Make Two 3D Vectors Parallel by Rotating Them Around Separate Axes

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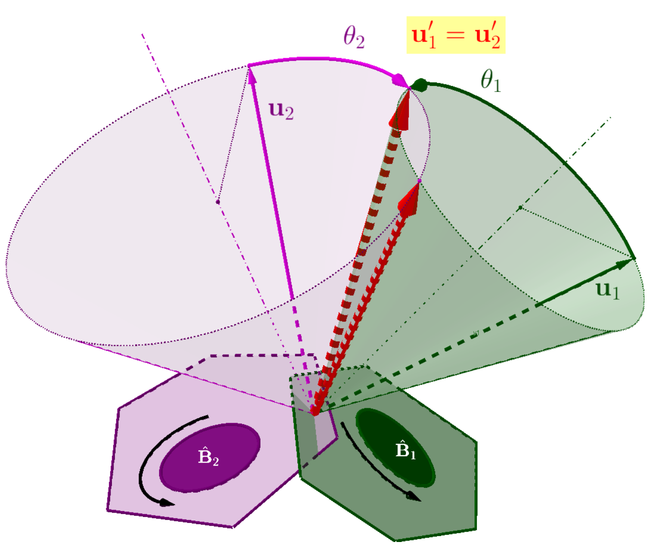
Abstract

To help fill the need for examples of introductory-level problems that have been solved via Geometric Algebra (GA), we show how to calculate the angle through which two unit vectors must be rotated in order to be parallel to each other. Among the ideas that we use are a transformation of the usual GA formula for rotations, and the use of GA products to eliminated variables in simultaneous equations. We will show the benefits of (1) examining an interactive GeoGebra construction before attempting a solution, and (2) considering a range of implications of given information.



1 Introduction

As an example of the sort of problems that can be solved by rotating vectors until they are parallel to each other, consider the three spheres in Fig. 1: the lines from the spheres' centers to the respective points of tangency are parallel to each other, because all three lines are perpendicular to the same plane. In this document, we will learn to use Geometric Algebra (GA) to rotate two vectors in 3D until they are parallel to each other (Fig. 2).

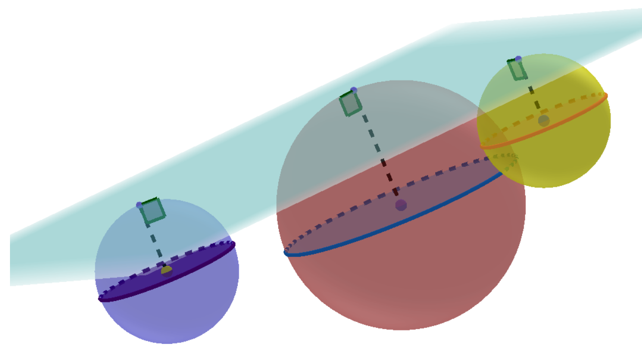


Figure 1: The lines from the spheres' centers to the respective points of tangency are parallel to each other, because all are perpendicular to the same plane.

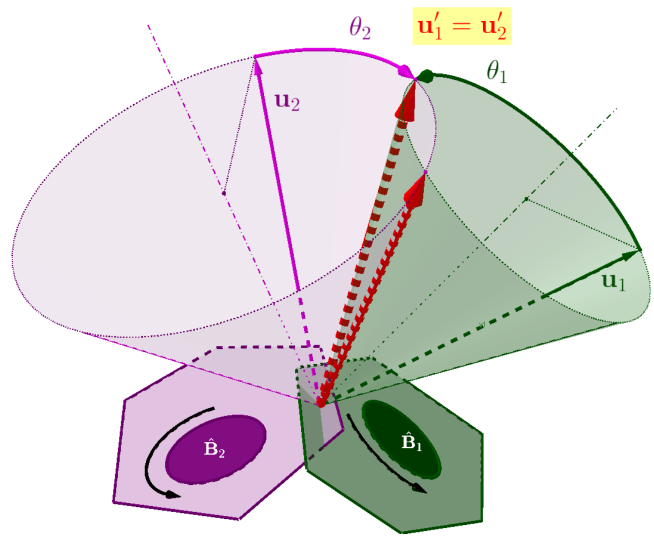


Figure 2: We want to rotate the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ to the positions $\mathbf{u}'$, where they will be parallel to each other. How can we determine the necessary rotation angles (i.e., θ_1 and θ_2)? $\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2$ are the unit bivectors of the respective planes of rotation. There will be two values of θ_1 (one for each position), and two values of θ_2 .

2 Preliminary Observations and Thoughts

From Fig. 2, we can see that there are only two positions at which the rotated vectors will be parallel. θ_1 —the angle through which $\mathbf{u}_1$ must be rotated to reach one of those positions—is unaffected by the initial position of $\mathbf{u}_2$. Similarly, θ_2 is unaffected by the initial position of $\mathbf{u}_2$. Thus, θ_1 and θ_2 are independent of each other. That independence should provide us with clues about how to solve this problem.

There will be two values of θ_1 (one for each position), and two values of θ_2 .

3 Ideas that We will Use

The ideas that we will use include . . .

1. The exponential form of the equation for the vector $\mathbf{w}'$ that results when the vector $\mathbf{w}$ is rotated through the bivector angle $\hat{\mathbf{B}}\phi$ is

$$\mathbf{w}' = \left[e^{-\hat{\mathbf{B}}\frac{\phi}{2}} \right] \mathbf{w} \left[e^{\hat{\mathbf{B}}\frac{\phi}{2}} \right].$$

   That equation reduces ([1], p. 89; [2]) to

$$\mathbf{w}' = \mathbf{w}_\perp + \mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w}) \cos\phi + \mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\, \hat{\mathbf{B}} \sin\phi,$$

   where (Fig. 3) $\mathbf{w}_\perp$ is the component of $\mathbf{w}$ perpendicular to $\hat{\mathbf{B}}$, and the vector $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})$ is the projection of $\mathbf{w}$ upon $\hat{\mathbf{B}}$. Note that the vector $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\, \hat{\mathbf{B}}$ is the 90°rotation of $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})$ in the sense of $\hat{\mathbf{B}}$. Thus, $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\, \hat{\mathbf{B}}$ is parallel to $\hat{\mathbf{B}}$, and $\mathbf{w}_\perp$ is perpendicular to both $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})$ and $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\, \hat{\mathbf{B}}$.

2. $\mathbf{w}_\perp = \left(\mathbf{w} \wedge \hat{\mathbf{B}} \right) \hat{\mathbf{B}}^{-1} = - \left(\mathbf{w} \wedge \hat{\mathbf{B}} \right) \hat{\mathbf{B}}$,
   and $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w}) = \left(\mathbf{w} \cdot \hat{\mathbf{B}} \right) \hat{\mathbf{B}}^{-1} = - \left(\mathbf{w} \cdot \hat{\mathbf{B}} \right) \hat{\mathbf{B}}$.

3. We can use GA products to eliminate variables in simultaneous equations. For example, to find the intersection of the parameterized lines

$$\mathcal{L}_1 : \mathbf{x}_1 = \mathbf{a}_1 + \lambda \hat{\mathbf{w}}_1,$$
$$\mathcal{L}_2 : \mathbf{x}_2 = \mathbf{a}_2 + \gamma \hat{\mathbf{w}}_2,$$

   we equate $\mathbf{x}_1$ and $\mathbf{x}_2$, for the specific values of λ and γ that correspond to the point of intersection:

$$\mathbf{a}_1 + \lambda^* \hat{\mathbf{u}}_1 = \mathbf{a}_2 + \gamma^* \hat{\mathbf{w}}_2.$$

   Then, we eliminate one of the unknowns (λ^* or γ^*) via the outer product. To eliminate γ^*, we would take the outer product of both sides with $\hat{\mathbf{u}}_2$:

$$[\mathbf{a}_1 + \lambda^* \hat{\mathbf{w}}_1] \wedge \hat{\mathbf{w}}_2 = [\mathbf{a}_2 + \gamma^* \hat{\mathbf{w}}_2] \wedge \hat{\mathbf{w}}_2;$$
$$\mathbf{a}_1 \wedge \hat{\mathbf{w}}_2 + \lambda^* \hat{\mathbf{w}}_1 \wedge \hat{\mathbf{w}}_2 = \mathbf{a}_2 \wedge \hat{\mathbf{w}}_2.$$

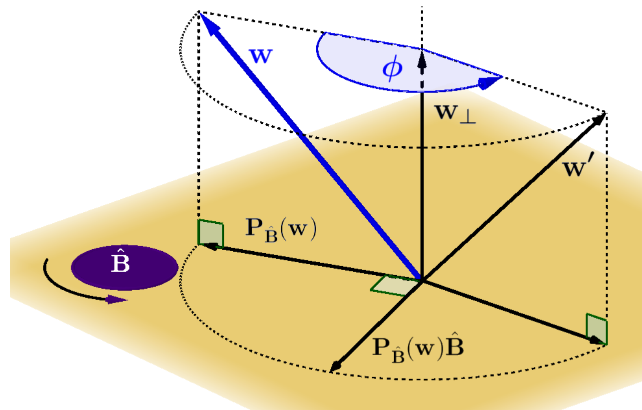


Figure 3: $\mathbf{w}_\perp$ is the component of $\mathbf{w}$ perpendicular to $\hat{\mathbf{B}}$, and $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})$ is the projection of $\mathbf{w}$ upon $\hat{\mathbf{B}}$. Note that $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\,\hat{\mathbf{B}}$ is the 90°rotation of $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})$ in the sense of $\hat{\mathbf{B}}$. Thus, $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\,\hat{\mathbf{B}}$ is parallel to $\hat{\mathbf{B}}$, and $\mathbf{w}_\perp$ is perpendicular to both $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})$ and $\mathbf{P}_{\hat{\mathbf{B}}}(\mathbf{w})\,\hat{\mathbf{B}}$.

4 Solution

We start by expressing the rotations of $\mathbf{u}_1$ and $\mathbf{u}_2$ as

$$\mathbf{u}_1' = \mathbf{u}_{1\perp} + \mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\cos\theta_1 + \mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\sin\theta_1;$$
$$\mathbf{u}_2' = \mathbf{u}_{2\perp} + \mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\cos\theta_2 + \mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2\sin\theta_2.$$

Next, we equate vectors $\mathbf{u}_1'$ and $\mathbf{u}_2'$:

$$\begin{aligned}&\mathbf{u}_{1\perp} + \mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\cos\theta_1 + \mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\sin\theta_1\\ &= \mathbf{u}_{2\perp} + \mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\cos\theta_2 + \mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2\sin\theta_2\,.\end{aligned} \tag{1}$$

The independence of θ_1 and θ_2 should make us look for a way to eliminate one or the other of those angles from Eq. (1). Is there a way to eliminate (for example) both of the θ_2 terms via a geometric product? We can't do so via the outer product, because there is no vector that is parallel to both of the vectors $\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)$ and $\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2$. To use the inner (dot) product, we would need to identify some vector that is *parallel* to both of those vectors. Is there such a vector? Yes: it's $\mathbf{u}_{2\perp}$. Thus, we "dot" both sides of Eq. (1) with $\mathbf{u}_{2\perp}$. After simplifying, we obtain

$$\left\{\mathbf{u}_{2\perp}\cdot\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\right\}\cos\theta_1 + \left\{\mathbf{u}_{2\perp}\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\right]\right\}\sin\theta_1 = \mathbf{u}_{2\perp}\cdot\left[\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}\right]. \tag{2}$$

We now have several options. The route that is perhaps most satisfactory (Ref. [3]) begins by recognizing that only the component of $\mathbf{u}_{2\perp}$ that is parallel to $\hat{\mathbf{B}}_1$ contributes to $\mathbf{u}_{2\perp}\cdot\mathbf{P}_{\hat{\mathbf{B}}_1}$ and $\mathbf{u}_{2\perp}\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\right]$. Thus, Eq. (2) becomes

$$\begin{aligned}&\left\{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\cdot\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\right\}\cos\theta_1 + \left\{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\right]\right\}\sin\theta_1\\ &\qquad= \mathbf{u}_{2\perp}\cdot\left[\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}\right]\,.\end{aligned}$$

Again following [3], we recognize that the coefficients of the trigonometric functions of θ_1 can be transformed into the sine and cosine of an angle that we shall call α_1. First, we divide both sides by the product $\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|$:

$$\frac{\left\{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\cdot\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\right\}}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\cos\theta_1+\frac{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\sin\theta_1 = \frac{\mathbf{u}_{2\perp}\cdot[\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}]}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\ .$$

Now, we recognize that $\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\| = \|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|$. We also recognize that we may define

$$\cos\alpha_1 = \frac{\left\{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\cdot\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\right\}}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\ ,$$

and

$$\sin\alpha_1 = \frac{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\,\hat{\mathbf{B}}_1\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\ .$$

Thus,

$$\cos\theta_1\cos\alpha_1+\sin\theta_1\sin\alpha_1 = \frac{\mathbf{u}_{2\perp}\cdot[\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}]}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\ ;$$

$$\cos(\theta_1-\alpha_1) = \frac{\mathbf{u}_{2\perp}\cdot[\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}]}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\ ;$$

A trigonometric identity: $\cos(\gamma-\beta) = \cos\gamma\cos\beta+\sin\gamma\sin\beta$.

and

$$\theta_1 = \alpha_1 \pm \cos^{-1}\left\{\frac{\mathbf{u}_{2\perp}\cdot[\mathbf{u}_{2\perp}-\mathbf{u}_{1\perp}]}{\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{2\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_1)\|}\right\}. \quad (3)$$

We've now identified the angles(s) θ_1. To find the two values of θ_2, we would "dot" both sides of Eq. (1) with $\hat{\mathbf{u}}_{1\perp}$ to eliminate the θ_1 terms, thereby obtaining

$$\left\{\mathbf{u}_{1\perp}\cdot\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\right\}\cos\theta_2+\left\{\mathbf{u}_{1\perp}\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2\right]\right\}\sin\theta_2 = \mathbf{u}_{1\perp}\cdot[\mathbf{u}_{1\perp}-\mathbf{u}_{2\perp}]\ . \quad (4)$$

We recognize that only the component of $\mathbf{u}_{1\perp}$ that is parallel to $\hat{\mathbf{B}}_2$ contributes to $\mathbf{u}_{1\perp}\cdot\mathbf{P}_{\hat{\mathbf{B}}_2}$ and $\mathbf{u}_{1\perp}\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2\right]$. Thus, Eq. 4 becomes

$$\left\{\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_{1\perp})\cdot\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\right\}\cos\theta_2+\left\{\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_{1\perp})\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2\right]\right\}\sin\theta_2 = \mathbf{u}_{1\perp}\cdot[\mathbf{u}_{1\perp}-\mathbf{u}_{2\perp}]\ .$$

We also recognize that $\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_2\| = \|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\|$, and proceed as we did in finding θ_1.

$$\frac{\left\{\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_{1\perp})\cdot\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\right\}}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_{1\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\|}\cos\theta_1+\frac{\mathbf{P}_{\hat{\mathbf{B}}_1}(\mathbf{u}_{1\perp})\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\,\hat{\mathbf{B}}_1\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_{1\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\|}\sin\theta_2 = \frac{\mathbf{u}_{1\perp}\cdot[\mathbf{u}_{1\perp}-\mathbf{u}_{2\perp}]}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_{1\perp})\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}(\mathbf{u}_2)\|}\ .$$

Now, we define

$$\cos\alpha_2 = \frac{\left\{\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\cdot\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\right\}}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\|},$$

and

$$\sin\alpha_2 = \frac{\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\cdot\left[\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\hat{\mathbf{B}}_2\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\|}.$$

Thus,

$$\cos\theta_2\cos\alpha_2+\sin\theta_2\sin\alpha_2 = \frac{\mathbf{u}_{1\perp}\cdot\left[\mathbf{u}_{1\perp}-\mathbf{u}_{2\perp}\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\|};$$

$$\cos\left(\theta_2-\alpha_2\right) = \frac{\mathbf{u}_{1\perp}\cdot\left[\mathbf{u}_{1\perp}-\mathbf{u}_{2\perp}\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\|};$$

and

$$\theta_2 = \alpha_2 \pm \cos^{-1}\left\{\frac{\mathbf{u}_{1\perp}\cdot\left[\mathbf{u}_{1\perp}-\mathbf{u}_{2\perp}\right]}{\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_{1\perp}\right)\|\|\mathbf{P}_{\hat{\mathbf{B}}_2}\left(\mathbf{u}_2\right)\|}\right\}. \quad (5)$$

5 Discussion

5.1 Observations and Lessons

This study has shown the benefits of constructing and exploring interactive visual representations of a problem before attempting a solution. For example, our examination of the GeoGebra construction that was used for Fig. 2 revealed the independence of θ_1 and θ_2. We also saw the benefits of considering a range of implications of given information. For example, we solved the problem readily when we equated the "expanded" versions of the rotations (Eq. (1)), but would have had much more difficulty making use of $\mathbf{u}_1' \wedge \mathbf{u}_2' = 0$.

5.2 Questions for Further Explanation

1. In Fig. 4, the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ cannot be made parallel. How would that situation be reflected in Eqs. (3) and (5)?

2. (Fig. 5) How can we use our solution to find the rotation needed to make $\mathbf{u}_1$ parallel to $\hat{\mathbf{M}}$?

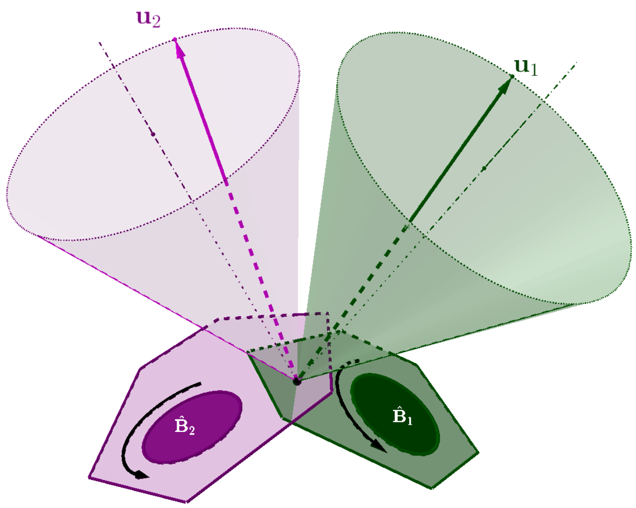


Figure 4: In the case shown here, the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ cannot be made parallel. How would that situation be reflected in Eqs. (3) and (5)?

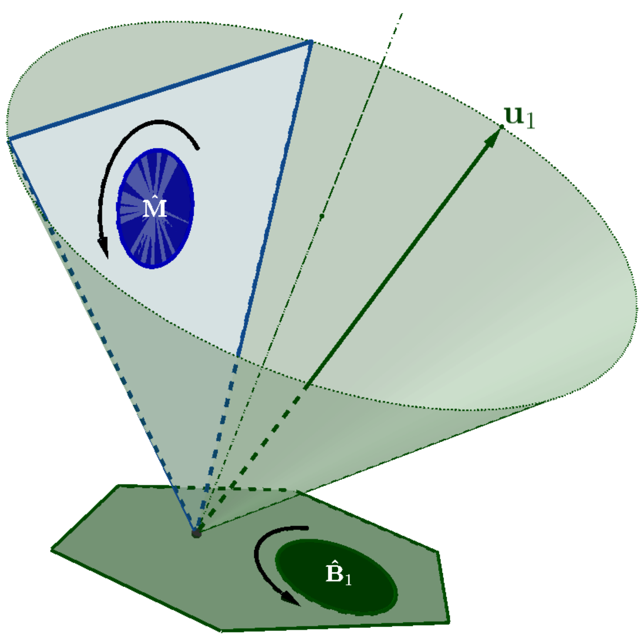


Figure 5: How can we use our solution to find the rotation needed to make $\mathbf{u}_1$ parallel to $\hat{\mathbf{M}}$?

References

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