In defense of Real Numbers A critique of finitism

Commie Cantor

No one shall expel us from the paradise that Cantor has created for us.

David Hilbert

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If you study mathematics you are probably aware of the foundational crises that mathematics went through at the beginning of the 20th century. The three broad schools of thought namely constructivism, intuitionism and formalism collided and judging by the approach used today by most mathematicians, we can easily say that formalism emerged victorious in some sense. However while debates regarding the foundations of mathematics have subsided over the years, they aren't dead. One such school of mathematics which still sees considerable traffic is finitism. In this article, we will be analysing the criticism of a finitist named Norman J Wildberger and trying to defend the current axiomatic mathematical systems against them.

1 History of finitism

Finitism as a philosophy derives many of its points from the school of Intuitionism introduced by L.E.J. Brouwer. During the Crisis of foundations, generally, all agreed with finite mathematical objects like Natural Numbers however they rejected infinite objects and Cantor's transfinite.
Leopold Kronecker famously said, "God created the integers; all else is the work of man.". Even Ludwig Wittgenstein's works have some affinity to finitism and his famous quote :

> "For if one person can see it as a paradise . . ., why should not another see it as a joke?"

Later, finitism was formalised a bit and there were several thoughts inside finitism itself. Here I am covering the two broad ones.

1.1 Classical Finitism vs Strict Finitism vs Ultrafinitism

$\rightarrow$ **Classical finitism** : Classical finitists allow for the Aristotelian "Potential infinity". Potential infinity is the concept that there are certain objects such as a line or a set of numbers which can go on and on in an infinite space. Finite space is capture only a finite number of them though. For example, a line can always be extended and given that we have an infinite amount of space, it can be extended infinitely. On earth, however, the space is finite, hence it can only be extended finitely. Historically, mathematics has had a classically finite point of view until Cantor transfinite cardinals.

$\rightarrow$ **Strict finitism or Ultrafinitism** Ultrafinitists reject the notion of potential infinities. But more than that, they only accept numbers that are physically realizable. For example, we cant construct a number like $1000000^{1000000^{1000000}} + 3221$ in practice or work with it computationally.
Alexander Esenin-Volpin, who is a Russian poet and a mathematician worked from 1059 to 2016 to make ZFC consistent with Ultrafinitism.

2 Who is Norman J Wildberger?

Norman J Wildberger is a retired professor of mathematics from the University of South Wales. He also has a Youtube channel named "Insights of Mathematics" [1] on which he has uploaded more than 1000 videos on various topics ranging from Foundation Of mathematics, Rational Geometry, Group theory, and Algebraic Topology.

3 Wildberger's views of the state of mathematics

Norman Wildberger is known mostly because of his unorthodox views regarding the foundations of mathematics. Even though he hasn't used a label to describe his school of thought, he is very close to an ultrafinitist. In his view, the "status quo" real number system is founded on very shaky foundations and doesn't actually exist. In any real usage, we simply use approximations of so-called real numbers. To quote him :

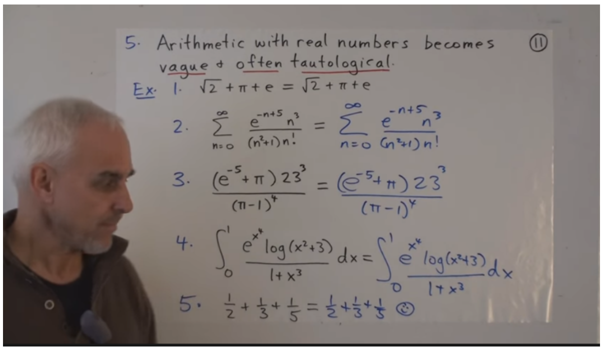
> " While engineers and scientists work primarily with finite decimal numbers in an approximate sense, "real numbers" as infinite decimals are idealized objects which attempt to extend the explicit finite but approximate numbers of engineers into a domain where infinite processes can be ostensibly be exactly evaluated. To make this magic work, mathematicians invoke a notion of 'equivalence classes of Cauchy sequences of rational numbers, or as 'Dedekind cuts'. [2] "

However, he doesn't stop here and goes one step forward to reject any sort of infinity whether it be a notion of a set of all natural numbers or infinite sequences.
Foundationally, Norman wants to alter the basic axioms to make maths a bit closer to computational science. This means getting rid of what he calls "uncomputable concepts" such as irrational numbers, infinite sequences and even "large" numbers such a

$$10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}} + 23$$

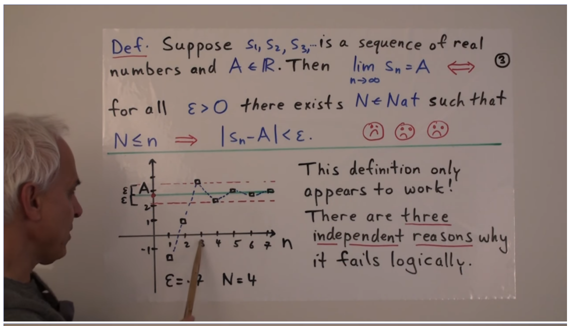
For the next section, I have gone over a lot of his videos and gathered some of his talking points. There are many videos which I couldn't go over, but I believe I will address the majority of his points by responding to these three criticisms that he has posed.

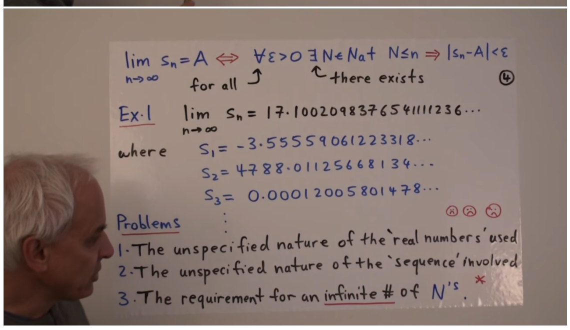
3.1 Criticism 1:Arithmetic with Real numbers is vague and tautological [3]



Norman points out that often irrational numbers lead to tautological arithmetic where they don't necessarily have a definite answer.
One of the most famous example is $\pi + e + \sqrt{2}$

3.2 Criticism 2: Supposed logical difficulties with the current definition of limits [4]





The three "logical inconsistencies" that Norman points out with the current definition of limits are as follows:

→ Real numbers in the sequence are unspecified since all the digits aren't written down and are just represented with three dots at the end.

→ Sequence itself is unspecified as they do not require a law, a procedure or an algorithm to generate it. (Outright false by definition)

→ We need to calculate the definition for an infinite number of N and an infinite number of epsilons which is computationally impossible.

3.3 Criticism 3: It is never possible to list all elements of infinite sequences [2]

Norman says and I quote :

> When we make the jump to "infinite sequences", such as a sequence somehow implied by the expression "m = 3,5,7," the situation changes dramatically. It is never possible to explicitly list "all the elements" of an infinite sequence. Instead, we are forced to rely on a rule generating the sequence to specify it. In this case perhaps: m is the list of all odd numbers starting with 3, or perhaps: m is the list of all odd primes. Without such a rule, a definition like "m = 3,5,7," is really rather meaningless.

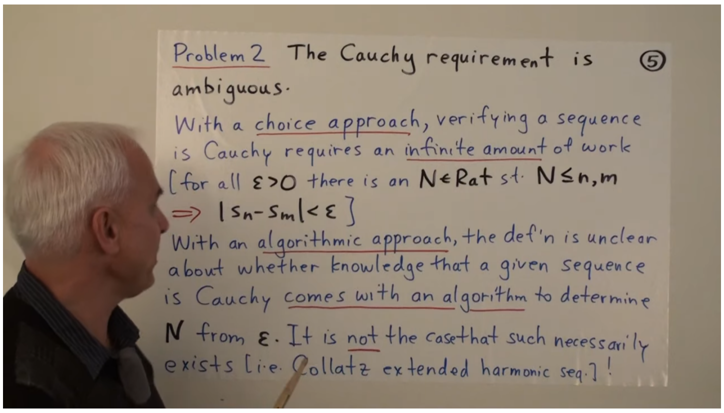
Here is another similar point about sets:

> An example might be: let S be the set of all odd perfect numbers less than 10^{100000}. [A perfect number, like 6 and 28, is the sum of those of its divisors less than itself, i.e. 6=1+2+3 and 28=1+2+4+7+14.] Such a description of S does not deserve to be called a specification of the set, at least not with our current understanding of perfect numbers, which doesn't even allow us to determine if S is empty or not.

3.4 Criticism 4: Cauchy sequences are ambiguous [5]

In the current system, we define real numbers as an equivalence class of Cauchy sequences. But Norman claims that Cauchy sequences themselves are ambiguous and have frail foundations. Here are his specific criticisms.

→ Proving a sequence is Cauchy requires an infinite amount of work.
→ The definition of the Cauchy sequence does not come with an algorithm to generate N from epsilon



4 My response

My response to Norman's objections to the current state of mathematics is broadly divided into two parts. First I want to go over the three specific criticisms, and then a general critique of Norman's theories.

4.1 Response to criticisms

$\rightarrow$ A broad theme among a lot of criticisms of Norman is his not understanding the idea of current mathematics is defining objects through their properties. If we believe in the theory of real numbers being an equivalence class of Cauchy sequences, we can easily prove that $\pi + e + \sqrt{2}$ is also an equivalence class of Cauchy sequence and hence a real number. Just because we can't explicitly list the number, doesn't stop us from making analytical deductions about the number.

z$\rightarrow$ I find his objection to the theory of limits most absurd simply because he misrepresents the current understanding of sequences as not requiring a procedure of algorithms to generate elements of sequences. Any good analysis text will tell you that a sequence is not a list of numbers written explicitly but in fact a function from $N \rightarrow R$ which produces n^{th} in the sequence.

$\rightarrow$ His next point about the definition asking us to check for infinite ϵ and hence asking us to do infinite operations is also wrong. He is writing about one thing which is that we can't do infinite operations. To get around this difficulty, we have to find an explicit relation between N and ϵ that satisfies the limit definition. The same goes for Cauchy sequences.

$\rightarrow$ Working with a rule to get the terms of the sequence doesn't have any disadvantage over explicitly listing them out. It makes the process of proving properties regarding the sequences much easier and convenient. So I don't get the gripe Norman has over accepting that sequences are in fact just functions from a set of Natural Numbers to a set of Real Numbers instead of believing that they should be explicitly listed out.

$\rightarrow$ Regarding the last point of Cauchy sequences not coming with an algorithm to find N from ϵ ; I would say that the construction of the Cauchy sequence does ensure that an N exists for each ϵ which means that there exists a function $\delta(\epsilon)$ such that δ takes ϵ and produces N. Finding that function might be tricky. It could be a piecewise function or a linear function, that depends on the sequence itself. But what we are sure of is that there does exist such function.

4.2 General critique of Normans theory

As much as Norman hates admitting it, I believe that this whole debate is very philosophical in nature. There is no practical advantage of restricting us in the finitist framework and bringing us closer to the "reality" of the world. The computer scientists and physicists of the world will continue using numerical methods and approximations to compute real-life problems. Pure mathematics as it is used today however provides a nice framework to solve physical problems analytically for things you can't simulate.
This brings us to our next question. What are the good foundations of mathematics? Does being closer to reality mean your foundations are better? Here as you might expect, I sorely disagree. In fact, I can go a step beyond and say that foundations in themselves are not very important. You can use whatever foundations, be it finitist, constructivist etc and the real measure of those foundations would be whether you are able to produce results from it that help you do things you want to do. And I believe the current infinitesimal analysis does exactly that.
Infinitesimal analysis and the theory of infinite sets is one of the most successful theories mankind has invented. It has provided us with sophisticated tools to understand pi, differential equations, fixpoint theorems, power series, convergence in power series etc.
Another reason why I don't believe that we should try to restrict ourselves is that the current system of analysis just works! We can do everything in it that we will be able to do in a finite system. So unless he is able to produce a serious crisis in the foundations, like the Russel paradox, it feels useless to talk about it. Also, when I say that there is a Cauchy sequence which is a solution to a particular equation, what he is really hearing is that we can come up with better and better approximations for the solutions. So, what serious benefit do we really get by changing the foundations to put this thought down formally when both of us can easily work in this system.

5 References

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