The Arrew theorem prover

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Abstract

Arrew (Arrow Rewriter) is a mathematical system (theorem prover) that allows expressing and working with formal systems. It relies on a simple substitution rule and set equality to derive theorems.

***K*eywords** Theorem prover, mathematical system

1 Mathematical system

A formal system is defined by the tuple $F = (R, V, T)$ together with the functions $subst^n_{rule}$ and $subst^n_{thm}$ where $R_n \in R$ is a set of rules of n-ary arguments, V is a set of variables, and T is a set of theorems. A rule $r = (r_1, \ldots, r_n) \in R_n$ is a sequence of string of symbols; it can be roughly interpreted as a function $r_1 \to \ldots \to r_n$, where the n-th argument represents a conclusion, and the others represent hypotheses.

Let $S \subseteq V \times T$ denote a set of substitutions, and $X[t/v]$ denote the expression X in which each occurrence of v is replaced with t. We define the following function which performs substitution on a rule's hypotheses and conclusion:

$$subst^n_{rule}(r, S) = \begin{cases} subst^n_{rule}(r_1[t/v], \ldots, r_n[t/v], S \setminus \{(v, t)\}), & r = (r_1, \ldots, r_n) \wedge (v, t) \in S \\ r & S = \emptyset \end{cases}$$

Let $h = (h_1, \ldots, h_{n-1})$ where $\forall i, h_i \in T$. The function $subst^{n-1}_{thm}(h, S)$ is defined similarly.

For deriving new theorems, we say that $t = subst^1_{rule}((r_n), S) \in T$ (i.e., t is a theorem) if and only if:

$$subst^{n-1}_{rule}((r_1, \ldots, r_{n-1}), S) = subst^{n-1}_{thm}(h, S)$$

Terms and axioms are represented as 1-ary rules; note that for $n = 1$ we have $subst^0_{rule}((), S) = () = subst^0_{thm}((), S)$ i.e. all 1-ary rules are theorems: $\forall r, r \in R_1 \to r \in T$.

2 Example

Let $R = \{\{\vdash \texttt{MI}, \texttt{I}\}, \{(\vdash \texttt{Mx}, \vdash \texttt{Mxx})\}\}$, $V = \{\texttt{x}\}$. The particular choice of R_1 allows us to pick $S = \{(\texttt{x}, \texttt{I})\}$; since $\texttt{I}$ is a 1-ary rule, $\texttt{I} \in T$. Similarly, $\vdash \texttt{MI} \in T$. To prove $\vdash \texttt{MII} \in T$, we use the rule within R_2 and since $(\texttt{x}, \texttt{I}) \in S$, we get that $subst^1_{rule}((\vdash \texttt{Mx}), S) =\vdash \texttt{MI} = subst^1_{thm}((\vdash \texttt{MI}), S)$. Since the rule's arguments match the theorem's hypotheses, $subst^1_{rule}((\vdash \texttt{Mxx}), S) =\vdash \texttt{MII} \in T$.

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