thus $\{x_i\}$ is closed, for all $i = 1, ..., n$.

Now, let $V \subseteq X$. It is known that:

$$X \setminus V = \bigcup_{x \in X \setminus V} \{x\}$$

And because we are dealing with a finite union of closed sets, the set $X \setminus V$ is closed, and thus the set $V$ is open. Because $V$ is arbitrary, we conclude that any subset is open. Therefore the space is discrete. ■

**Proposition 2.13:** If $G$ is a finite topological group, then $G$ is complete.

**Proof:** Let $(a_n)_n$ be a Cauchy sequence on $G$. According to lemma 2.12, $G$ is a discrete space and hence the set $\{e\}$ is an open neighbourhood of the identity element. Because $(a_n)_n$ is a Cauchy sequence:

$$\exists N \in \mathbb{N} : n, m \geq N \implies a_n \cdot a_m^{-1} \in \{e\} \iff a_n \cdot a_m^{-1} = e$$

Now, fix $n = N$. It follows that

$$m \geq N \implies a_N \cdot a_m^{-1} = e \iff a_m = a_N$$

Thus, if $m \geq N$, the sequence stays constant. Now, let $V$ be an open neighbourhood of $a_N$:

$$m \geq N \implies a_m = a_N \in V$$

Thus, according to the definition of limit, $\lim a_n = a_N$ and thus every Cauchy sequence converges. ■

Note that, on this proof, we never actually used the fact that $G$ is finite, we just used that fact that it was a discrete space, thus we can generalize this as follows:

**Proposition 2.14:** Let $G$ be a discrete topological group. Then, $G$ is complete.

**Proof:** The proof is analogous to the one in proposition 2.13. ■

**Proposition 2.15:** Let $G$ be a topological group and let $H$ be an open subgroup of $G$. Then $H$ is also closed.

**Proof:** We will start by partitioning the group $G$ into left cosets of $H$

$$G = \bigcup_{i \in I} x_i H$$

for some representatives of the cosets $\{x_i\}_{i \in I}$. Note that, for some $i_0 \in I$, we have that $x_{i_0} H = H$, so we have the following decomposition of $G$:

$$G = H \cup \left( \bigcup_{i \in J} x_i H \right)$$

With $J = I \setminus \{i_0\}$. Using the fact that, for any $g \in G$, the map $f : G \to G$, $x \mapsto g \cdot x$ is a homeomorphism [3], we can conclude that every $x_i H$ is an open set (because the set $H$ is in itself open) meaning that $\bigcup_{i \in J} x_i H$ is an open set. Now, note that $x_i H \cap x_j H = \emptyset$, for $i \neq j$, so this means that $H \cap \left( \bigcup_{i \in J} x_i H \right) = \emptyset$. With this in mind we can write $G \setminus H$ as:

$$G \setminus H = \bigcup_{i \in J} x_i H$$

Therefore, $G \setminus H$ is indeed open and thus $H$ is also a closed subgroup. ■

This proposition allows us to conclude the following interesting corollaries:

**Corollary 2.16:** Let $G$ be a connected topological group. Then $G$ has no proper open subgroups.

**Proof:** Let's assume that $H$ is a proper open subgroup of $G$. Then $H$ is also closed, according to proposition 2.15. But, because $G$ is connected, the only clopen sets are $G$ and $\emptyset$ [4]. The empty set is not a subgroup of $G$, so we conclude that $H = G$, which is a contradiction since $H$ is a proper subgroup of $G$. ■

**Corollary 2.17:** Let $G$ be a complete topological group. Then, any open subgroup of $G$ is also complete.

**Proof:** Let $H$ be an open subgroup of $G$. According to proposition 2.15, $H$ is closed and, according to proposition 2.9, $H$ being a closed subgroup of a complete group implies that $H$ itself is complete. ■

We will now see how different topological groups interact with the notion of completeness. In particular, we will see how completeness is preserved under homeomorphic isomorphisms:

**Proposition 2.18:** Let $G$ and $H$ be two homeomorphic and isomorphic topological groups. If $G$ is complete, then $H$ is complete.

**Proof:** Let $f : G \to H$ be a homeomorphic isomorphism , i.e $f$ is a group isomorphism and a homeomorphism, and let $(a_n)_n$ be a Cauchy sequence on $H$. Our main goal of the proof is:

1. Proving that $\left(f^{-1}(a_n)\right)_n$ is a Cauchy sequence on $G$ and thus a convergent sequence;
2. Proving that $\lim a_n = f(a)$, where $a = \lim f^{-1}(a_n)$.

1- Let $V$ be an open neighbourhood of $e_G$ in $G$, where $e_G$ is the identity element in $G$. Then $f(V)$ is an open neighbourhood of $e_H$ in $H$, where $e_H$ is the identity element in $H$. Because $(a_n)_n$ is a Cauchy sequence on $H$, there exists a $N \in \mathbb{N}$ such that:

$$n, m \geq N \implies a_n \cdot a_m^{-1} \in f(V) \implies f^{-1}(a_n \cdot a_m^{-1}) \in f^{-1}(f(V))$$

Because $f$ is a group isomorphism, then $f^{-1}$ is also an isomorphism, so have that: $f^{-1}(a_n \cdot a_m^{-1}) = f^{-1}(a_n) \cdot f^{-1}(a_m)^{-1}$ and because $f$ is bijective, $f^{-1}(f(V)) = V$. So

$$n, m \geq N \implies f^{-1}(a_n) \cdot f^{-1}(a_m)^{-1} \in V$$

,and thus $\left(f^{-1}(a_n)\right)_n$ is a Cauchy sequence on $G$ .

2- Because $G$ is complete, the sequence $\left(f^{-1}(a_n)\right)_n$ converges. Let $a = \lim f^{-1}(a_n)$ and $V$ be any open neighbourhood of $f(a)$ in $H$. Because $f$ is continuous, $f^{-1}(V)$ is an open neighbourhood of $a$. Using the fact that $\lim f^{-1}(a_n)$ exists, we know that there is a $N \in \mathbb{N}$ such that:

$$n \geq N \implies f^{-1}(a_n) \in f^{-1}(V) \implies f(f^{-1}(a_n)) \in f(f^{-1}(V))$$

$f(f^{-1}(a_n))$ is simply $a_n$ and $f(f^{-1}(V)) = V$, because $f$ is bijective. So we conclude that:

$$\exists N \in \mathbb{N} : n \geq N \implies a_n \in V$$

Therefore, $\lim a_n = f(a)$ and the topological group $H$ is complete. ■

In this part of the section, we will see a generalization of the previous result:

**Proposition 2.19:** Let $G$ and $H$ be two homeomorphic Hausdorff topological spaces. If there is a binary operation on $G$ that makes $G$ a complete topological group, then there is one binary operation that turns $H$ into a complete topological group.

**Proof:** Let $f : G \to H$ be an homeomorphism between these two topological spaces, let $\cdot : G \times G \to G$ be a binary operation such that $(G, \cdot)$ is a complete topological group and let $\otimes : H \times H \to H$ be a binary operation on $H$ defined as:

$$a \otimes b = f(f^{-1}(a) \cdot f^{-1}(b))$$

We will start by verifying that $(H, \otimes)$ is indeed a group:

1. $(H, \otimes)$ has an identity element:

Let $a \in H$ and $e$ be the identity element on $G$. Then

$$f(e) \otimes a = f(e \cdot f^{-1}(a)) = f(f^{-1}(a)) = a$$

We can prove analogously that $a \otimes f(e) = a$.
So we have that $f(e)$ is the identity element on $(H, \otimes)$.

2. Every element of $(H, \otimes)$ has an inverse:

Let $a \in H$. Because $f$ is bijective, there is a unique $b \in G$ such that $a = f(b)$. Now:

$$a \otimes f(b^{-1}) = f(b) \otimes f(b^{-1}) = f(b \cdot b^{-1}) = f(e)$$

we can prove in the exact same way that $f(b^{-1}) \otimes a = f(e)$. So the inverse of any element $a$ $(= f(b))$ is a $a^{-1\otimes} = f(b^{-1})$. (I will use $a^{-1\otimes}$ to denote inversion on $(H, \otimes)$ )

3. $\otimes$ is associative:

Let $a, b, c \in H$. Then, because $f$ is bijective, there are unique $a', b', c' \in G$ such that $a = f(a'), b = f(b')$ and $c = f(c')$. Now:

$$\begin{aligned}(a \otimes b) \otimes c &= (f(a') \otimes f(b')) \otimes f(c') = (f(a' \cdot b')) \otimes f(c') = f((a' \cdot b') \cdot c')) \\ &= f(a' \cdot (b' \cdot c')) = f(a') \otimes (f(b' \cdot c')) = f(a') \otimes (f(b') \otimes f(c')) = a \otimes (b \otimes c)\end{aligned}$$

So we have that $(H, \otimes)$ is indeed a group.

Note that, because $\otimes$ is the composition of continuous maps, it is continuous. As we denoted in (2.), the inversion map can be defined as: ${}^{-1_\otimes} : H \to H$ such that $a^{-1_\otimes} = f(f^{-1}(a)^{-1})$ , which is also a composition of continuous maps, and therefore the inversion map on $H$ is also continuous. So we have that $(H, \otimes)$ is a topological group.

Let $a, b \in G$ be any two elements. Note that

$$f(a \cdot b) = f(f^{-1}(f(a)) \cdot f^{-1}(f(b))) = f(a) \otimes f(b)$$

So $f$ is not only an homeomorphism, but also an isomorphism between the groups $(G, \cdot)$ and $(H, \otimes)$.

Thus we have two topological groups that are homeomorphic and isomorphic to one another and $(G, \cdot)$ is complete and so, according to Proposition 2.18, $(H, \otimes)$ is complete as well. ■

In this section, we defined what the limit of a sequence on a topological group is, which is purely a topological concept and only depends on the topological structure of the space.

We saw that this topological concept seems very compatible with the group structure of the space, for example (proposition 2.3). Afterwards, we generalized the concept of a Cauchy sequence, using not only the topological but also the group structure of the space and, with that, we were able to generalize the concept of a complete space that is so important in the study of metric spaces.

In the next chapter, we will see how we can use these concepts to construct new groups from non-complete topological groups- for example, how we can build $\mathbb{R}$ from $\mathbb{Q}$.

# 3 Cauchy extension of topological groups

One very important result in constructive mathematics is the construction of real numbers. There are various ways of constructing the real numbers but the two most straightforward and simple ones are Dedekind cuts and the construction from Cauchy sequences of rational numbers, this last one usually credited to Georg Cantor. In this section, we will see how we can use the generalization we made in the previous section of Cauchy sequences to generalize this last construction even further.

On top of assuming that all topological groups we are working with are Hausdorff (which we were doing since the beginning of the second section), only in this section of the paper we will also assume that every topological group we are working with is Abelian.

Let's start by defining some key concepts that we will use:

**Definition 3.1:** Let $G$ be a topological group, then we will define:

1. $\mathrm{Seq}(G)$ as the set of all Cauchy sequences on $G$;
2. $\sim$ as the equivalence relation on $\mathrm{Seq}(G)$ defined as:

$$(a_n)_n \sim (b_n)_n \iff \lim(a_n \cdot b_n^{-1}) = e$$

Let's start by proving that $\sim$ is actually an equivalence relation on $\mathrm{Seq}(G)$:

**Proof:**

1. Let $(a_n)_n \in \mathrm{Seq}(G)$, then $\lim(a_n \cdot a_n^{-1}) = \lim(e) = e$, so we have that $(a_n)_n \sim (a_n)_n$ ($\sim$ is reflexive)
2. Let $(a_n)_n$, $(b_n)_n \in \mathrm{Seq}(G)$ such that $(a_n)_n \sim (b_n)_n$, then $\lim(b_n \cdot a_n^{-1}) = \lim\big((a_n \cdot b_n^{-1})^{-1}\big)$. Because of Proposition 2.3, this is just $\big(\lim a_n \cdot b_n^{-1}\big)^{-1} = e^{-1} = e$, so we conclude that $(b_n)_n \sim (a_n)_n$ ($\sim$ is symmetric )
3. Now, let $(a_n)_n, (b_n)_n, (c_n)_n \in \mathrm{Seq}(G)$ such that: $(a_n)_n \sim (b_n)_n$ and $(b_n)_n \sim (c_n)_n$. Note that $\lim(a_n \cdot c_n^{-1}) = \lim((a_n \cdot b_n^{-1}) \cdot (b_n \cdot c_n^{-1}))$ and, by Proposition 2.3 this is equal to $\lim(a_n \cdot b_n^{-1}) \cdot \lim(b_n \cdot c_n^{-1}) = e \cdot e = e$. So we have that $(a_n)_n \sim (c_n)_n$ (so $\sim$ is transitive) ■

This next lemma simply states that the group operations are compatible with the notion of Cauchy sequences. This lemma will be very useful on justifying some constructions we will do in further propositions and definitions.

**Lemma 3.2:** Let $G$ be a topological group, and let $(a_n)_{n\in\mathbb{N}}$,$(b_n)_{n\in\mathbb{N}}$ be two Cauchy sequences on $G$, then:

1. $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence
2. $(a_n^{-1})_{n\in\mathbb{N}}$ is a Cauchy sequence

**Proof:**

1. Let $V$ be an open neighbourhood of $e$ and $n, m \in \mathbb{N}$. Firstly, note that:

$$a_n \cdot b_n \cdot (a_m \cdot b_m)^{-1} = \big(a_n \cdot a_m^{-1}\big) \cdot \big(b_n \cdot b_m^{-1}\big)$$

and that $e \cdot e$ is an element of $V$ so, according to proposition 1.2, there are open neighbourhoods $U$ and $W$ of $e$ such that:

$$U \cdot W \subseteq V$$

Because $(a_n)_n$ and $(b_n)_n$ are both Cauchy sequences, there are some constants $N_1, N_2 \in \mathbb{N}$ such that:

$$n, m \geq N_1 \implies a_n \cdot a_m^{-1} \in U$$

$$n, m \geq N_2 \implies b_n \cdot b_m^{-1} \in W$$

Now, let $N = \max\{N_1, N_2\}$. Then:

$$n, m \geq N \implies a_n \cdot a_m^{-1} \in U \ \wedge \ b_n \cdot b_m^{-1} \in W \implies \left(a_n \cdot a_m^{-1}\right) \cdot \left(b_n \cdot b_m^{-1}\right) \in V$$

Since $(a_n \cdot b_n) \cdot (a_m \cdot b_m)^{-1} = \left(a_n \cdot a_m^{-1}\right) \cdot \left(b_n \cdot b_m^{-1}\right)$, we conclude that

$$\exists N \in \mathbb{N} : n, m \geq N \implies (a_n \cdot b_n) \cdot (a_m \cdot b_m)^{-1} \in V$$

Thus $(a_n \cdot b_n)_{n \in \mathbb{N}}$ is a Cauchy sequence on $G$.

2. Let $V$ be an open neighbourhood of $e$. Because $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence:

$$\exists N \in \mathbb{N} : n, m \geq N \implies a_n \cdot a_m^{-1} \in V \iff (a_n^{-1})^{-1} \cdot (a_m)^{-1} \in V$$

So $(a_n^{-1})_n$ is indeed Cauchy. ■

Now we are ready to define how we can generalize Cantor's construction of the real numbers to any general abelian Hausdorff topological group $G$:

**Definition 3.3:** Let $G$ be a topological group. We define the Cauchy extension, or simply the extension of $G$, denoted as $\mathrm{ext}(G)$ as the set of all equivalence classes of $\mathrm{Seq}(G)$ with respect to the equivalence relation $\sim$, as defined earlier. In other words: $\mathrm{ext}(G) := \mathrm{Seq}(G)/\sim$.

Before, we simply constructed the set $\mathrm{ext}(G)$. Now, in this next proposition, we shall give it a natural group structure induced by the group operation present in $G$.

**Proposition 3.4:** Let $(G, \cdot)$ be a topological group. Then $\mathrm{ext}(G)$ together with the following binary operation:

$$\begin{gathered} \otimes : \mathrm{ext}(G) \times \mathrm{ext}(G) \to \mathrm{ext}(G) \\ ([(a_n)_n], [(b_n)_n]) \mapsto [(a_n \cdot b_n)_n] \end{gathered}$$

is an abelian group. Here, $[(a_n)_n]$ denotes the equivalence class of $(a_n)_n$. Note that because of lemma 3.2, the sequence $(a_n \cdot b_n)_n$ is also Cauchy so it makes sense to talk about its equivalence class.

**Proof:** First, we need to verify that the operation is well defined. Let $[(a_n)_n], [(b_n)_n] \in \mathrm{ext}(G)$ and let $(c_n)_n$ and $(d_n)_n$ be other representatives of the equivalence classes $[(a_n)_n], [(b_n)_n]$. Our goal is to show the following:

$$[(a_n)_n] \otimes [(b_n)_n] = [(c_n)_n] \otimes [(d_n)_n] \iff [(a_n \cdot b_n)_n] = [(c_n \cdot d_n)_n]$$

Because we are working with equivalence classes, it suffices to show that $(a_n \cdot b_n)_n \sim (c_n \cdot d_n)_n$. Consider the following limit: $\lim((a_n \cdot b_n) \cdot (c_n \cdot d_n)^{-1})$. Using the fact that $G$ is Abelian, we can rearrange this as follows: $\lim((a_n \cdot c_n^{-1}) \cdot (b_n \cdot d_n^{-1}))$ and, according to proposition 2.3, this is: $\lim(a_n \cdot c_n^{-1}) \cdot \lim(b_n \cdot d_n^{-1}) = e \cdot e = e$. So, $(a_n \cdot b_n)_n \sim (c_n \cdot d_n)_n \iff [(a_n \cdot b_n)_n] = [(c_n \cdot d_n)_n] \iff [(a_n)_n] \otimes [(b_n)_n] = [(c_n)_n] \otimes [(d_n)_n]$, therefore the operation is well defined.

Now, let's show that the set together with the operation satisfies the group axioms:

1. Let $(e_n)_n$ denote the constant sequence $e_n = e\,, \forall n \in \mathbb{N}$ and let $[(a_n)_n] \in \text{ext}(G)$ . Then:

$$[(e_n)_n] \otimes [(a_n)_n] = [(e_n \cdot a_n)_n] = [(a_n)_n] = [(a_n \cdot e_n)_n] = [(a_n)_n] \otimes [(e_n)_n]$$

So $[(e_n)_n]$ is the identity element.

2. Let $[(a_n)_n] \in \text{ext}(G)$. According to lemma 3.2, $(a_n^{-1})_n$ is a Cauchy sequence, so it makes sense to consider the equivalence class of $(a_n^{-1})_n$. Then

$$[(a_n)_n] \otimes [(a_n^{-1})_n] = [(a_n \cdot a_n^{-1})_n] = [(e_n)_n] = [(a_n^{-1} \cdot a_n)_n] = [(a_n^{-1})_n] \otimes [(a_n)_n]$$

So $[(a_n^{-1})_n]$ is the inverse of $[(a_n)_n]$.

3. Let $[(a_n)_n], [(b_n)_n], [(c_n)_n] \in \text{ext}(G)$. Then:

$$([(a_n)_n] \otimes [(b_n)_n]) \otimes [(c_n)_n] = [((a_n \cdot b_n) \cdot c_n)_n] = [(a_n \cdot (b_n \cdot c_n))_n] = [(a_n)_n] \otimes ([(b_n)_n] \otimes [(c_n)_n])$$

4. Finally, let $[(a_n)_n], [(b_n)_n] \in \text{ext}(G)$. The only thing left to check is that $(\text{ext}(G), \otimes)$ is Abelian:

$$[(a_n)_n] \otimes [(b_n)_n] = [(a_n \cdot b_n)_n] = [(b_n \cdot a_n)_n] = [(b_n)_n] \otimes [(a_n)_n]$$

So we conclude that $(\text{ext}(G), \otimes)$ is an Abelian group. ■

Now that we have a group structure to $\text{ext}(G)$, we can see how this new group relates to the original group, $G$. As we shall see, we have two main cases:

- if $G$ is already complete, then $\text{ext}(G)$ will be isomorphic to $G$;
- if $G$ is not complete, then $G$ will be isomorphic to some subgroup of $\text{ext}(G)$ and, in that way, we can look at $G$ as a subgroup of $\text{ext}(G)$.

**Proposition 3.5:** Let $G$ be a complete topological group, then $\text{ext}(G)$ is isomorphic to $G$.

**Proof:** Consider the following map

$$\begin{aligned} f : G &\to \text{ext}(G) \\ a &\mapsto [(a_n)_n] \end{aligned}$$

where $(a_n)_n$ is the constant sequence $a_n = a, \forall n \in \mathbb{N}$.
Let's start by proving that the map $f$ is an homomorphism:

- Let $a, b \in G$, then $f(a \cdot b) = [(a \cdot b)_n]$ where $(a \cdot b)_n$ is the constant sequence $(a \cdot b)_n = a \cdot b, \forall n \in \mathbb{N}$. But note that this is just the product of the two constant sequences $(a_n)_n \cdot (b_n)_n = (a_n \cdot b_n)_n$, with $a_n = a, b_n = b, \forall n \in \mathbb{N}$. Then,

$$f(a \cdot b) = [(a_n \cdot b_n)_n]$$

But if you recall the definition from proposition 3.4 $\text{ext}(G)$, $[(a_n \cdot b_n)_n]$ is just $[(a_n)_n] \otimes [(b_n)_n]$. But $f(a) = [(a_n)_n]$ and $f(b) = [(b_n)_n]$. So

$$f(a \cdot b) = f(a) \otimes f(b)$$

and $f$ is indeed a group homomorphism.

- Let $[(b_n)_n] \in \text{ext}(G)$. Because $(b_n)_n$ is Cauchy on a complete topological group $G$, it converges. Let i's limit be $a \in G$ and let $(a_n)_n$ be the constant sequence $a_n = a, \forall n \in \mathbb{N}$. We'll start by proving that $[(b_n)_n] = [(a_n)_n]$ using that fact that this is true if and only it $(b_n)_n \sim (a_n)_n$. Consider the following limit: $\lim(b_n \cdot a_n^{-1})$. By proposition 2.3, this limit is equal to

$$\lim(b_n) \cdot \lim(a_n)^{-1} = a \cdot a^{-1} = e$$

so $(b_n)_n \sim (a_n)_n$ and therefore $[(b_n)_n] = [(a_n)_n]$. Now note that $f(a) = [(a_n)_n] = [(b_n)_n]$, so the $f$ is surjective.

- Now to prove that $f$ is injective, let $a, b \in G$. Assume that $f(a) = f(b)$, this is $[(a_n)_n] = [(b_n)_n]$, where both $(a_n)_n$ and $(b_n)_n$ are the constant sequences $a_n = a$ and $b_n = b$, for all $n \in \mathbb{N}$. We want to show that this implies that $a = b$. The fact that $[(a_n)_n] = [(b_n)_n]$ means that $(a_n)_n \sim (b_n)_n$, and by the definition of $\sim$ this means that $\lim(a_n \cdot b_n^{-1}) = e$. According to proposition 2.3, this is the same as $\lim(a_n) \cdot \lim(b_n)^{-1} = e$. Because $(a_n)_n$ and $(b_n)_n$ are constant, we have that $\lim a_n = a$ and $\lim b_n = b$. So it follows that

$$a \cdot b^{-1} = e \iff a = b$$

Therefore $f$ is injective and therefore a bijection. So $G \simeq \text{ext}(G)$. ■

We can use this proposition together with propositions 2.13 and 2.14 to easily prove the following:

**Corollary 3.6:** If $G$ is either finite or a discrete topological group, then $G \simeq \text{ext}(G)$.

**Proof:** If $G$ is finite then, according to proposition 2.13, it is also complete, and therefore $G \simeq \text{ext}(G)$ (proposition 3.5).

If $G$ is discrete, according to proposition 2.14, it is also complete, and therefore, due to proposition 3.5, $G \simeq \text{ext}(G)$. ■

**Proposition 3.7:** Let $G$ be a topological group. Then $G$ is isomorphic to some subgroup of $\text{ext}(G)$.

**Proof:** Consider the same map $f : G \to \text{ext}(G)$ that we used to prove proposition 3.5.

The proof that $f$ is an injective homomorphism doesn't depend on whether $G$ is complete or not, so we can use what we did when proving proposition 3.5 to conclude that $f$ is an injective group homomorphism. Then, because $f$ is an homomorphism, $f(G)$ is a subgroup of $\text{ext}(G)$. If we restrict the codomain of $f$ to $f(G)$ we get a bijective group homomorphism from $G$ to $f(G)$, so we conclude that $G \simeq f(G) \leq \text{ext}(G)$. ■

Note that using the map $f : G \hookrightarrow \text{ext}(G)$ we identify each element of $G$ with one from $\text{ext}(G)$, so, in that sense, we can use some abuse of notation to write $G \leq \text{ext}(G)$.

We will now see that the extensions of two isomorphic and homeomorphic groups are itself isomorphic.

**Proposition 3.8:** Let $G$ and $H$ be two isomorphic and homeomorphic topological groups. Then, $\text{ext}(G) \simeq \text{ext}(H)$.

**Proof:** Let $f : G \to H$ be both an isomorphism and a homeomorphism. Consider the following map

$$g : \text{ext}(G) \to \text{ext}(H)$$
$$[(a_n)_n] \mapsto [(f(a_n))_n]$$

Note that this operation is well defined because, according to proposition 2.7, $f(a_n)_n$ is also Cauchy, so it makes sense to talk about its equivalence class in $\text{ext}(H)$. Let's first prove that $g$ is a group homomorphism. Let $[(a_n)_n], [(b_n)_n] \in \text{ext}(G)$. We want to show that $g([(a_n)_n] \otimes [(b_n)_n]) = g([(a_n)_n]) \otimes g([(b_n)_n])$. Using the definition of $g$ and $\otimes$, we have that

$$g([(a_n)_n] \otimes [(b_n)_n]) = g([(a_n \cdot b_n)_n]) = [(f(a_n \cdot b_n))_n]$$

Because $f$ is a group homomorphism, we can rewrite this as

$$[(f(a_n) \cdot f(b_n))_n] = [(f(a_n))_n] \otimes [(f(b_n))_n] = g([(a_n)_n]) \otimes g([(b_n)_n])$$

So it follows that $g$ is a group homomorphism.

Let's now prove that $g$ is surjective. Let $[(b_n)_n] \in \text{ext}(H)$. According to proposition 2.7, $(f^{-1}(b_n))_n$ is a Cauchy sequence on $G$, because $f^{-1}$ is both an isomorphism and a homeomorphism. Note that $g([(f^{-1}(b_n))_n]) = [(f(f^{-1}(b_n)))_n] = [(b_n)_n]$, therefore $g$ is surjective.

To prove that $g$ is injective, let $[(a_n)_n], [(b_n)_n] \in \text{ext}(G)$ such that $g([(a_n)_n]) = g([(b_n)_n])$. Thus $[(f(a_n))_n] = [(f(b_n))_n]$. It follows that $(f(a_n))_n \sim (f(b_n))_n$, i.e:

$$\lim(f(a_n) \cdot f(b_n)^{-1}) = e_H$$

Using the fact that $f$ is an isomorphism:

$$\lim(f(a_n \cdot b_n^{-1})) = e_H$$

Because $f(a_n \cdot b_n^{-1})$ converges on $H$, and $f^{-1} : H \to G$ is an isomorphism and a homeomorphism, according to proposition 2.4, $f^{-1}(f(a_n \cdot b_n^{-1})) = a_n \cdot b_n^{-1}$ converges on $G$ and its limit is $f^{-1}(\lim(f(a_n \cdot b_n^{-1}))) = f^{-1}(e_H) = e_G$. So, $\lim(a_n \cdot b_n^{-1}) = e_G$ and therefore $(a_n)_n \sim (b_n)_n$, i.e, $[(a_n)_n] = [(b_n)_n]$, so we conclude that $f$ is injective.

Therefore $\text{ext}(G) \simeq \text{ext}(H)$. ■

Now with all this we are ready to prove the cantor construction of the real numbers using Cauchy extensions as we defined them

**Theorem 3.9 (Cantor construction of the Real numbers):** Let $\mathbb{Q}$ denote the additive group of rational numbers with the subspace topology induced by $\mathbb{R}$. Then $\text{ext}(\mathbb{Q}) \simeq \mathbb{R}$, where $\mathbb{R}$ is the additive group of real numbers.

**Proof:** Consider the following map $f : \mathbb{R} \to \text{ext}(\mathbb{Q})$ that maps every real number $a$ with decimal expansion $r_1...r_k, a_1...a_n...$ to the equivalence class of the sequence: $(r_1...r_k, a_1...a_n)_n$ (here we use , to denote the decimal point so we can use ... freely without causing any confusion), in which $r_1$ can be negative, and all the other digits being non-negative. For example, $f(\pi)$ is the equivalence class of the sequence $3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415, ....$

We shall begin by proving that for any real number $f(a)$ is indeed defined (this is, we need to show that $(r_1...r_k, a_1...a_n)_n$ is indeed a Cauchy sequence on $\mathbb{Q}$).

Let $a \in \mathbb{R}$ with decimal expansion $r_1...r_k, a_1...a_n...$and let $m \geq n$, then $|(r_1...r_k, a_1...a_m) - (r_1...r_k, a_1...a_n)| = |0, 0...a_{n+1}...a_m| \leq 10^{-n}$.

Now, let $V$ be any open neighbourhood of $0 \in \mathbb{Q}$. This means that $V = A \cap \mathbb{Q}$, for some open set $A \subset \mathbb{R}$. Because $A$ is open in $\mathbb{R}$ and $0 \in A$, there is some $\varepsilon > 0$ such that $0 \in (-\varepsilon, \varepsilon) \subseteq A$, so: $0 \in (-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq V$.

Because $\lim 10^{-n} = 0$, there is some $q \in \mathbb{N}$ such that: $n \geq q \implies 10^{-n} \leq \varepsilon$ which means that $m \geq n \geq q \implies |(r_1...r_k, a_1...a_m) - (r_1...r_k, a_1...a_n)| \leq \varepsilon \iff (r_1...r_k, a_1...a_m) - (r_1...r_k, a_1...a_n) \in (-\varepsilon, \varepsilon)$. We also have that $(r_1...r_k, a_1...a_m) - (r_1...r_k, a_1...a_n) \in \mathbb{Q}$, thus

$$m \geq n \geq q \implies (r_1...r_k, a_1...a_m) - (r_1...r_k, a_1...a_n) \in (-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq V$$

Then, $(r_1...r_k, a_1...a_n)_n$ is a Cauchy sequence and $[(r_1...r_k, a_1...a_n)_n]$ is indeed a well-defined element of $\text{ext}(\mathbb{Q})$.

Now we need to prove that $f$ is a group homomorphism.

Let $a, b \in \mathbb{R}$ with $a$ and $b$ having decimal expansions $r_1...r_k, a_1...a_n...$ and $l_1...l_k, b_1...b_n....$ Then $a+b = r_1...r_k, a_1...a_n...+l_1...l_k, b_1...b_n...$ and $f(a+b) = [(r_1...r_k, a_1...a_n+l_1...l_k, b_1...b_n)_n] = [(r_1...r_k, a_1...a_n)_n] \otimes [(l_1...l_k, b_1...b_n)_n]$, according to the definition of the group operation of $\text{ext}(\mathbb{Q})$. But $[(r_1...r_k, a_1...a_n)_n] = f(a)$ and $[(l_1...l_k, b_1...b_n)_n] = f(b)$, therefore:

$$f(a + b) = f(a) \otimes f(b)$$

And this allows us to conclude that $f$ is indeed a group homomorphism.

Knowing that $f$ is an homomorphism, we can apply the first homomorphism theorem and obtain the following:

$$\mathbb{R}/\ker(f) \simeq \text{Im}(f)$$

Let's now find the kernel of $f$.

As we saw in proposition 3.4, the identity element of $\text{ext}(\mathbb{Q})$ is the equivalence class $[(0_n)_n]$ where $0_n = 0,\ \forall n \in \mathbb{N}$ is a constant sequence. Let $a \in \mathbb{R}$ be a real number with decimal expansion $r_1...r_k, a_1...a_n...$ such that $f(a) =[(0_n)_n]$. This would imply that $(r_1...r_k, a_1...a_n)_n \sim (0_n)_n \iff \lim(r_1...r_k, a_1...a_n - 0_n) = \lim(r_1...r_k, a_1...a_n) = 0$, which means that, for every $\varepsilon > 0$ , there is some $N \in \mathbb{N}$ such that:

$$n \geq N \implies |r_1...r_k, a_1...a_n| < \varepsilon$$

So, setting $\varepsilon = 0.99 < 1$, then $|r_1...r_k, a_1...a_n| < \varepsilon = 0.99$ if and only if $r_1, ..., r_k = 0$, so we get that: $a = 0, a_1...a_n...$

Let $\varepsilon = 10^{-l}$, for $l \in \{1, 2, 3, ...\}$. Then $|r_1...r_k, a_1...a_n| < \varepsilon = 10^{-l} \implies a_1, a_2, ..., a_l =$ 0- This is true for all $l \in \{1, 2, 3, ...\}$, so, for all $n \in \mathbb{N}$, $a_n = 0$.Therefore, the decimal expansion of $a$ is $0,000...$ or, in other words, $a = 0$.

Thus we have that $\ker(f) = \{0\}$. Because $\mathbb{R}/\{0\} \simeq \mathbb{R}$ (this is easy to verify by considering the isomorphism $f : \mathbb{R} \to \mathbb{R}/\{0\}$ given by $f(a) = a + \{0\}$), according to the first isomorphism theorem:

$$\mathbb{R} \simeq \mathrm{Im}(f)$$

The final part of this proof is $f$ being surjective.

Let $[(a_n)_n] \in \mathrm{ext}(\mathbb{Q})$. Because $(a_n)_n \subseteq \mathbb{Q} \subseteq \mathbb{R}$ is a Cauchy sequence and because $\mathbb{R}$ is complete, $(a_n)_n$ converges on $\mathbb{R}$. Let $b \in \mathbb{R}$ be its limit with decimal expansion $r_1...r_k, b_1...b_n...$ .

We want to show that $f(b) = [(a_n)_n]$. For that we need to verify that $(r_1...r_k, b_1...b_n)_n \sim (a_n)_n \iff \lim(r_1...r_k, b_1...b_n - a_n) = 0$. Due to proposition 2.3, $\lim(r_1...r_k, b_1...b_n - a_n) = \lim(r_1...r_k, b_1...b_n) - \lim(a_n) = b - b = 0$, and so, indeed, $(r_1...r_k, b_1...b_n)_n \sim (a_n)_n \iff f(b) = [(a_n)_n]$.

Being proven that $f$ is surjective, we have shown that $\mathrm{Im}(f) = \mathrm{ext}(\mathbb{Q})$. So, according to the isomorphism theorem:

$$\mathbb{R} \simeq \mathrm{Im}(f) = \mathrm{ext}(\mathbb{Q})$$

which concludes our proof. ■

In this paper we generalized important notions of Analysis and metric spaces such as limits and Cauchy sequences to the realm of topological groups.

I tried, yet failed, to give $\mathrm{ext}(G)$ a meaningful topology induced by the topological structure of $G$. If I make any progress on the matter and find its implications interesting a continuation of this paper will be posted in the future.

## References


[1] Sidney A. Morris, "Topology Without Tears", https://www.topologywithouttears.net/topbook.pdf, p. 195 - proposition 8.1.2

[2] https://personal.math.ubc.ca/~gor/Math220_2016/functions_notes.pdf, p. 1 - theorem 12.4 (b)

[3] https://www.uni-muenster.de/AGKramer/content/ch1.pdf, pag. 4

[4] https://www.math.wustl.edu/~freiwald/ch5.pdf, pag. 2 - theorem 2.3