Least Common Multiple and Optimization

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ABSTRACT We reduce finding of Least Common Multiple of integer numbers to polynomial-time integer optimization problems and to NP-hard integer optimization problems.

1. Introduction

In arithmetic and number theory, the least common multiple, lowest common multiple, or smallest common multiple of two integers p and q, usually denoted by lcm(p, q), is the smallest positive integer that is divisible by both p and q (see e.g. [8]), e.g., lcm(6, 9) = 6×3 = 9×2 = 18. Correspondingly, lowest common multiple, or smallest common multiple of three integers p, q and r, usually denoted by lcm(p, q, r), is the smallest positive integer that is divisible by p, q and r, e.g., lcm(3, 6, 9) = 3×6 = 6×3 = 9×2 = 18, etc.

We will reduce the problem of finding of the Least Common Multiple of three integer numbers to the following two integer minimization problems, wherein the first one is polynomial-time problem and another one is NP-hard problem (see also [1, 4]).

2. Reducing to polynomial-time linear minimization problem

Theorem 1. *The problem of finding the Least Common Multiple of three integer numbers: $p > 0$, $q > 0$ and $r > 0$ can be reduced to the following three-dimensional linear integer minimization problem:*

$$lcm(p, q, r) = \{ \min \ (px - qy) + (px - rz) + px, \quad \text{subject to} \quad px - qy \geq 0, \ px - rz \geq 0, \quad x \leq s, \ y \leq s, \ z \leq s, \ s = p + q + r, \quad x, y, z, p, q, r \in \mathbf{N} \}. \tag{1}$$

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Proof. It follows from the Least Common Multiple's definition. □

Due to Lenstra [10], minimizing a linear function over the integer points in a polyhedron is solvable in polynomial time provided that the number of integer variables is a constant(see also Del Pia et al. [2, 3], Hemmecke et al. [7]). Thus, we can obtain the following

Theorem 2. Problem (1) is a polynomial-time problem.

Remark 1. Note, that results, similar to Theorem 1 and Theorem 2 can be obtained for finding Least Common Multiples of more than three integer numbers. In case of two integer numbers, we can obtain the following

Theorem 3. *The problem of finding the Least Common Multiple of two integer numbers: $p > 0$, $q > 0$ can be reduced to the following two-dimensional linear integer minimization problem:*

$$lcm(p, q) = \{ \min\ (px - qy) + px, \quad \text{subject to} \quad px - qy \geq 0, \quad x \leq s,\ y \leq s,\ s = p + q,\ x, y, p, q \in \mathbf{N} \}. \tag{2}$$

Proof. It follows from the Least Common Multiple's definition. □

Correspondingly, we can obtain the following

Theorem 4. Problem (2) is a polynomial-time problem.

The following can be obtained as well:

Theorem 5. *The problem of finding the Least Common Multiple of two integer numbers: $p > 0$, $q > 0$ can be reduced to the following two-dimensional integer quadratic minimization problem:*

$$lcm(p, q) = \{ \min\ (px - qy)^2 + p^2x^2, \quad \text{subject to} \quad x \leq s,\ y \leq s,\ s = p + q,\ x, y, p, q \in \mathbf{N} \}, \tag{3}$$

Proof. It follows from the Least Common Multiple's definition. and due to monotonicity(strictly increasing) of power function. □

Correspondingly, we can obtain the following

Theorem 6. *Problem (3) is a polynomial-time problem.*
Proof. It follows due to Del Pia and Weismantel [2], where they show that Integer Quadratic Programming can be solved in polynomial time in the plane. □

3. Reducing to NP-hard nonlinear minimization problem

On the other hand, the problem of finding the Least Common Multiple of three integer numbers: $p > 0$, $q > 0$ and $r > 0$ can be reduced to the following nonlinear three-dimensional integer minimization problem:

Theorem 7. *The problem of finding the Least Common Multiple of three integer numbers: $p > 0$, $q > 0$ and $r > 0$ can be reduced to the following three-dimensional nonlinear integer minimization problem:*

$$lcm(p, q, r) = \{ \min \ (p^2x^2 - q^2y^2)^2 + (p^2x^2 - r^2z^2)^2 + p^4x^4 , \quad \text{subject to} \quad x \le s,\ y \le s,\ z \le s,\ \ s = p + q + r,\quad x, y, z, p, q, r \in \mathbf{N} \}. \tag{4}$$

Proof. It follows from the Least Common Multiple's definition and monotonicity(strictly increasing) of power function. □

Remark 2. Note, that results, similar to Theorem 7 can be obtained for finding Least Common Multiples of more than three integer numbers. In case of two integer numbers, we can correspondingly obtain the following

Theorem 8. *The problem of finding the Least Common Multiple of two integer numbers: $p > 0$, $q > 0$ can be reduced to the following two-dimensional nonlinear integer minimization problem:*

$$lcm(p, q) = \{ \min \ \ (p^2x^2 - q^2y^2)^2 + p^4x^4, \tag{5}$$
$$\text{subject to}$$

$$x \leq s, y \leq s, \; s = p + q, x, y, p, q \in \boldsymbol{N}\}.$$

Proof. It follows from the Least Common Multiple's definition and monotonicity(strictly increasing) of power function. □

Theorem 9. *Problem (5) is a polynomial-time problem.*
Proof. It follows due to Del Pia et al. [3], where they show that the problem of minimizing a homogeneous polynomial of any fixed degree over the integer points in a bounded rational polyhedron is solvable in polynomial time in the plane(Theorem 1.6). □

Despite in general, Integer Programming is NP-hard or even incomputable (see, e.g., Hemmecke et al. [7]), for some subclasses of target functions and constraints it can be computed in time polynomial.

Note that the dimension of the problem (4) is fixed and is equal to 3.

A fixed-dimensional polynomial minimization in integer variables, where the objective function is a convex polynomial and the convex feasible set is described by arbitrary polynomials can be solved in time polynomial(see, e.g ., Khachiyan and Porkolab [8]).

A fixed-dimensional polynomial minimization over the integer variables, where the objective function is a quasiconvex polynomial with integer coefficients and where the constraints are inequalities with quasiconvex polynomials of degree at most ≥ 2 with integer coefficients can be solved in time polynomial in the degrees and the binary encoding of the coefficients(see, e.g., Heinz [6], Hemmecke et al. [7], Lee [9]).

Minimizing a convex function over the integer points of a bounded convex set is polynomial in fixed dimension, according to Oertel et al. [11].

Del Pia and Weismantel [2] showed that Integer Quadratic Programming can be solved in polynomial time in the plane.

It was further generalized for cubic and homogeneous polynomials in Del Pia et al. [3].

However, according to

Theorem 10 (Hemmecke et al. [7], Del Pia et al. [3]). *The problem of minimizing a degree-4 polynomial over the lattice points of a convex polygon is NP-hard.*

Furthermore,

Proposition 1 (Del Pia et al. [3]). *The problem of minimizing a function f over the integer points in n-dimensional rational polyhedron is NP-hard when f is a homogeneous polynomial of degree d with integer coefficients,* $n \geq 3$ *and* $d \geq 4$ *are fixed, even when rational polyhedron is bounded.*

Thus, we can obtain the following

Theorem 11. *Problem (4) is NP-hard problem.*
Proof. It follows from the Proposition 1, since problem (4) is a problem of minimizing a degree $d = 4$ homogeneous polynomial with integer coefficients over the integer points in a bounded three-dimensional ($n = 3$) rational polyhedron. □

As a result, since we reduced the same problem to polynomial-time problem (1) and to NP-hard problem (4), we would conclude that P = NP, as since if there is a polynomial-time algorithm for any NP-hard problem then there are polynomial-time algorithms for all problems in NP (see [1, 4]).

4. Conclusion

We reduced the problem of finding of the Least Common Multiple of three integer numbers to two integer minimization problems: linear three-dimensional polynomial-time integer minimization problem and nonlinear three-dimensional NP-hard integer minimization problem and concluded that it would mean that P = NP.

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