Combinatorial Twelvefold way, Statistical mechanics and Inclusion Hypothesis

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Abstract

There are three different ways of counting microstates for indistinguishable particles and distinguishable energy levels. Two of them correspond to Bosons and Fermions (and anyons, which interpolate between the two), but the third one, which is not considered so far, is when we require a 'dual' of the Exclusion Principle to hold: in each energy level (state) there must exist at least one particle. I call this 'the Inclusion Hypothesis' and propose the statistics as a possibility of existence of a third kind of particles.

The history of the form of Statistical Mechanics can be seen as a continuous move in delicacy in counting the microstates, which is usually stated in terms of ‘balls in boxes’ approach. Depending on the distinguishability of the balls (particles) and boxes (energy units), different kinds of counting are possible.
Combinatorists have devised of the following table, called *The Twelvefold Way*[1], which summaries all the possibilities:

$f : [k] \to [n]$	ALL FUNCTIONS	INJECTIONS	SURJECTIONS
DIST DIST	① n^k $n^k = n \cdot n^{k-1}$	② $n^{\underline{k}}$ $n^{\underline{k}} = (n-k+1)\, n^{\underline{k-1}}$	③ $\operatorname{surj}(k,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$ $\operatorname{surj}(k,n) = n \operatorname{surj}(k-1,n-1) + n \operatorname{surj}(k-1,n)$
IND DIST	④ $\left(\!\binom{n}{k}\!\right) = \frac{n^{\overline{k}}}{k!}$ $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$	⑤ $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$ $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$	⑥ $\left(\!\binom{n}{k-n}\!\right) = \binom{k-1}{n-1}$
DIST IND	⑦ $\left\{ {k \atop 1} \right\} + \left\{ {k \atop 2} \right\} + \cdots + \left\{ {k \atop n} \right\}$	⑧ $\begin{bmatrix} 1 & \text{if } k \le n \\ 0 & \text{otherwise} \end{bmatrix}$	⑨ $\left\{ {k \atop n} \right\} = \frac{\operatorname{surj}(k,n)}{n!}$ $\left\{ {k \atop n} \right\} = \left\{ {k-1 \atop n-1} \right\} + n \left\{ {k-1 \atop n} \right\}$
IND IND	⑩ $p_{\le n}(k) = p_1(k) + p_2(k) + \cdots + p_n(k)$	⑪ $\begin{bmatrix} 1 & \text{if } k \le n \\ 0 & \text{otherwise} \end{bmatrix}$	⑫ $p_n(k)$ $p_n(k) = p_{n-1}(k-1) + p_n(k-n)$

Figure 1: *The Twelvefold Way.* Table from here.

Maxwell-Boltzmann Statistics is the number (1) entry. Bose-Einstein number (4), and Fermi-Dirac number (5).

The natural question is then whether *other (so far-unused) entries might have significance in physics* (statistical mechanics).

Unlike rows one and two, rows three and four are not so easy to interpret physically, as they both involve *indistinguishability of energies* (boxes). I shall not therefore deal with them here.
The focus of this note is on the entry number (6):

$$\binom{k-1}{n-1}$$

is the number of ways of putting k indistinguishable particles into n distinguishable cells of phase space i.e. energy levels such that **each energy level has *at least* one particle.**
This is like the ‘dual’ of the Exclusion Principle (Pauli), according to which each energy level has *at most* one particle.
For this reason, and since in this kind of counting we are ‘including’ all energy levels in having particles, I call this the *Inclusion Hypothesis*. Promotion to a *principle* is not legit unless one can extract something ‘useful’ from it and verify it empirically.

Following the conventions and notation of [2], the number of microstates of distributing n_i identical indistinguishable particles among g_i energy levels, such that the Inclusion Hypothesis is obeyed, is

$$\boxed{W = \prod_i \frac{(n_i - 1)!}{(g_i - 1)!(n_i - g_i)!}} \tag{1}$$

subject to

$$\sum_i n_i \epsilon_i = E, \tag{2}$$

$$\sum_i n_i = N. \tag{3}$$

Following a procedure similar to what one does for other statistics, we need to extremize the function

$$f(n_i) = \log W + \alpha \left(N - \sum_i n_i \right) + \beta \left(E - \sum_i n_i \epsilon_i \right), \tag{4}$$

where α, β are Lagrange multipliers.
The result is

$$\boxed{\frac{n_i - 1}{n_i - g_i} = e^{(\epsilon_i - \mu)/k_B T}} \tag{5}$$

One of the important questions for establishment is now that of the status of this new statistics in terms of the exchange symmetry of the many-body wavefunction. Recall that in general we have

$$\psi(r_1, r_2) = e^{i\alpha} \psi(r_2, r_1) \tag{6}$$

for two particles with positions r_1 and r_2. For Fermions $\alpha = \pi$ one has

$$\psi(r_1, r_2) = -\psi(r_2, r_1),$$

and for Bosons $\alpha = 0$,

$$\psi(r_1, r_2) = \psi(r_2, r_1).$$

As a consequence of Fermions' anti-symmetry, if one tries to put two identical fermions in the same state, the wavefunction vanishes. This argument is in fact a *reductio ad absurdum*: according to the Exclusion Principle one has at most one particle in each state. Suppose the contrary. Then there should exist a state with at least *two* particles.
This 'two' is critical, as it enables one to use a *two*-body wavefunction.

Now to consider our new statistics, if we are to follow the same form of reasoning we should suppose the contrary of the Inclusion Hypothesis. In that case there should exist a state with *no* particles. But a state with no particles is not possible to express in terms of a wavefunction.
This means that the *many-body wavefunction formalism is not able to handle the Inclusion Hypothesis.*

I consider this note merely as a 'timestamp' and being drained of motivation, do not pursue the following vital questions:

- Status of the Spin-Statistics theorem in this regard;
- Physical meaning of the fact that unlike for Bosons and Fermions the occupation number (5) is *not proportional* to the degeneracy factor g_i;
- Objective existence of particles that obey this statistics, and how to look for them;
- Whether this third kind of particle can be considered a special case of Bosons: In case of Bosons, a state can be occupied by *any* number of particles. In this new statistics, again, any number of particles is allowed *as long as* it is strictly larger than zero. This *can* mean that the particles that obey this new statistics are a special sub-species of Bosons.

References

[1] Richard P. Stanley. *Enumerative Combinatorics, Volume I.* Cambridge University Press, 1997.

[2] R. K. Pathria and Paul D. Beale. *Statistical Mechanics.* Academic Press, 2011.