

THE RIEMANN HYPOTHESIS PROVED

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Abstract

The Riemann hypothesis is proved through a theorem on the nature of points critics of the real part and the imaginary part $u(x, y)$, $v(x, y)$ of a holomorphic function having the same zeros of the Riemann zeta function. Precisely, the zeros of these functions are saddle points, and furthermore in these points the partial derivatives of odd order. From this derives a system of infinite identities that are check if and only if the real part of the zeros of the zeta function is equal to $1/2$.

Introduction

This work is structured as follows:

1. Section 1 recalls the definition of *non-trivial zeros* of the Riemann zeta function, after which the *Riemann hypothesis* is stated. Technically this is a conjecture, and is currently the most important open problem in mathematics. Using the functional equation found by Riemann, the integral representation of a holomorphic function having the same non-trivial zeros as the zeta function is obtained.
2. In section 2 the functional equation referred to in point 1, is written in the framework of group theory, in particular of the subgroup $HL(2, \mathbb{R})$ of the antisymmetric matrices of order 2 on the real field. This purpose is achieved by demonstrating a theorem that allows us to algebraize the symmetry of non-trivial zeros with respect to the critical line whose equation in the complex plane is $\text{Re } z = 1/2$, and to be able to express the values assumed by $u(x, y)$ e $v(x, y)$ in a region of the critical strip with respect to the values assumed by the same functions in the symmetrical region with respect to the critical line.
3. In section 3 we introduce the *harmonic functions* on the real field and their properties.
4. Nella sezione 4 we study *conjugate harmonic functions*, an important class of harmonic functions in \mathbb{R}^2 , stating and proving important theorems.
5. In section 5 we prove a theorem according to which the real and imaginary parts of the zeta function have a saddle point in every non-trivial zero.
6. In section 6 we prove the Riemann hypothesis, using part of the proof of the theorem referred to in the previous point.

1 The non-trivial zeros of the zeta function and the Riemann hypothesis

Denoting by $z = x + iy$ the usual complex variable, an integral representation of the Riemann zeta function $\zeta(z)$ in the half plane $\text{Re } z > 0$ excluding $z = 1$ where this function has a simple pole, is [1]-[2]

$$\zeta(z) = \frac{1}{(1 - 2^{1-z}) \Gamma(z)} \int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1}, \quad (\forall z \in \mathbb{C} \mid \text{Re } z > 0, z \neq 1) \quad (1)$$

where

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad \text{Re } z > 0 \quad (2)$$

is an integral representation of the *Euler gamma function* in the half plane $\text{Re } z > 0$. Turns out [3]:

$$\Gamma(z) = \sum_{n=0}^{+\infty} \left[\frac{(-1)^n}{n!} \frac{1}{z+n} \right] + \int_0^{+\infty} t^{z-1} e^{-t} dt \quad (3)$$

It follows that $\Gamma(z)$ is meromorphic with simple poles in $z_n = -n$ for every $n \in \mathbb{N}$ with residue $\frac{(-1)^n}{n!}$. Also [3], $\Gamma(z)^{-1}$ is a transcendent integer, so $\Gamma(z)$ has no zeros.

Riemann derived [2] the following functional equation for the $\zeta(z)$

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z) \quad (4)$$

The *trivial zeros* (i.e. with null imaginary part) of the function $\zeta(z)$ are $z_n = -2n$, with $n = 1, 2, 3, \dots$, and the graph of fig. 1 illustrates some of them.

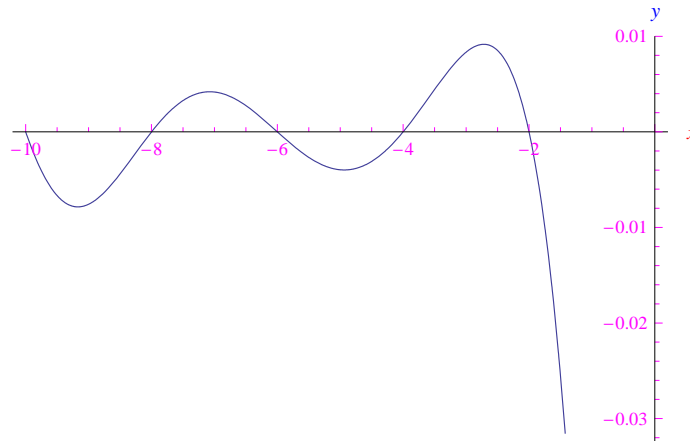


Figure 1: Some trivial zeros of the Riemann zeta function. The curve is the graph of the function $\zeta(x)$.

Non-trivial zeros fall into the critical strip [1]:

$$S_{crit} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, -\infty < y < +\infty\}$$

In fig. 2 shows the graph of $|\zeta(z)|$, in which we see some non-trivial zeros.

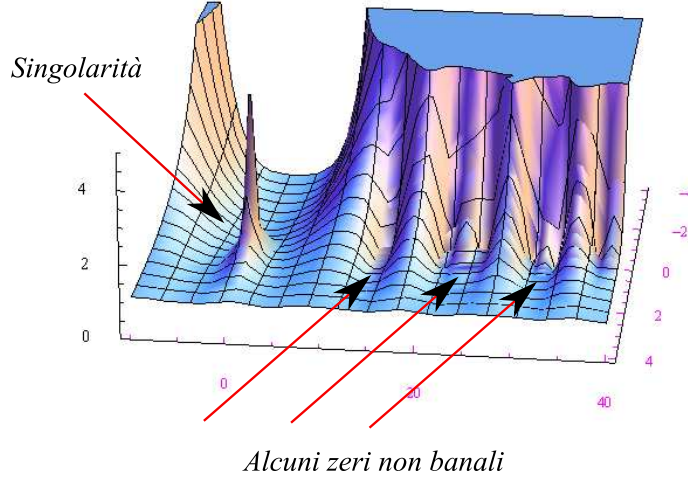


Figure 2: Some non-trivial zeros of the function $\zeta(z)$. Notice the polar singularity at $z = 1$.

Conjecture 1 (Riemann Hypothesis – RH)

The non-trivial zeros of the function $\zeta(z)$ belong to the critical line $\text{Re } z = \frac{1}{2}$.

From (4) it follows

$$z_0 \in S_{crit} \mid \zeta(z_0) = 0 \implies \zeta(1 - z_0) = 0 \tag{5}$$

that is, the distribution of non-trivial zeros is symmetrical with respect to the *critical line* $\text{Re } z = 1/2$. It is also known that if $z_0 = x_0 + iy_0$ is a zero, so is its conjugate complex $z_0^* = x_0 - iy_0$. So we have further symmetry (with respect to the real axis). This suggests redefining the region S_{crit} as the *critical half-strip*:

$$S_{crit} = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < +\infty\} \tag{6}$$

From the (1):

$$\zeta(1 - z) = \frac{1}{(1 - 2^z) \Gamma(1 - z)} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}, \quad \text{Re}(1 - z) > 0 \implies \text{Re } z < 1 \tag{7}$$

and even more so

$$\zeta(1 - z) = \frac{1}{(1 - 2^z) \Gamma(1 - z)} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}, \quad \forall z \in S_{crit} \tag{8}$$

The integral converges. In fact from the (1):

$$\left| \int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1} \right| < +\infty, \quad \text{Re } z > 0, \quad z \neq 1$$

By placing $z' = 1 - z$

$$\int_0^{+\infty} \frac{t^{-z'} dt}{e^t + 1} = \int_0^{+\infty} \frac{t^{z'-1} dt}{e^t + 1}$$

hence the convergence for $\text{Re}(1 - z') > 0$ i.e. $\text{Re } z' < 1$ and therefore, in S_{crit} .

Writing the (4) in S_{crit} and using the integral representations of $\zeta(z)$ and $\zeta(1-z)$, i.e. the (1)-(8), we get:

$$\int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1} = \chi(z) \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1} \quad (9)$$

being

$$\chi(z) \stackrel{def}{=} \pi^{z-\frac{1}{2}} \frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \frac{(1-2^{1-z})\Gamma(z)}{(1-2^z)\Gamma(1-z)} \quad (10)$$

It is easy to believe that $\chi(z)$ is holomorphic in S_{crit} and is zero-zero there. Let's start with the study of the behavior of the G function Γ

$$z' = \frac{1-z}{2} \implies \Gamma(z') \text{ has singularity in } z'_n = -n \implies z_n = 1 + 2n > 1, \quad \forall n \in \mathbb{N}$$

$$\Gamma\left(\frac{z}{2}\right) \text{ has singularity in } z_n = -2n < 0, \quad \forall n \in \mathbb{N}$$

$$\Gamma(1-z) \text{ has singularity in } z_n = 1 + n \geq 1, \quad \forall n \in \mathbb{N}$$

and it is immediate to verify the absence of zeros.

Proposition 2 *The function*

$$\Phi(z) \stackrel{def}{=} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}, \quad z \in S_{crit} \quad (11)$$

has the same zeros as the Riemann zeta function $\zeta(z)$.

Proof. Multiplying first and second members of the (9) for $\frac{1}{(1-2^{1-z})\Gamma(z)}$:

$$\underbrace{\frac{1}{(1-2^{1-z})\Gamma(z)} \int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1}}_{=\zeta(z)} = \frac{\chi(z)}{(1-2^{1-z})\Gamma(z)} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}$$

from which

$$\zeta(z) = \underbrace{\frac{\chi(z)}{(1-2^{1-z})\Gamma(z)}}_{\neq 0, \forall z \in S_{crit}} \Phi(z)$$

and therefore the assertion. ■

2 The subgroup $HL(2, \mathbb{R})$ of order 2 antisymmetric matrices on the real field

The functional equation (9) written as:

$$\chi(z) \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1} = \int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1},$$

can be interpreted as the result of applying $\chi(z)$ on the holomorphic function $\int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}$, giving rise to $\int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1}$. This is corroborated by the following theorem:

Theorem 3

$$\chi(z) \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1} = \int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1} \iff \Lambda(x, y) \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} u(1-x, y) \\ v(1-x, y) \end{pmatrix}, \quad \forall (x, y) \in S_{crit} \quad (12)$$

where

$$\begin{aligned} \Lambda(x, y) &= \begin{pmatrix} f(x, y) & -g(x, y) \\ g(x, y) & f(x, y) \end{pmatrix} & (13) \\ u(x, y) &= \operatorname{Re} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}, \quad v(x, y) = \operatorname{Im} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1} \\ f(x, y) &= \operatorname{Re} \chi(x + iy), \quad g(x, y) = \operatorname{Im} \chi(x + iy) \end{aligned}$$

Proof. Let's say

$$\Psi(z) \stackrel{def}{=} \int_0^{+\infty} \frac{t^{z-1} dt}{e^t + 1}, \quad z \in S_{crit} \quad (14)$$

the (9) becomes

$$\Psi(z) = \chi(z) \Phi(z) \quad (15)$$

In the second member, separating the real part and the imaginary part and then developing the product, we easily obtain:

$$\begin{aligned} \chi(z) \Phi(z) &= [\operatorname{Re} \chi(z) \operatorname{Re} \Phi(z) - \operatorname{Im} \chi(z) \operatorname{Im} \Phi(z)] + \\ &+ i [\operatorname{Re} \chi(z) \operatorname{Im} \Phi(z) + \operatorname{Im} \chi(z) \operatorname{Re} \Phi(z)] \end{aligned} \quad (16)$$

Likewise for the first member, noting that

$$t^{z-1} = \frac{1}{t^{1-x}} [\cos(y \ln t) + i \sin(\ln t)]$$

we obtain

$$\begin{aligned} \Psi(x + iy) &= \int_0^{+\infty} \frac{\cos(y \ln t) + i \sin(\ln t)}{t^{1-x} (e^t + 1)} dt \\ &= \underbrace{\int_0^{+\infty} \frac{\cos(y \ln t)}{t^{1-x} (e^t + 1)} dt}_{\operatorname{Re} \Psi} + i \underbrace{\int_0^{+\infty} \frac{\sin(\ln t)}{t^{1-x} (e^t + 1)} dt}_{\operatorname{Im} \Psi} \end{aligned}$$

From the (15)

$$\operatorname{Re} \Psi(z) = \operatorname{Re} [\chi(z) \Phi(z)], \quad \operatorname{Im} \Psi(z) = \operatorname{Im} [\chi(z) \Phi(z)]$$

from which

$$\begin{aligned} \int_0^{+\infty} \frac{\cos(y \ln t) dt}{t^{1-x} (e^t + 1)} &= \operatorname{Re} [\chi(z) \Phi(z)] = \operatorname{Re} \chi(z) \operatorname{Re} \Phi(z) - \operatorname{Im} \chi(z) \operatorname{Im} \Phi(z) \\ \int_0^{+\infty} \frac{\sin(\ln t) dt}{t^{1-x} (e^t + 1)} &= \operatorname{Im} [\chi(z) \Phi(z)] = \operatorname{Re} \chi(z) \operatorname{Im} \Phi(z) + \operatorname{Im} \chi(z) \operatorname{Re} \Phi(z) \end{aligned} \quad (17)$$

The function we are interested in is $\Phi(z)$ (eq. (11))

$$\begin{aligned} \Phi(x + iy) &= \int_0^{+\infty} \frac{t^{-x-iy} dt}{e^t + 1} = \int_0^{+\infty} \frac{\cos(y \ln t) - i \sin(\ln t)}{t^x (e^t + 1)} dt \\ &= \int_0^{+\infty} \frac{\cos(y \ln t) dt}{t^x (e^t + 1)} - i \int_0^{+\infty} \frac{\sin(y \ln t) dt}{t^x (e^t + 1)} \end{aligned} \quad (18)$$

For the position 13 (second equation):

$$\Phi(x + iy) = u(x, y) + iv(x, y)$$

which compared to (18) returns:

$$u(x, y) = \int_0^{+\infty} \frac{\cos(y \ln t) dt}{t^x (e^t + 1)}, \quad v(x, y) = - \int_0^{+\infty} \frac{\sin(y \ln t) dt}{t^x (e^t + 1)} \quad (19)$$

For the position 13 (third equation):

$$f(x, y) = \operatorname{Re} \chi(x + iy), \quad g(x, y) = \operatorname{Im} \chi(x + iy) \quad (20)$$

so that the (17) become:

$$\begin{aligned} u(1-x, y) &= f(x, y) u(x, y) - g(x, y) v(x, y) \\ v(1-x, y) &= g(x, y) u(x, y) + f(x, y) v(x, y) \end{aligned} \quad (21)$$

which can be written in matrix form:

$$\Lambda(x, y) \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} u(1-x, y) \\ v(1-x, y) \end{pmatrix} \quad (22)$$

being $\Lambda(x, y)$ the square matrix of order 2 on the real field:

$$\Lambda(x, y) = \begin{pmatrix} f(x, y) & -g(x, y) \\ g(x, y) & f(x, y) \end{pmatrix}, \quad \forall (x, y) \in S_{crit} \quad (23)$$

■

The determinant of $\Lambda(x, y)$ is

$$\det \Lambda(x, y) = f(x, y)^2 + g(x, y)^2 = |\chi(x + iy)|^2 \neq 0, \quad \forall (x, y) \in S_{crit} \quad (24)$$

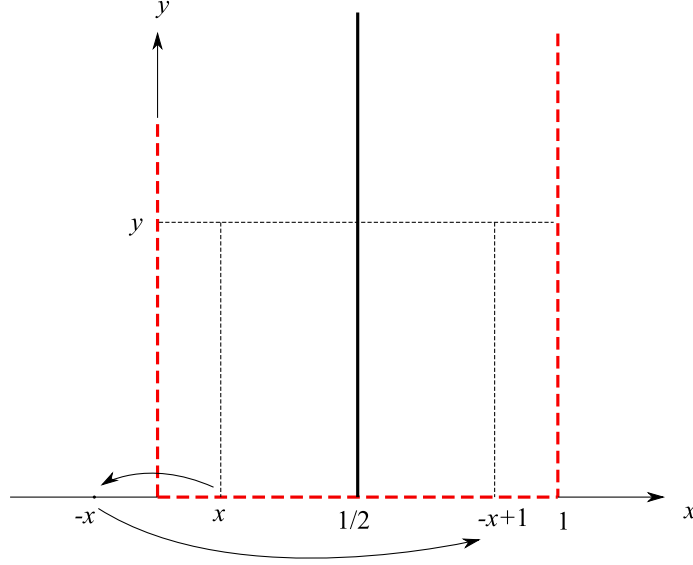


Figure 3: The linear transformation (22) is the composition of the spatial reflection $x \rightarrow -x$ and of the translation along the x axis of amplitude $+1$.

Regarding the antisymmetry of Λ , we observe that this translates the symmetry with respect to the critical line $\text{Re } z = 1/2$. In fact, the linear transformation (22) is the composition of a spatial reflection with respect to x , cioè $x \rightarrow -x$, and of a translation along the x axis of amplitude $+1$, as illustrated in fig. 3.

Hence $\Lambda(x, y) \in GL(2, \mathbb{R})$ that is, it is an element of the linear group of order 2 on the real field. As is known, the latter is the set of square matrices of order 2 on the real field and with a non-zero determinant (therefore with an inverse). The algebraic structure is given by the law of internal composition “product row by column”. In fact, the group axioms are verified:

G1 Associative property

$$(AB)C = A(BC), \quad \forall A, B, C \in GL(2, \mathbb{R})$$

G2 Existence of the neutral element

$$\bar{I}A = A\bar{I}, \quad \forall A \in GL(2, \mathbb{R})$$

where \bar{I} is the identity matrix of order 2:

$$\bar{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

G3 Existence of the inverse

$$\forall A \in GL(2, \mathbb{R}), \quad \exists A^{-1} \in GL(2, \mathbb{R}) \mid AA^{-1} = A^{-1}A = \bar{I}$$

The group $GL(2, \mathbb{R})$ is *non-commutative* (or *non-Abelian*) since in general this is the product row by column. The matrix (23) is a functional matrix in the sense that its elements are functions of the variables x, y . Furthermore, this matrix is antisymmetric and we denote

by $HL(2, \mathbb{R})$ the subset of $GL(2, \mathbb{R})$ whose elements are precisely the antisymmetric matrices of the type:

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \forall a, b \in \mathbb{R}$$

It follows

$$AA' = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a' & -b' \\ b' & a' \end{pmatrix} = \begin{pmatrix} aa' - bb' & -ab' - ba' \\ ab' + ba' & aa' - bb' \end{pmatrix} \implies AA' \in HL(2, \mathbb{R}) \quad (25)$$

Furthermore

$$A^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \implies A^{-1} \in HL(2, \mathbb{R}) \quad (26)$$

From (25)-(26) it follows for a well-known theorem [5], the group structure for $HL(2, \mathbb{R})$ and we will say that it is a subgroup of $GL(2, \mathbb{R})$. Also, from a simple calculation

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a' & -b' \\ b' & a' \end{pmatrix} = \begin{pmatrix} a' & -b' \\ b' & a' \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \forall a, b, a', b' \in \mathbb{R}$$

for which the subgroup $HL(2, \mathbb{R})$ is commutative (or abelian).

For the above:

$$\begin{aligned} [\Lambda(1-x, y) \Lambda(x, y)] \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} &= \Lambda(1-x, y) \left[\Lambda(x, y) \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \right] \\ &= \Lambda(1-x, y) \begin{pmatrix} u(1-x, y) \\ v(1-x, y) \end{pmatrix} \\ &= \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}, \quad \forall \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \end{aligned}$$

so that

$$\Lambda(1-x, y) \Lambda(x, y) = \bar{1}$$

As the matrices switch:

$$\Lambda(x, y) \Lambda(1-x, y) = \bar{1}$$

That is

$$\Lambda(1-x, y) = \Lambda^{-1}(x, y) \quad (27)$$

illustrated in fig. 4.

A direct computation of the inverse matrix starting from (23) gives:

$$\Lambda^{-1}(x, y) = \frac{1}{f(x, y)^2 + g(x, y)^2} \begin{pmatrix} f(x, y) & g(x, y) \\ -g(x, y) & f(x, y) \end{pmatrix} \quad (28)$$

which compared with the (28) returns:

$$f(1-x, y) = \frac{f(x, y)}{f(x, y)^2 + g(x, y)^2}, \quad g(1-x, y) = -\frac{g(x, y)}{f(x, y)^2 + g(x, y)^2} \quad (29)$$

Theorem 4

$$\Lambda\left(\frac{1}{2}, y\right) = \bar{1}, \quad \forall y \in (0, +\infty) \quad (30)$$

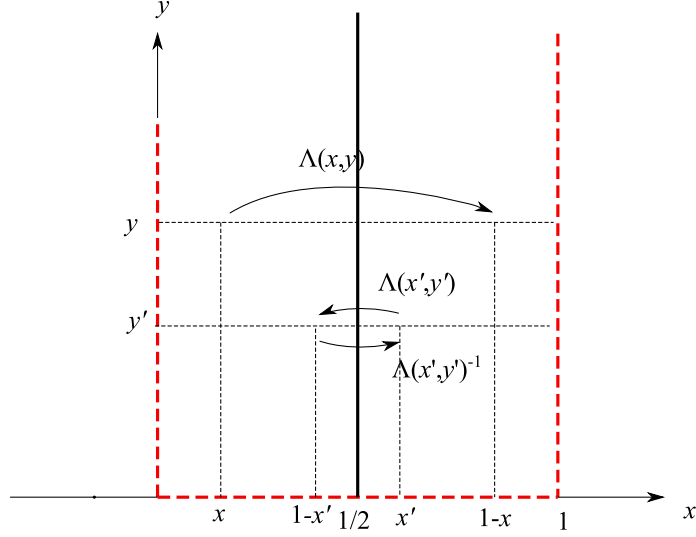


Figure 4: Geometric interpretation of the (27).

Proof. From (28) it is easy to get

$$f\left(\frac{1}{2}, y\right) = 1, \quad g\left(\frac{1}{2}, y\right) = 0, \quad \forall y \in (0, +\infty)$$

that is the assertion. Alternatively, from (22)

$$\Lambda\left(\frac{1}{2}, y\right) \begin{pmatrix} u\left(\frac{1}{2}, y\right) \\ v\left(\frac{1}{2}, y\right) \end{pmatrix} = \begin{pmatrix} u\left(\frac{1}{2}, y\right) \\ v\left(\frac{1}{2}, y\right) \end{pmatrix}$$

i.e. $\begin{pmatrix} u\left(\frac{1}{2}, y\right) \\ v\left(\frac{1}{2}, y\right) \end{pmatrix}$ is eigenvector of $\Lambda\left(\frac{1}{2}, y\right)$ with eigenvalue $+1$, and this for all $y \in (0, +\infty)$.

But the only matrix with this property is the identical matrix. ■

3 The harmonic functions. Properties and theorems

Definition 5 A real function of n real variables x_1, x_2, \dots, x_n , is called **harmonic** in a field A of \mathbb{R}^n , if it is continuous there with first and second partial derivatives, and solves Laplace's equation in A :

$$\nabla^2 u = 0 \tag{31}$$

being

$$\nabla^2 = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \tag{32}$$

the Laplacian.

We are interested in $n = 2$, so $u(x, y)$ is harmonic in a field A if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{33}$$

resulting continuous in A together with the first and second partial derivatives. Real part and imaginary part of a holomorphic function in a field A are harmonic functions there [3]-[4]:

$$f(x + iy) = u(x, y) + iv(x, y)$$

Remember that u and v are not independent, in the sense that if u, v harmonics in A are taken as arbitrary, the composition $u + iv$ is not in general a holomorphic function. In fact, the real part and the imaginary part are linked by the Cauchy-Riemann equations [3]-[4]

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \tag{34}$$

In this regard, the following definition exists:

Definition 6 If $u(x, y)$ is harmonic in A , we say **conjugate** of $u(x, y)$, every function $v(x, y)$ is harmonic in A and such that the complex function $u + iv$ is holomorphic in A .

It follows that given a harmonic function $u(x, y)$ in A , any conjugate functions are the solutions of the system of partial differential equations (34). The existence of such solutions is guaranteed by the theorem:

Theorem 7 In a simply connected field A every harmonic function admits a conjugate function defined up to an additive constant. Precisely:

$$v(x, y) = c + \int_{(x_0, y_0)}^{(x, y)} \left(-\frac{\partial u}{\partial \xi} d\xi + \frac{\partial u}{\partial \eta} d\eta \right), \quad \forall c \in \mathbb{R}$$

where the curvilinear integral is extended to an arc of a regular curve of extremes an arbitrary point (x_0, y_0) and the point (x, y) .

Proof. Please refer to [4]. ■

Here are some notable theorems that we will use later:

Theorem 8 *A harmonic function in a field A is analytic therein, i.e. developable at any point in power series:*

$$u(x, y) = \sum_{h=0}^{+\infty} \sum_{k=0}^{+\infty} a_{hk} (x - x_0)^h (y - y_0)^k \quad (35)$$

Proof. Please refer to [4]. ■

From this the corollary immediately follows:

Corollary 9 *Any harmonic function in a field A is endowed with partial derivatives of any high order, and each derivative is in turn a harmonic function in A .*

Theorem 10 (Gauss' mean theorem)

If $u(x, y)$ is continuous in the domain

$$C_R(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 \leq R^2\}$$

and harmonic in $C_R(x_0, y_0) \setminus \partial C_R(x_0, y_0)$

$$u(x_0, y_0) = \frac{1}{2\pi R} \oint_{+\partial C_R(x_0, y_0)} u(x, y) ds \quad (36)$$

that is $u(x_0, y_0)$ is the average value of the values assumed by $u(x, y)$ on $\partial C_R(x_0, y_0)$.

Proof. Please refer to [4]. ■

From this theorem follows this other:

Theorem 11 *A harmonic function $u(x, y)$ in a connected field A and which is not constant there, is devoid of relative maxima and minima.*

Proof. Please refer to [4]. ■

Theorem 12 (Areal averages theorem)

In the same hypotheses as the theorem 10:

$$\begin{aligned} u(x_0, y_0) &= \frac{1}{\pi R^2} \iint_{C_R(x_0, y_0)} u(x, y) dx dy \\ u_x(x_0, y_0) &= \frac{3}{\pi R^3} \iint_{C_R(x_0, y_0)} u(x, y) \frac{x - x_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} dx dy \\ u_y(x_0, y_0) &= \frac{3}{\pi R^3} \iint_{C_R(x_0, y_0)} u(x, y) \frac{y - y_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} dx dy \end{aligned} \quad (37)$$

whose interpretation is immediate.

Proof. Please refer to [4]. ■

Theorem 13 (Theorem of means for derivatives)

Nelle stesse ipotesi del teorema 10:

$$\begin{aligned} u_x(x_0, y_0) &= \frac{1}{\pi R^3} \oint_{+\partial C_R(x_0, y_0)} u(x, y) (x - x_0) ds \\ u_y(x_0, y_0) &= \frac{1}{\pi R^3} \oint_{+\partial C_R(x_0, y_0)} u(x, y) (y - y_0) ds \end{aligned} \quad (38)$$

One notable theorem that we will apply later is

Theorem 14 *The critical points of a harmonic function in a field A are critical points of any conjugate function.*

Proof. If $(x_0, y_0) \in A$ is a critical point of $u(x, y)$, we have

$$u_x(x_0, y_0) = 0, \quad u_y(x_0, y_0) = 0$$

From the Cauchy-Riemann equations (34)

$$v_x(x_0, y_0) = 0, \quad v_y(x_0, y_0) = 0$$

■

4 Place of the zeros of the conjugate harmonic functions

We premise the following theorem that characterizes the zeros of a holomorphic function:

Theorem 15 *The derivative of the set of zeros of a holomorphic function in a field A which is not identically null therein is contained in ∂A .*

Proof. Please refer to [4]. ■

This means that any accumulation points of the set of zeros fall on the frontier of the holomorphic field of the assigned function. Put another way, the zeros of a holomorphic function are isolated points. At the same time, they are the common zeros of the real part and the imaginary part of the function itself. Precisely, let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function in a field A . The locus of zeros of the harmonic function $u(x, y)$ is a curve of the xy coordinate plane of implied representation:

$$\gamma_u : u(x, y) = 0$$

In the same way for the conjugate

$$\gamma_v : v(x, y) = 0$$

It follows that the zeros of $f(z)$ are the points of intersection of the aforesaid plane curves which for a known property [3] are orthogonal therein. We prove the following theorem:

Theorem 16 *The zeros of a holomorphic function $f(z)$ in a field A , are critical points for the real function $|f(z)|^2$.*

Proof. Writing $f(z) = u(x, y) + iv(x, y)$, we have the real function

$$\psi(x, y) \stackrel{def}{=} |f(z)|^2 = u(x, y)^2 + v(x, y)^2$$

It follows

$$\begin{aligned} \psi_x(x, y) &= 2[u(x, y)u_x(x, y) + v(x, y)v_x(x, y)] \\ \psi_y(x, y) &= 2[u(x, y)u_y(x, y) + v(x, y)v_y(x, y)] \end{aligned} \quad (39)$$

If $(x_0, y_0) \in A$ is a zero of $f(z)$, from (39):

$$\psi_x(x_0, y_0) = \psi_y(x_0, y_0) = 0$$

■

Corollary 17 *The critical points of any pair (u, v) of conjugate functions are critical points of the holomorphic functions $u + iv$ and $v + iu$.*

Proof. Immediately follows from (39). ■

Always from (39):

$$(x_0, y_0) \text{ is critical point of } u, v \begin{matrix} \implies \\ \nLeftarrow \end{matrix} (x_0, y_0) \text{ is critical point of } |f(z)|^2 \quad (40)$$

$$\begin{aligned} (x_0, y_0) \text{ is critical point of } f(z) &\implies (x_0, y_0) \text{ critical point of } |f(z)|^2 \\ &\nRightarrow (x_0, y_0) \text{ is critical point of } u, v \end{aligned}$$

We conclude: being a critical point for u, v is a sufficient but not necessary condition to be a critical point for $|f(z)|^2 = u^2 + v^2$.

5 The zeros of the zeta function are saddle points for the real part and the imaginary part

Theorem 18 *The zeros of the function $\Phi(z) = \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}$ (and therefore of the $\zeta(z)$) are saddle points of the real part and the imaginary part.*

Proof. Let's rewrite the (22):

$$\Lambda(x, y) \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} u(1-x, y) \\ v(1-x, y) \end{pmatrix} \quad (41)$$

remembering that

$$\Lambda(x, y) = \begin{pmatrix} f(x, y) & -g(x, y) \\ g(x, y) & f(x, y) \end{pmatrix}, \quad \forall (x, y) \in S_{crit} \quad (42)$$

As seen in the section 2, tge (41) tells us how the column vector $\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ when we pass from one side of the critical strip to the other. Let's examine how derivatives are transformed. To do this, we apply the operator $\frac{\partial}{\partial x}$ to the first and second members of the (41):

$$\frac{\partial}{\partial x} \left[\Lambda(x, y) \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \right] = \frac{\partial}{\partial x} \begin{pmatrix} u(1-x, y) \\ v(1-x, y) \end{pmatrix}$$

It follows

$$\frac{\partial \Lambda}{\partial x} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + \Lambda(x, y) \begin{pmatrix} u_x(x, y) \\ v_x(x, y) \end{pmatrix} = \begin{pmatrix} -u_x(1-x, y) \\ -v_x(1-x, y) \end{pmatrix} \quad (43)$$

being

$$\frac{\partial \Lambda}{\partial x} = \begin{pmatrix} f_x(x, y) & -g_x(x, y) \\ g_x(x, y) & f_x(x, y) \end{pmatrix} \quad (44)$$

If (x_0, y_0) is a zero of the zeta function, with $x_0 \neq 1/2$ the (43) becomes

$$\Lambda(x_0, y_0) \begin{pmatrix} u_x(x_0, y_0) \\ v_x(x_0, y_0) \end{pmatrix} = \begin{pmatrix} -u_x(1-x_0, y_0) \\ -v_x(1-x_0, y_0) \end{pmatrix} \quad (45)$$

Deriving with respect to y first and second members of the (41)

$$\frac{\partial \Lambda}{\partial y} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + \Lambda(x, y) \begin{pmatrix} u_y(x, y) \\ v_y(x, y) \end{pmatrix} = \begin{pmatrix} u_y(1-x, y) \\ v_y(1-x, y) \end{pmatrix} \quad (46)$$

As usual, if (x_0, y_0) is a zero

$$\Lambda(x_0, y_0) \begin{pmatrix} u_y(x_0, y_0) \\ v_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} u_y(1-x_0, y_0) \\ v_y(1-x_0, y_0) \end{pmatrix} \quad (47)$$

Developing in (45)-(47) the product rows by columns, we obtain

$$\begin{cases} f(x_0, y_0) u_x(x_0, y_0) - g(x_0, y_0) v_x(x_0, y_0) = -u_x(1-x_0, y_0) \\ g(x_0, y_0) u_x(x_0, y_0) + f(x_0, y_0) v_x(x_0, y_0) = -v_x(1-x_0, y_0) \\ f(x_0, y_0) u_y(x_0, y_0) - g(x_0, y_0) v_y(x_0, y_0) = u_y(1-x_0, y_0) \\ g(x_0, y_0) u_y(x_0, y_0) + f(x_0, y_0) v_y(x_0, y_0) = v_y(1-x_0, y_0) \end{cases} \quad (48)$$

which is a system of four linear equations in the unknowns $u_x(x_0, y_0), u_y(x_0, y_0), v_x(x_0, y_0), v_y(x_0, y_0)$. This system can be reduced to two equations, thanks to the Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_x$$

It follows

$$\begin{cases} f(x_0, y_0) u_x(x_0, y_0) + g(x_0, y_0) u_y(x_0, y_0) = -u_x(1 - x_0, y_0) \\ g(x_0, y_0) u_x(x_0, y_0) - f(x_0, y_0) u_y(x_0, y_0) = u_y(1 - x_0, y_0) \\ f(x_0, y_0) u_y(x_0, y_0) - g(x_0, y_0) u_x(x_0, y_0) = u_y(1 - x_0, y_0) \\ g(x_0, y_0) u_y(x_0, y_0) + f(x_0, y_0) u_x(x_0, y_0) = u_x(1 - x_0, y_0) \end{cases} \quad (49)$$

By adding the first and fourth, and then subtracting the third from the second, we obtain the homogeneous linear system

$$\begin{cases} f(x_0, y_0) u_x(x_0, y_0) + g(x_0, y_0) u_y(x_0, y_0) = 0 \\ -g(x_0, y_0) u_x(x_0, y_0) + f(x_0, y_0) u_y(x_0, y_0) = 0 \end{cases} \quad (50)$$

The determinant of the coefficients is

$$\begin{vmatrix} f(x_0, y_0) & g(x_0, y_0) \\ -g(x_0, y_0) & f(x_0, y_0) \end{vmatrix} = f(x_0, y_0)^2 + g(x_0, y_0)^2 = |\chi(x_0 + iy_0)|^2 \neq 0,$$

for which the aforesaid system admits only the trivial solution $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$. It follows that zero (x_0, y_0) is a critical point for $u(x, y)$ and it is also a critical point for $v(x_0, y_0)$ by virtue of the theorem 14.

If $x_0 = 1/2$ la (45) is written

$$\begin{aligned} \Lambda\left(\frac{1}{2}, y_0\right) \begin{pmatrix} u_x\left(\frac{1}{2}, y_0\right) \\ v_x\left(\frac{1}{2}, y_0\right) \end{pmatrix} &= \begin{pmatrix} -u_x\left(\frac{1}{2}, y_0\right) \\ -v_x\left(\frac{1}{2}, y_0\right) \end{pmatrix} \xrightarrow{\Lambda\left(\frac{1}{2}, y_0\right)=\bar{1}} \begin{pmatrix} u_x\left(\frac{1}{2}, y_0\right) \\ v_x\left(\frac{1}{2}, y_0\right) \end{pmatrix} = \begin{pmatrix} -u_x\left(\frac{1}{2}, y_0\right) \\ -v_x\left(\frac{1}{2}, y_0\right) \end{pmatrix} \\ &\implies \begin{pmatrix} u_x\left(\frac{1}{2}, y_0\right) \\ v_x\left(\frac{1}{2}, y_0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Likewise, from the (45):

$$\begin{pmatrix} u_y\left(\frac{1}{2}, y_0\right) \\ v_y\left(\frac{1}{2}, y_0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let us now study the behavior of the second order partial derivatives of $u(x, y)$ e $v(x, y)$ respectively. For this purpose we derive with respect to x , the first and second members of (43), obtaining

$$\frac{\partial^2 \Lambda}{\partial x^2} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + 2 \frac{\partial \Lambda}{\partial x} \begin{pmatrix} u_x(x, y) \\ v_x(x, y) \end{pmatrix} + \Lambda(x, y) \begin{pmatrix} u_{xx}(x, y) \\ v_{xx}(x, y) \end{pmatrix} = \begin{pmatrix} u_{xx}(1 - x, y) \\ v_{xx}(1 - x, y) \end{pmatrix}$$

which, evaluated on zero (x_0, y_0) of the functions u, v (and as far as we have seen, of their first derivatives), returns:

$$\Lambda(x_0, y_0) \begin{pmatrix} u_{xx}(x_0, y_0) \\ v_{xx}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} u_{xx}(1 - x_0, y_0) \\ v_{xx}(1 - x_0, y_0) \end{pmatrix} \quad (51)$$

To obtain the mixed derivative of the second order, derived with respect to y first and second members of (43), obtaining

$$\frac{\partial^2 \Lambda}{\partial x \partial y} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + \frac{\partial \Lambda}{\partial x} \begin{pmatrix} u_x(x, y) \\ v_x(x, y) \end{pmatrix} + \frac{\partial \Lambda}{\partial y} \begin{pmatrix} u_x(x, y) \\ v_x(x, y) \end{pmatrix} + \Lambda(x, y) \begin{pmatrix} u_{xy}(x, y) \\ v_{xy}(x, y) \end{pmatrix} = \begin{pmatrix} -u_{xy}(1 - x, y) \\ -v_{xy}(1 - x, y) \end{pmatrix}$$

Calculating on (x_0, y_0)

$$\Lambda(x_0, y_0) \begin{pmatrix} u_{xy}(x_0, y_0) \\ v_{xy}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} -u_{xy}(1 - x_0, y_0) \\ -v_{xy}(1 - x_0, y_0) \end{pmatrix} \quad (52)$$

Even if the hypotheses of Schwartz's theorem (on the invertibility of the partial order of derivation) are verified, it is worthwhile to calculate the mixed derivatives u_{yx}, v_{yx} . Then we derive with respect to y , first and second members of (46), obtaining:

$$\frac{\partial^2 \Lambda}{\partial y \partial x} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + \frac{\partial \Lambda}{\partial y} \begin{pmatrix} u_x(x, y) \\ v_x(x, y) \end{pmatrix} + \frac{\partial \Lambda}{\partial x} \begin{pmatrix} u_y(x, y) \\ v_y(x, y) \end{pmatrix} + \Lambda(x, y) \begin{pmatrix} u_{yx}(x, y) \\ v_{yx}(x, y) \end{pmatrix} = \begin{pmatrix} -u_{yx}(1 - x, y) \\ -v_{yx}(1 - x, y) \end{pmatrix}$$

In (x_0, y_0) reduces to

$$\Lambda(x_0, y_0) \begin{pmatrix} u_{yx}(x_0, y_0) \\ v_{yx}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} -u_{yx}(1 - x_0, y_0) \\ -v_{yx}(1 - x_0, y_0) \end{pmatrix} \quad (53)$$

For the partial derivatives of the second order with respect to y , we are going to derive with respect to this variable, the first and second members of the (46), obtaining:

$$\frac{\partial^2 \Lambda}{\partial y^2} \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + 2 \frac{\partial \Lambda}{\partial x} \begin{pmatrix} u_y(x, y) \\ v_y(x, y) \end{pmatrix} + \Lambda(x, y) \begin{pmatrix} u_{yy}(x, y) \\ v_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} u_{yy}(1 - x, y) \\ v_{yy}(1 - x, y) \end{pmatrix}$$

In (x_0, y_0) reduces to

$$\Lambda(x_0, y_0) \begin{pmatrix} u_{yy}(x_0, y_0) \\ v_{yy}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} u_{yy}(1 - x_0, y_0) \\ v_{yy}(1 - x_0, y_0) \end{pmatrix} \quad (54)$$

Before discussing the equations found, let's rewrite them (??)-(47):

$$\begin{aligned} \Lambda(x_0, y_0) \begin{pmatrix} u_x(x_0, y_0) \\ v_x(x_0, y_0) \end{pmatrix} &= \begin{pmatrix} -u_x(1 - x_0, y_0) \\ -v_x(1 - x_0, y_0) \end{pmatrix} \\ \Lambda(x_0, y_0) \begin{pmatrix} u_y(x_0, y_0) \\ v_y(x_0, y_0) \end{pmatrix} &= \begin{pmatrix} u_y(1 - x_0, y_0) \\ v_y(1 - x_0, y_0) \end{pmatrix} \end{aligned} \quad (55)$$

which as seen lead to the cancellation of the derivatives, that is

$$\begin{aligned} u_x(x_0, y_0) &= 0, & u_y(x_0, y_0) &= 0 \\ v_x(x_0, y_0) &= 0, & v_y(x_0, y_0) &= 0 \end{aligned}$$

This derives from the inversion of the sign of the derivative with respect to x (eq. 55). On the other hand, this clearly does not happen for the partial derivative of the second order and more generally of the partial (not mixed) derivatives of even order. In fact, for the second order the (51)-(54) are valid. It follows that these derivatives do not cancel out in any non-trivial zero. Hence (x_0, y_0) is a critical point, and by the theorem 11 it is a saddle point for $u(x, y)$ and $v(x, y)$. ■

6 Proof of the Riemann hypothesis

We are now in a position to prove the conjecture 1.

Proof. We rewrite the second of (13), remembering that $u(x, y)$ and $v(x, y)$ have the same zeros of the real part and the imaginary part of the zeta function:

$$u(x, y) = \operatorname{Re} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}, \quad v(x, y) = \operatorname{Im} \int_0^{+\infty} \frac{t^{-z} dt}{e^t + 1}$$

During the proof of the theorem 3 we obtained:

$$u(x, y) = \int_0^{+\infty} \frac{\cos(y \ln t) dt}{t^x (e^t + 1)}, \quad v(x, y) = - \int_0^{+\infty} \frac{\sin(y \ln t) dt}{t^x (e^t + 1)} \quad (56)$$

Deriving $u(x, y)$ respect to x under the integral sign:

$$u_x(x, y) = \frac{\partial}{\partial x} \int_0^{+\infty} \frac{\cos(y \ln t) dt}{t^x (e^t + 1)} = \int_0^{+\infty} \frac{\partial}{\partial x} \frac{\cos(y \ln t) dt}{t^x (e^t + 1)}$$

That is

$$u_x(x, y) = - \int_0^{+\infty} \frac{\ln t \cos(y \ln t) dt}{t^x (e^t + 1)} \quad (57)$$

Deriving $u(x, y)$ respect to y under the integral sign:

$$u_y(x, y) = - \int_0^{+\infty} \frac{\ln t \cos(y \ln t) dt}{t^x (e^t + 1)} \quad (58)$$

Similarly (or by applying the (34)):

$$v_x(x, y) = \int_0^{+\infty} \frac{\ln t \sin(y \ln t) dt}{t^x (e^t + 1)} \quad (59)$$

$$v_y(x, y) = - \int_0^{+\infty} \frac{\ln t \sin(y \ln t) dt}{t^x (e^t + 1)} \quad (60)$$

Deriving again:

$$u_{xx}(x, y) = \int_0^{+\infty} \frac{\ln^2 t \cos(y \ln t) dt}{t^x (e^t + 1)}, \quad u_{xy}(x, y) = \int_0^{+\infty} \frac{\ln^2 t \sin(y \ln t) dt}{t^x (e^t + 1)} \quad (61)$$

$$u_{yy}(x, y) = - \int_0^{+\infty} \frac{\ln^2 t \cos(y \ln t) dt}{t^x (e^t + 1)}$$

and the like for v_{xx}, v_{xy}, v_{yy} . We separate the derivatives of even order from those of odd order, obtaining with obvious meaning of the symbols:

$$\pm \int_0^{+\infty} \frac{(\ln t)^{2n} \begin{cases} \cos(y \ln t) \\ \sin(y \ln t) \end{cases} dt}{t^x (e^t + 1)}, \quad \pm \int_0^{+\infty} \frac{(\ln t)^{2n+1} \begin{cases} \cos(y \ln t) \\ \sin(y \ln t) \end{cases} dt}{t^x (e^t + 1)}, \quad \forall n \in \mathbb{N} \quad (62)$$

During the proof of the theorem 18 we have seen that in any zero (x_0, y_0) of the zeta function, the partial derivatives of odd order, of the real part and of the imaginary part, must cancel out. From (62)

$$\int_0^{+\infty} \frac{(\ln t)^{2n+1} \begin{cases} \cos(y_0 \ln t) \\ \sin(y_0 \ln t) \end{cases} dt}{t^{x_0} (e^t + 1)} = 0, \quad \forall n \in \mathbb{N}$$

It follows

$$\int_0^{+\infty} \frac{(\ln t)^{2n+1} \cos(y_0 \ln t) dt}{t^{x_0} (e^t + 1)} = 0, \quad \int_0^{+\infty} \frac{(\ln t)^{2n+1} \sin(y_0 \ln t) dt}{t^{x_0} (e^t + 1)} = 0, \quad \forall n \in \mathbb{N}$$

For the symmetry of zeros $(1 - x_0, y_0)$ is also a zero, so:

$$\int_0^{+\infty} \frac{(\ln t)^{2n+1} \cos(y_0 \ln t) dt}{t^{1-x_0} (e^t + 1)} = 0, \quad \int_0^{+\infty} \frac{(\ln t)^{2n+1} \sin(y_0 \ln t) dt}{t^{1-x_0} (e^t + 1)} = 0, \quad \forall n \in \mathbb{N}$$

We have thus obtained the system of infinite identities

$$\left\{ \begin{array}{l} \int_0^{+\infty} \frac{(\ln t)^{2n+1} \cos(y_0 \ln t) dt}{t^{x_0} (e^t + 1)} = \int_0^{+\infty} \frac{(\ln t)^{2n+1} \cos(y_0 \ln t) dt}{t^{1-x_0} (e^t + 1)} \\ \int_0^{+\infty} \frac{(\ln t)^{2n+1} \sin(y_0 \ln t) dt}{t^{x_0} (e^t + 1)} = \int_0^{+\infty} \frac{(\ln t)^{2n+1} \sin(y_0 \ln t) dt}{t^{1-x_0} (e^t + 1)} \end{array} \right., \quad \forall n \in \mathbb{N}$$

It necessarily follows

$$x_0 = 1 - x_0 \implies x_0 = \frac{1}{2}$$

■

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