# A LOWER BOUND FOR MULTIPLE INTEGRAL OF NORMALIZED LOG DISTANCE FUNCTION IN $\mathbb{R}^n$

T. AGAMA

Abstract. In this note we introduce the notion of the local product on a sheet and associated space. As an application, we prove that for $\langle a,b\rangle > e^e$ then

$$\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\left|\log\left(i\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\right)\right|dx_1dx_2\cdots dx_n \geq \frac{\left|\prod_{j=1}^{n}|b_j|-|a_j|\right|}{\log\log(\langle a,b\rangle)}$$

for all $s\in\mathbb{N}$, where $\langle,\rangle$ denotes the inner product and $i^2=-1$.

## 1. Introduction

There is hardly no formal introduction to the concept of an inner product and associated space in the literature. The inner product space is usually a good place to go for a wide range of mathematical results, from identities to inequalities. In this situation, the best potential result is frequently obtained. The Cauchy-Schwartz inequality obtained in the case of the Hilbert space [1] is a good example. The notion of the local product and the induced local product space are introduced in this study. This space reveals itself to be a unique form of complicated inner product space. The following inequality is obtained by utilizing this space.

**Theorem 1.1.** *Let $\vec{a}=(a_1,a_2,\ldots,a_n),\vec{b}=(b_1,b_2,\ldots,b_n)\in\mathbb{R}^n$ such that $e^e<\langle\vec{a},\vec{b}\rangle$ and $\langle\vec{a},\vec{b}\rangle\neq 1$, then we have*

$$\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\left|\log\left(i\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\right)\right|dx_1dx_2\cdots dx_n \geq \frac{\left|\prod_{j=1}^{n}|b_j|-|a_j|\right|}{\log\log(\langle a,b\rangle)}$$

*for all $s\in\mathbb{N}$, where $\langle,\rangle$ denotes the inner product and $i^2=-1$.*

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## 2. **The local product and associated space**

In this section, we introduce and study the notion of the **local product** and associated space.

**Definition 2.1.** Let $\vec{a},\vec{b}\in \mathbb{C}^n$ and $f:\mathbb{C}\longrightarrow \mathbb{C}$ be continuous on $\cup_{j=1}^{n}[|a_j|,|b_j|]$. Let $(\mathbb{C}^n,\langle,\rangle)$ be a complex inner product space. Then by the $k^{th}$ local product of $\vec{a}$ with $\vec{b}$ on the sheet $f$, we mean the bi-variate map $\mathcal{G}_f^k:(\mathbb{C}^n,\langle,\rangle)\times(\mathbb{C}^n,\langle,\rangle)\longrightarrow \mathbb{C}$ such that

$$\mathcal{G}_f^k(\vec{a};\vec{b})=f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ \mathbf{e}\bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n$$

where $\langle,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi i q}$. We denote an inner product space with a $k^{th}$ **local product** defined over a sheet $f$ as the $k^{th}$ local product space over a sheet $f$. We denote this space with the triple $(\mathbb{C}^n,\langle,\rangle,\mathcal{G}_f^k(;))$.

In certain ways, the $k^{th}$ local product is a universal map induced by a sheet. To put it another way, a local product can be made by carefully selecting the sheet. We get the local product by making our sheet the constant function $f:=1$

$$\begin{aligned}\mathcal{G}_1^k(\vec{a};\vec{b})&=\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}dx_1dx_2\cdots dx_n\\&=\prod_{i=1}^{n}|b_i|-|a_i|.\end{aligned}$$

Similarly, if we take our sheet to be $f=\log$, then under the condition that $\langle\vec{a},\vec{b}\rangle\neq 0$, we obtain the induced local product

$$\mathcal{G}_{\log}^k(\vec{a};\vec{b})=2\pi\times(i)^{k+1}\frac{\log(\langle\vec{a},\vec{b}\rangle)}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[k]{\sum_{j=1}^{n}x_j^k}dx_1dx_2\cdots dx_n.$$

By taking the sheet $f=\mathrm{Id}$ to be the identity function, then we obtain in this setting the associated local product

$$\mathcal{G}_{\mathrm{Id}}^k(\vec{a};\vec{b})=\langle\vec{a},\vec{b}\rangle\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(\frac{(i)^k\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n.$$

Again, by taking the sheet $f=\mathrm{Id}^{-1}$ with $\langle a,b\rangle\neq 0$, then we obtain the corresponding induced $k^{th}$ local product

$$\mathcal{G}_{\mathrm{Id}^{-1}}^k(\vec{a};\vec{b})=\frac{1}{\langle\vec{a},\vec{b}\rangle}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(-\frac{(i)^k\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n.$$

Also by taking the sheet $f = \log\log$, then we have the associated $k^{th}$ local product

$$\mathcal{G}^k_{\log\log}(\vec{a};\vec{b}) = \log\log(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\log\left(i\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\right)dx_1dx_2\cdots dx_n.$$

## 3. Properties of the local product product

In this section we study some properties of the local product on a fixed sheet.

**Proposition 3.1.** *The following holds*

(i) *If $f$ is linear such that $\langle a,b\rangle = -\langle b,a\rangle$ then*

$$\mathcal{G}^k_f(\vec{a};\vec{b}) = (-1)^{n+1}\mathcal{G}^k_f(\vec{b};\vec{a}).$$

(ii) *Let $f,g : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $f(t) \leq g(t)$ for any $t \in [1,\infty)$. Then $|\mathcal{G}_f(\vec{a};\vec{b})| \leq |\mathcal{G}_g(\vec{a};\vec{b})|$.*

*Proof.* (i) By the linearity of $f$, we can write

$$\begin{aligned}
\mathcal{G}^k_f(\vec{a};\vec{b}) &= f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ\mathbf{e}\Bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n\\
&= f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ\mathbf{e}\Bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n\\
&= f(-\langle b,a\rangle)(-1)^n\int\limits_{|b_n|}^{|a_n|}\int\limits_{|b_{n-1}|}^{|a_{n-1}|}\cdots\int\limits_{|b_1|}^{|a_1|} f\circ\mathbf{e}\Bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n\\
&= (-1)^{n+1}f(\langle b,a\rangle)\int\limits_{|b_n|}^{|a_n|}\int\limits_{|b_{n-1}|}^{|a_{n-1}|}\cdots\int\limits_{|b_1|}^{|a_1|} f\circ\mathbf{e}\Bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\Bigg)dx_1dx_2\cdots dx_n\\
&= (-1)^{n+1}\mathcal{G}^k_f(\vec{b};\vec{a}).
\end{aligned}$$

(ii) Property (ii) follows very easily from the inequality $f(t) \leq g(t)$.

□

## 4. Applications of the local product

In this section we explore some applications of the local product.

**Theorem 4.1.** *Let $\vec{a}=(a_1,a_2,\dots,a_n),\vec{b}=(b_1,b_2,\dots,b_n)\in\mathbb{R}^n$ such that $e^e<\langle\vec{a},\vec{b}\rangle$, then the lower bound holds*

$$\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\left|\log\left(i\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\right)\right|dx_1dx_2\cdots dx_n$$

$$\geq\frac{\left|\prod_{j=1}^{n}|b_j|-|a_j|\right|}{\log\log(\langle a,b\rangle)}$$

*for all $s\in\mathbb{N}$, where $\langle,\rangle$ denotes the inner product and $i^2=-1$.*

*Proof.* Let $f:\mathbb{R}\longrightarrow\mathbb{R}^+$ and $\vec{a},\vec{b}\in\mathbb{R}^n$ such that $e^e<\langle\vec{a},\vec{b}\rangle$. We note that

$$\mathcal{G}_{\log\log}^{4s}(\vec{a};\vec{b})=\log\log(\langle\vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\log\left(i\frac{\sqrt[4s]{\sum\limits_{j=1}^{n}x_j^{4s}}}{||\vec{a}||^{4s+1}+||\vec{b}||^{4s+1}}\right)dx_1dx_2\cdots dx_n$$

by taking $k=4s$ for any $s\in\mathbb{N}$. Also by taking the sheet $f:=1$ to be the constant function, then we obtain in this setting the associated local product

$$\mathcal{G}_1^{4s}(\vec{a};\vec{b})=\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}dx_1dx_2\cdots dx_n$$

$$=\prod_{i=1}^{n}|b_i|-|a_i|.$$

Since $|\log it|=|\log t+i\frac{\pi}{2}|\geq 1$ on $\mathbb{R}^+$ the claim inequality is a consequence by appealing to Proposition 3.1. □

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## References

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Department of Mathematics, African Institute for mathematical sciences, Accra, Ghana.

*E-mail address*: Theophilus@aims.edu.gh/emperordagama@yahoo.com