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Special affine Fourier transform for space-time algebra signals in detail

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Abstract. We generalize the space-time Fourier transform (SFT) [10] to a special affine Fourier transform (SASFT, also known as offset linear canonical transform) for 16-dimensional space-time multivector $Cl(3,1)$-valued signals over the domain of space-time (Minkowski space) $\mathbb{R}^{3,1}$. We establish how it can be computed in terms of the SFT, and introduce its properties of multivector coefficient linearity, shift and modulation, inversion, Rayleigh (Parseval) energy theorem, partial derivative identities, a directional uncertainty principle and its specialization to coordinates. All important results are proven in full detail.

1. Introduction

In signal processing and optics the special affine Fourier transforms have been introduced in 1994 [1, 2] as a vast generalization of Fourier transforms, and other known signal transforms like the fractional Fourier transform. A further notable trend has been the introduction of hypercomplex (quaternionic and Clifford) Fourier transforms [13, 15, 18]. A species of Fourier transform particularly relevant to signal processing, navigation and physics is the space-time Fourier transform (SFT) [10, 11, 6, 7] that transforms signals defined on the domain of (special relativistic) space-time (Minkowski space) $\mathbb{R}^{3,1}$ with range in the corresponding geometric (Clifford) algebra of space-time (space-time algebra STA) $Cl(3,1)$. Apart from electromagnetic waves and

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Soli Deo Gloria. This work is dedicated to Robert F. Kennedy Jr. of Children's Health Defense, who recently published *The Real Anthony Fauci – Bill Gates, Big Pharma, and the Global War on Democracy and Public Health* [19], writing: *... [on] the historical underpinnings of the bewildering cataclysm that began in 2020. ... Across Western nations, shell-shocked citizens experienced all the well-known tactics of rising totalitarianism – mass propaganda and censorship, the orchestrated promotion of terror, the manipulation of science, the suppression of debate, the vilification of dissent, and use of force to prevent protest. Conscientious objectors who resisted these unwanted, experimental, zero-liability medical interventions faced orchestrated gaslighting, marginalization, and scapegoating.* [19], p. XIV. Please note that this research is subject to the Creative Peace License [14].

light it can be applied in any area of physics, including electro-magnetism, special relativity, satellite navigation, optics and quantum mechanics (e.g. to spinor wave functions). It is therefore on the applied side also of interest to quantum bit computations and quantum computing in general. A steerable split of quaternions [16] is found to have a natural algebraic analogue in STA as space-time split, defined by the time vector and its dual trivector (the three-dimensional space volume element). Applied to any space-time signal, it naturally generates two wave packages, one traveling to the left and one to the right, classical solutions of relativistic wave equations.

With this backdrop we undertake to generalize the SFT to a special affine Fourier transform. This automatically generates (for special parameter settings) a fractional SFT, a linear canonical SFT, a lens transformation SFT, a free-space propagation SFT and a magnification SFT, amongst others. We thus create a set of potentially powerful new tools for physics, signal processing, optics, quantum mechanics, quantum computing, space navigation, etc. The current work is an extension of [17], with special emphasize on providing detailed proof of all important new results.

This paper is structured as follows. Section 2 introduces the necessary background of space-time algebra and the space-time Fourier transform (SFT). Section 3 then defines the special affine space-time Fourier transform (SASFT) generalization and establishes several of its properties. This includes in Subsection 3.3 the uncertainty principle for the SASFT in a general directional form and a specific coordinate system related form. The paper concludes with Section 4 and a list of references.

2. Background

For a general introduction to Clifford's geometric algebras see [12].

2.1. Space-Time Algebra

Space-time of Einstein's special relativity is a four-dimensional non-Euclidean quadratic space $\mathbb{R}^{3,1}$ equipped with the orthonormal vector basis[1]

$$\{e_t, e_1, e_2, e_3\}, \qquad -e_t^2 = e_1^2 = e_2^2 = e_3^2 = 1. \tag{2.1}$$

Space-time algebra [9, 5] (isomorphic to Dirac algebra of quantum mechanics) is Clifford's geometric algebra $Cl(3,1)$ generated by all geometric products of $\mathbb{R}^{3,1}$ vectors, also called (flat) Minkowski space, with a 16-dimensional algebra basis of scalar, vector, bivector, trivector, and pseudoscalar elements:

$$\begin{aligned}\{1, e_t, e_1, e_2, e_3, e_{t1}, e_{t2}, e_{t3}, e_{23}, e_{31}, e_{12},\\ e_{123} = i_3, e_{t31}, e_{t23}, e_{t12}, e_{t123} = i_{st}\},\end{aligned} \tag{2.2}$$

[1]The signature $(+ - --)$ chosen in [9] would also be possible, but then the important quaternionic subalgebra (2.4) would be absent. The possibility of our $(- + ++)$ is also indicated in Footnote 15 on page 22 of [9].

where we used the conventional index notation $e_{t1} = e_t e_1$, $e_{123} = e_1 e_2 e_3$, etc. The even subalgebra $Cl^+(3,1)$ of the space-time algebra has the eight-dimensional basis

$$\{1, e_{t1}, e_{t2}, e_{t3}, e_{12}, e_{23}, e_{31}, e_{t123} = i_{st}\}, \tag{2.3}$$

and is isomorphic to (both Hamilton's complex biquaternions and) the Clifford algebra of space $Cl(3,0)$, also known as Pauli algebra, if we identify (denoting $Cl(3,0)$ elements with bold $\vec{e}$)

$$1 = 1, \quad \vec{e}_k = e_{tk}, \quad \vec{e}_{jk} = e_{jk}, \quad \vec{e}_{123} = i_{st}, \quad k, j \in \{1,2,3\}, k \neq j. \tag{2.4}$$

Furthermore we have the important four-dimensional subalgebra of space-time algebra isomorphic to quaternions with basis[2]

$$\{1, e_t, i_3, i_{st}\}, \qquad e_t^2 = i_3^2 = i_{st}^2 = -1. \tag{2.5}$$

Left multiplication with e_t and right multiplication with its space-time dual $i_3 = e_t^* = e_t i_{st}^{-1}$ is a form of *space-time duality*[3] mapping for basis elements of grade k to grade $4-k$, $0 \leq k \leq 4$:

$$e_t\{(2.2)\}i_3 =$$
$$\{i_{st}, -i_3, e_{t23}, e_{t31}, e_{t12}, -e_{23}, -e_{31}, -e_{12}, -e_{t1}, -e_{t2}, -e_{t3}, -e_t, e_1, e_2, e_3, 1\}, \tag{2.6}$$

A useful tool for us will be the following split [10] of space-time multivectors and multivector functions[4]

$$M_\pm = \frac{1}{2}(M \pm e_t M i_3), \qquad M = M_+ + M_-, \tag{2.7}$$

with the convenient property that

$$e_t M_\pm = \mp M_\pm i_3. \tag{2.8}$$

Using the (anti-involution) principal reverse (reverse product order combined with changing the sign of every basis vector of negative square) [20], we have for any two multivectors $M, N \in Cl(3,1)$ that

$$M * \overline{N} = \langle M\overline{N} \rangle_0 = \sum_A M_A N_A, \tag{2.9}$$

where index A ranges over all indexes in (2.2) and in addition index 0 is for the scalar part: $M_0 = \langle M \rangle_0$, $N_0 = \langle N \rangle_0$. Then we can define the *multivector norm*

$$|N| = \sqrt{N * \overline{N}} = \sqrt{\sum_A N_A^2}. \tag{2.10}$$

[2]Note that this four-dimensional subalgebra of STA is spatially isotropic, i.e. invariant under spatial rotations.

[3]Note that as for bivectors, the space-time duality map (2.6) exchanges relative vectors e_{t1}, e_{t2}, e_{t3}, with pure space bivectors e_{12}, e_{23}, e_{31}, and vice versa. For Maxwell's theory [5, 9] this means to exchange electric and magnetic fields.}

[4]Regarding physics, the split is *determined by the time vector* and its dual trivector (the three-dimensional space volume element). Applied to any space-time signal, it naturally generates two wave packages, one traveling to the left and one to the right, classical solutions of relativistic wave equations. [10]

Example 2.1. We compute the norm of e_tM for $M \in Cl(3,1)$. We have

$$\begin{aligned}|e_tM| &= \sqrt{\langle e_tM\overline{e_tM}\rangle_0} = \sqrt{\langle e_tM\overline{M}\overline{e_t}\rangle_0} = \sqrt{\langle M\overline{M}\overline{e_t}e_t\rangle_0} = \sqrt{\langle M\overline{M}\rangle_0} \\ &= |M|, \end{aligned} \tag{2.11}$$

where we used the fact that the principal reverse is an anti-involution, the cyclic factor permutation within the scalar grade bracket $\langle\ldots\rangle_0$, and that $\overline{e_t}e_t = -e_te_t = 1$.

Note that the split (2.7) results in the norm identity

$$|M|^2 = |M_+|^2 + |M_-|^2. \tag{2.12}$$

2.2. Space-Time Fourier Transform (SFT)

For background on Fourier transforms we refer to [3, 8] and for Clifford Fourier transforms to [13, 18].

For functions $f : \mathbb{R}^{3,1} \to Cl(3,1)$ and $1 \leq a < \infty$, we introduce the linear spaces

$$\begin{aligned}&L^a(\mathbb{R}^{3,1}; Cl(3,1)) \\ &\quad = \Big\{\mathbb{R}^{3,1} \to Cl(3,1) : ||f||_a = \Big(\int_{\mathbb{R}^{3,1}} |f(\mathbf{x})|^a \, d^4x\Big)^{1/a} < \infty\Big\}. \end{aligned} \tag{2.13}$$

Definition 2.2. The space-time Fourier transform (SFT) [10] maps 16-dimensional space-time algebra functions $f \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$ to 16-dimensional spectrum functions $\mathcal{F}_{SFT}\{f\} : \mathbb{R}^{3,1} \to Cl(3,1)$. It is defined as

$$\mathcal{F}_{SFT}\{f\}(\omega) = \int_{\mathbb{R}^{3,1}} e^{-e_t t\omega_t} f(\mathbf{x}) e^{-i_3 \vec{x}\cdot\vec{\omega}} d^4x, \tag{2.14}$$

with space-time vectors $\mathbf{x} = te_t + \vec{x} \in \mathbb{R}^{3,1}$, $\vec{x} = xe_1 + ye_2 + ze_3 \in \mathbb{R}^3$, infinitesimal space-time volume $d^4x = dtdxdydz$ and space-time frequency vector $\omega = \omega_t e_t + \vec{\omega} \in \mathbb{R}^{3,1}$, $\vec{\omega} = \omega_1 e_1 + \omega_2 e_2 + \omega_e e_3 \in \mathbb{R}^3$.

We have the following lemma.

Lemma 2.3. *$\mathcal{F}_{SFT}\{f\}$ is a continuous and hence measurable function.*

Proof. The proof can be done analogously to the proof of Proposition 3.1, property (ix) of the QFT in [4]. □

The SFT is invertible, compare (72) of [10] or Lemma 2.5 of [7].

Theorem 2.4. For $f, \mathcal{F}_{SFT}\{f\} \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, the inverse of the SFT is

$$\begin{aligned}f(\mathbf{x}) &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} e^{e_t t\omega_t} \mathcal{F}_{SFT}\{f\}(\omega) e^{i_3 \vec{x}\cdot\vec{\omega}} d^4\omega, \\ d^4\omega &= d\omega_t d\omega_1 d\omega_2 d\omega_3. \end{aligned} \tag{2.15}$$

We will need the following anisotropic scaling property, a special case of the general linear transformation property Theorem 5.4 of [10].

Lemma 2.5. *For* $\alpha_t, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \setminus \{0\}$, $f \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, *we have*

$$\begin{aligned} &\mathcal{F}_{SFT}\{f(\alpha_t t e_t + \alpha_1 x_1 e_1 + \alpha_2 x_2 e_2 + \alpha_3 x_3 e_3)\}(\omega) \\ &\quad = \frac{1}{|\alpha_t \alpha_1 \alpha_2 \alpha_3|} \mathcal{F}_{SFT}\{f\}(\frac{\omega_t}{\alpha_t} e_t + \frac{\omega_1}{\alpha_1} e_1 + \frac{\omega_2}{\alpha_2} e_2 + \frac{\omega_3}{\alpha_3} x_3). \end{aligned} \tag{2.16}$$

We will furthermore need the SFT Rayleigh (or SFT Parseval) energy theorem, see Corollary 16 of [6].

Lemma 2.6. *For* $f \in L^2(\mathbb{R}^{3,1}; Cl(3,1))$, *we have*

$$\int_{\mathbb{R}^{3,1}} |\mathcal{F}_{SFT}\{f\}(\omega)|^2 d^4\omega = (2\pi)^4 \int_{\mathbb{R}^{3,1}} |f(\mathbf{x})|^2 d^4x. \tag{2.17}$$

We have the following *Riemann-Lebesgue lemma* for the SFT.

Lemma 2.7. *If* $f \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, *then* $\mathcal{F}_{SFT}\{f\}(\omega) \to 0$, *as* $\omega \to 0$.

Proof. The proof is analogous to that of Theorem 3.1 in [4]. □

We also need the following lemma (Lemma 13 in [6]) on partial derivatives.

Lemma 2.8. *For* $f, \frac{\partial}{\partial t} f, \frac{\partial}{\partial x_k} f \in L^1(\mathbb{R}^{3,1};\, Cl(3,0))$, $k = 1,2,3$, *provided that the derivatives exist, we obtain*

$$\mathcal{F}_{SFT}\Big\{\frac{\partial}{\partial t} f(\mathbf{x})\Big\}(\omega) = \omega_t e_t \mathcal{F}_{SFT}\{f(\mathbf{x})\}(\omega), \tag{2.18}$$

$$\mathcal{F}_{SFT}\Big\{\frac{\partial}{\partial x_k} f(\mathbf{x})\Big\}(\omega) = \mathcal{F}_{SFT}\{f(\mathbf{x})\}(\omega) i_3 \omega_k. \tag{2.19}$$

We note the following directional uncertainty principle for the SFT (Theorem 7.2 of [11]).

Theorem 2.9. For two arbitrary space-time vectors $\mathbf{c}, \mathbf{d} \in \mathbb{R}^{3,1}$ and f, $|\mathbf{x}|^{1/2} f \in L^2(\mathbb{R}^{3,1}; Cl(3,0))$ we have

$$\begin{aligned} &\int_{\mathbb{R}^{3,1}} (c_t t - \overrightarrow{c} \cdot \overrightarrow{x})^2 |f(\mathbf{x})|^2 d^4x \int_{\mathbb{R}^{3,1}} (d_t \omega_t - \overrightarrow{d} \cdot \overrightarrow{\omega})^2 |\mathcal{F}_{SFT}\{f\}(\omega)|^2 d^4\omega \\ &\quad \geq \frac{(2\pi)^4}{4} \Big[(c_t d_t - \overrightarrow{c} \cdot \overrightarrow{d})^2 F_-^2 + (c_t d_t + \overrightarrow{c} \cdot \overrightarrow{d})^2 F_+^2 \Big], \end{aligned} \tag{2.20}$$

with energies of the left- and right traveling wavepackets

$$F_\pm = \int_{\mathbb{R}^{3,1}} |f_\pm(\mathbf{x})|^2 d^4x, \qquad \int_{\mathbb{R}^{3,1}} |f(\mathbf{x})|^2 d^4x = F_- + F_+. \tag{2.21}$$

3. Special Affine Space-Time Fourier Transform (SASFT)

3.1. Defining the SASFT

We now want to vastly extend the SFT to a special affine Fourier transform (SASFT) following the approach in [1, 2]. For this purpose we introduce two special affine phase space transformations given by matrixes and vectors

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} m \\ n \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} \overrightarrow{M} \\ \overrightarrow{N} \end{pmatrix},$$
$$a, b, c, d, m, n, A, B, C, D \in \mathbb{R}, \qquad \overrightarrow{M}, \overrightarrow{N} \in \mathbb{R}^3, \tag{3.1}$$

with the lossless area- and power-preserving unit determinants

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1, \qquad \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC = 1. \tag{3.2}$$

We modify the left and the right kernel factors of the SFT (2.14) to[5]

$$k(t, \omega_t) = \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at^2 + 2mt + d\omega_t^2 - 2(t + dm - bn)\omega_t\right)},$$
$$K(\overrightarrow{x}, \overrightarrow{\omega}) = \frac{1}{(2\pi B)^{3/2}} e^{\frac{i_3}{2B}\left(A\overrightarrow{x}^2 + 2\overrightarrow{M}\cdot\overrightarrow{x} + D\overrightarrow{\omega}^2 - 2(\overrightarrow{x} + D\overrightarrow{M} - B\overrightarrow{N})\cdot\overrightarrow{\omega}\right)}, \tag{3.3}$$

Remark 3.1. Note that we need to choose the quantities $\overrightarrow{M}, \overrightarrow{N}$ as vectors in $\mathbb{R}^3$, otherwise we would not be able to obtain a scalar phase factor in the exponent of $K(\overrightarrow{x}, \overrightarrow{\omega})$. A more elaborate alternative would be to construct $K(\overrightarrow{x}, \overrightarrow{\omega})$ as the product of three kernel factors $K_1(x, \omega_1)$, $K_2(y, \omega_2)$, $K_3(z, \omega_3)$, and use three (2×2) matrices and three (2×1) vectors for their construction in analogy to $k(t, \omega_t)$. But this approach would introduce eight extra parameters.

We now define the special affine space-time Fourier transform.

Definition 3.2. The special affine space-time Fourier transform (SASFT) maps 16-dimensional space-time algebra functions $f \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$ to 16-dimensional spectrum functions $\mathcal{F}\{f\} : \mathbb{R}^{3,1} \to Cl(3,1)$. It is defined as

$$\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}^{3,1}} k(\omega_t, t) f(\mathbf{x}) K(\overrightarrow{x}, \overrightarrow{\omega}) d^4x, \tag{3.4}$$

with kernel factors (3.3).

Remark 3.3. Note that we obtain the SFT [10] by setting $a = d = m = n = 0$, $b = -c = 1$ and $A = D = 0$, $B = -C = 1$, $\overrightarrow{M} = \overrightarrow{N} = \overrightarrow{0}$. By setting only $m = n = 0$ and $\overrightarrow{M} = \overrightarrow{N} = \overrightarrow{0}$ we obtain a *linear canonical transform*[6] generalization of the SFT without offsets. By choosing $b = -c = -\sin\vartheta$,

[5] Note that Abe and Sheridan adopt in their 1994 papers that introduce the SAFT slightly different sign conventions in (61) of [1] and in (3) of [2]. For consistency, we use the conventions specified in (3) of [2].

[6] The SASFT is therefore more general than the linear canonical SFT, obtained by setting for the SASFT the translation offsets to zero: $m = n = 0$ and $\overrightarrow{M} = \overrightarrow{N} = \overrightarrow{0}$.

$a = d = \cos\vartheta$, $m = n = 0$ and $B = -C = -\sin\theta$, $A = D = \cos\theta$, $\overrightarrow{M} = \overrightarrow{N} = \overrightarrow{0}$ we get a generalization of the SFT to a *fractional Fourier transform*. Including translations $m, n \neq 0$ and $\overrightarrow{M}, \overrightarrow{N} \neq \overrightarrow{0}$, we obtain SASFT generalizations of special known cases of interest [2] in optics as the *lens transformation*[7] with $a = d = 1$, $c = \xi$, $b = 0$ and $A = D = 1$, $C = \Xi$, $B = 0$; the *free-space propagation* with $a = d = 1$, $c = 0$, $b = \eta$ and $A = D = 1$, $C = 0$, $B = \Gamma$; and *magnification*[8] with $a = e^{\alpha}$, $d = e^{-\alpha}$, $c = b = 0$ and $A = e^{\Lambda}$, $D = e^{-\Lambda}$, $C = B = 0$. Any property of the SASFT of Definition 3.2 will therefore automatically apply to all the special cases listed in this remark.

3.2. Properties of the SASFT

Convenient for computations, it is possible to pull out parts of the kernel factors to the left and right of the integral in Definition 3.2 that do not depend on the integration variable $\mathbf{x} \in \mathbb{R}^{3,1}$.

Lemma 3.4.

$$\begin{aligned}\mathcal{F}\{f\}(\omega) =& \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)} \\ & \int_{\mathbb{R}^{3,1}} e^{\frac{e_t}{2b}\left(at^2+2(m-\omega_t)t\right)} f(\mathbf{x}) e^{\frac{i_3}{2B}\left(A\vec{x}^2+2(\overrightarrow{M}-\vec{\omega})\cdot\vec{x}\right)} d^4x \\ & e^{\frac{i_3}{2B}\left(D\vec{\omega}^2+2(B\overrightarrow{N}-D\overrightarrow{M})\cdot\vec{\omega}\right)} \frac{1}{(2\pi B)^{3/2}}. \end{aligned} \tag{3.5}$$

The next lemma shows, that we can reduce the computation of the SASFT to that of the SFT. By defining the function

$$g(\mathbf{x}) = e^{\frac{e_t}{2b}(at^2+2mt)} f(\mathbf{x})\, e^{\frac{i_3}{2B}(A\vec{x}^2+2\overrightarrow{M}\cdot\vec{x})}, \tag{3.6}$$

we can reduce the computation of the SAFT to

$$\begin{aligned}\mathcal{F}\{f\}(\omega) =& \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)} \int_{\mathbb{R}^{3,1}} e^{-\frac{e_t}{b}\omega_t t} g(\mathbf{x})\, e^{-\frac{i_3}{B}\vec{\omega}\cdot\vec{x}} d^4x \\ & e^{\frac{i_3}{2B}\left(D\vec{\omega}^2+2(B\overrightarrow{N}-D\overrightarrow{M})\cdot\vec{\omega}\right)} \frac{1}{(2\pi B)^{3/2}}. \end{aligned} \tag{3.7}$$

Lemma 3.5. *The SASFT can be computed from the SFT of $g(\mathbf{x})$, defined in (3.6), by*

$$\begin{aligned}\mathcal{F}\{f\}(\omega) =& \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)}\ \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t + \frac{\vec{\omega}}{B}\Big) \\ & e^{\frac{i_3}{2B}\left(D\vec{\omega}^2+2(B\overrightarrow{N}-D\overrightarrow{M})\cdot\vec{\omega}\right)} \frac{1}{(2\pi B)^{3/2}}. \end{aligned} \tag{3.8}$$

The following derivatives of g will be needed later.

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[7]As [2] points out on page 1802, for the lens transformation a degenerate version of the SAFT is required, see also [1].

[8]As pointed out related to equation (13) on page 1802 of [2], a special limit for $b \to 0$ formula will need to be used in this case.

Lemma 3.6. *The time derivative of g is given by*

$$\frac{\partial}{\partial t}g(\mathbf{x}) = e^{\frac{e_t}{2b}(at^2+2mt)}\Big[e_t(\frac{a}{b}t+\frac{m}{b})f(\mathbf{x})+\frac{\partial}{\partial t}f(\mathbf{x})\Big]\,e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}. \quad (3.9)$$

The derivative in the x_k-coordinate direction of g is given by

$$\frac{\partial}{\partial x_k}g(\mathbf{x}) = e^{\frac{e_t}{2b}(at^2+2mt)}\Big[(\frac{A}{B}t+\frac{M_k}{B})f(\mathbf{x})i_3+\frac{\partial}{\partial x_k}f(\mathbf{x})\Big] \\ e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}. \quad (3.10)$$

Proof. We compute only the time derivative, the derivative in the x_k-coordinate direction can be found analogously. We use the product rule to obtain

$$\begin{aligned}\frac{\partial}{\partial t}g(\mathbf{x}) &= \frac{\partial}{\partial t}\Big[e^{\frac{e_t}{2b}(at^2+2mt)}f(\mathbf{x})\,e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\Big]\\ &= \Big[\frac{\partial}{\partial t}e^{\frac{e_t}{2b}(at^2+2mt)}\Big]f(\mathbf{x})\,e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\\ &\quad + e^{\frac{e_t}{2b}(at^2+2mt)}\frac{\partial}{\partial t}f(\mathbf{x})\,e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\\ &= e^{\frac{e_t}{2b}(at^2+2mt)}e_t(\frac{a}{b}t+\frac{m}{b})f(\mathbf{x})\,e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\\ &\quad + e^{\frac{e_t}{2b}(at^2+2mt)}\frac{\partial}{\partial t}f(\mathbf{x})\,e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\\ &= e^{\frac{e_t}{2b}(at^2+2mt)}\Big[e_t(\frac{a}{b}t+\frac{m}{b})f(\mathbf{x})+\frac{\partial}{\partial t}f(\mathbf{x})\Big]e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}. \quad (3.11)\end{aligned}$$

□

We further note useful relationships between the functions f and g of (3.6).

Lemma 3.7. *For g defined according to* (3.6) *in terms of f, we have*

$$|g(\mathbf{x})| = |f(\mathbf{x})|, \qquad |g_\pm(\mathbf{x})| = |f_\pm(\mathbf{x})|. \quad (3.12)$$

Proof. For the proof we observe that by its definition $g = e^{e_t\alpha}fe^{i_3\beta}$ for some scalar functions α, β, see (3.6). Therefore

$$\begin{aligned}|g(\mathbf{x})|^2 &= \langle g(\mathbf{x})\overline{g(\mathbf{x})}\rangle_0 = \langle e^{e_t\alpha}f(\mathbf{x})e^{i_3\beta}e^{-i_3\beta}\overline{f(\mathbf{x})}e^{-e_t\alpha}\rangle_0\\ &= \langle f(\mathbf{x})\overline{f(\mathbf{x})}\rangle_0 = |f(\mathbf{x})|^2, \quad (3.13)\end{aligned}$$

where we used for the third identity the cyclic commutation property of the scalar part $\langle abc\rangle_0 = \langle cab\rangle_0$, for any $a, b, c \in Cl(3,1)$. Because by construction

$$g_\pm = \left(e^{e_t\alpha}fe^{i_3\beta}\right)_\pm = e^{e_t\alpha}f_\pm e^{i_3\beta}, \quad (3.14)$$

we also have $|g_\pm(\mathbf{x})| = |f_\pm(\mathbf{x})|$. □

In analogy to Theorem 3.3 of [7], where in the quaternionic setting the SAFT is referred to by another popular name of *offset linear canonical transform*, we can establish the following properties.

Theorem 3.8. The SASFT is left linear for multivector coefficients that commute with e_t

$$\alpha = \alpha_0+\alpha_t e_t+\alpha_{12}e_{12}+\alpha_{23}e_{23}+\alpha_{31}e_{31}+\alpha_{t12}e_{t12}+\alpha_{t23}e_{t23}+\alpha_{t31}e_{t31} \quad (3.15)$$

and right linear for coefficients that commute with i_3

$$\beta = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_{12}e_{12} + \beta_{23}e_{23} + \beta_{31}e_{31} + \beta_{123}i_3. \quad (3.16)$$

That is

$$\begin{aligned} \mathcal{F}\{\alpha f + \alpha' g\} &= \alpha\mathcal{F}\{f\} + \alpha'\mathcal{F}\{g\}, \\ \mathcal{F}\{f\beta + g\beta'\} &= \mathcal{F}\{f\}\beta + \mathcal{F}\{g\}\beta'. \end{aligned} \quad (3.17)$$

Furthermore, we have

$$\lim_{|\omega|\to 0} |\mathcal{F}\{f\}(\omega)| = 0. \quad (3.18)$$

Finally, $\mathcal{F}\{f\}$ is uniformly continuous on $\mathbb{R}^{3,1}$.

Proof. We first proof left linearity (3.17). For that we note that due to the assumed commutation property of α and α' with e_t:

$$e_t\alpha = \alpha e_t, \qquad e_t\alpha' = \alpha' e_t, \quad (3.19)$$

we also have commutation of the left kernel factor (3.3) with α and α':

$$k(t,\omega_t)\alpha = \alpha k(t,\omega_t), \qquad k(t,\omega_t)\alpha' = \alpha' k(t,\omega_t). \quad (3.20)$$

Therefore,

$$\begin{aligned} &\mathcal{F}\{\alpha f + \alpha' g\}(\omega) = \int_{\mathbb{R}^{3,1}} k(\omega_t,t)[\alpha f(\mathbf{x}) + \alpha' g(\mathbf{x})]K(\overrightarrow{x},\overrightarrow{\omega})d^4x \\ &= \int_{\mathbb{R}^{3,1}} \alpha k(\omega_t,t)f(\mathbf{x})K(\overrightarrow{x},\overrightarrow{\omega})d^4x + \int_{\mathbb{R}^{3,1}} \alpha' k(\omega_t,t)g(\mathbf{x})K(\overrightarrow{x},\overrightarrow{\omega})d^4x \\ &= \alpha\mathcal{F}\{f\}(\omega) + \alpha'\mathcal{F}\{g\}(\omega). \end{aligned} \quad (3.21)$$

Next, we proof right linearity (3.17). We note that due to the assumed commutation property of β and β' with i_3:

$$\beta i_3 = i_3\beta, \qquad \beta' i_3 = i_3\beta', \quad (3.22)$$

we also have commutation of the right kernel factor (3.3) with β and β':

$$\beta K(\overrightarrow{x},\overrightarrow{\omega}) = K(\overrightarrow{x},\overrightarrow{\omega})\beta, \qquad \beta' K(\overrightarrow{x},\overrightarrow{\omega}) = K(\overrightarrow{x},\overrightarrow{\omega})\beta'. \quad (3.23)$$

Therefore,

$$\begin{aligned} &\mathcal{F}\{f\beta + g\beta'\}(\omega) = \int_{\mathbb{R}^{3,1}} k(\omega_t,t)[f(\mathbf{x})\beta + g(\mathbf{x})\beta']K(\overrightarrow{x},\overrightarrow{\omega})d^4x \\ &= \int_{\mathbb{R}^{3,1}} k(\omega_t,t)f(\mathbf{x})K(\overrightarrow{x},\overrightarrow{\omega})\beta d^4x + \int_{\mathbb{R}^{3,1}} k(\omega_t,t)g(\mathbf{x})K(\overrightarrow{x},\overrightarrow{\omega})\beta' d^4x \\ &= \mathcal{F}\{f\}(\omega)\beta + \mathcal{F}\{g\}(\omega)\beta'. \end{aligned} \quad (3.24)$$

Equation (3.18) follows from the computation of the SASFT from the SFT by Lemma 3.5, and by applying Lemma 2.7 for the SFT.

The uniform continuity of $\mathcal{F}\{f\}$ on $\mathbb{R}^{3,1}$ follows from the computation of the SASFT from the SFT by Lemma 3.5, and from Lemma 2.3 for the SFT. $\square$

We further obtain the following shift and modulation properties and the inverse transform.

Theorem 3.9. For $f \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, $\mathbf{x}, \omega \in \mathbb{R}^{3,1}$, and constant vector $\mathbf{k} = ke_t + \overrightarrow{K} \in \mathbb{R}^{3,1}$ we have

$$\mathcal{F}\{f(\mathbf{x}-\mathbf{k})\}(\omega) = e^{e_t(-\frac{1}{2}ack^2 - cmk + ck\omega_t + ank)}\ \mathcal{F}\{f(\mathbf{x})\}\big(\omega - (ake_t + A\overrightarrow{K})\big) \\ e^{i_3(-\frac{1}{2}AC\overrightarrow{K}^2 - C\overrightarrow{M}\cdot\overrightarrow{K} + C\overrightarrow{K}\cdot\overrightarrow{\omega} + A\overrightarrow{N}\cdot\overrightarrow{K})}. \tag{3.25}$$

Proof. We compute, using substitution $\mathbf{x}' = \mathbf{x} - \mathbf{k}$, $d^4x = d^4x'$, and therefore $\mathbf{x} = \mathbf{x}' + \mathbf{k}$, or in components $t = t' + k$, $\overrightarrow{x} = \overrightarrow{x}' + \overrightarrow{K}$, that

$$\begin{aligned}\mathcal{F}\{f(\mathbf{x}-\mathbf{k})\}(\omega) &= \int_{\mathbb{R}^{3,1}} k(t,\omega_t) f(\mathbf{x}-\mathbf{k}) K(\overrightarrow{x}, \overrightarrow{\omega}) d^4x \\ &= \int_{\mathbb{R}^{3,1}} k(t'+k,\omega_t) f(\mathbf{x}') K(\overrightarrow{x}' + \overrightarrow{K}, \overrightarrow{\omega}) d^4x' \end{aligned} \tag{3.26}$$

Now we compute the kernel factors separately.

$$\begin{aligned} k(t'+k,\omega_t) &= \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(a(t'+k)^2 + 2m(t'+k) + d\omega_t^2 - 2(t'+k+dm-bn)\omega_t\right)} \\ &= \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at'^2 + 2akt' + ak^2 + 2mt' + 2mk + d\omega_t^2 - 2(t'+dm-bn)\omega_t - 2k\omega_t\right)} \\ &= e^{\frac{e_t}{2b}\left(ak^2 + 2mk - 2k\omega_t\right)} \frac{1}{\sqrt{2\pi b}} \\ &\quad e^{\frac{e_t}{2b}\left(at'^2 + 2mt' + d(\omega_t - ak)^2 + 2dak\omega_t - da^2k^2 - 2(t'+dm-bn)(\omega_t - ak) - 2(dm-bn)ak\right)} \\ &= e^{\frac{e_t}{2b}\left(ak^2 - da^2k^2 - 2k\omega_t + 2dak\omega_t + 2mk - 2(dm-bn)ak\right)} \\ &\quad \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at'^2 + 2mt' + d(\omega_t - ak)^2 - 2(t'+dm-bn)(\omega_t - ak)\right)} \\ &= e^{e_t\left(\frac{a}{2b}(1-da)k^2 + \frac{2}{2b}(-1+da)k\omega_t + \frac{2}{2b}(1-da)mk + \frac{2b}{2b}nak\right)} k(t', \omega_t - ak) \\ &= e^{e_t\left(-\frac{1}{2}ack^2 + ck\omega_t - cmk + nak\right)} k(t', \omega_t - ak), \end{aligned} \tag{3.27}$$

where in the fifth equality we have replaced the line before by $k(t', \omega_t - ak)$, according to (3.3), and for the last equality we have used the determinant $ad - bc = 1$ of (3.2), i.e. $c = (ad-1)/b$. We can similarly compute for the right kernel factor

$$K(\overrightarrow{x}' + \overrightarrow{K}, \overrightarrow{\omega}) = K(\overrightarrow{x}', \overrightarrow{\omega} - A\overrightarrow{K}) \\ e^{i_3\left(-\frac{1}{2}AC\overrightarrow{K}^2 + C\overrightarrow{K}\cdot\overrightarrow{\omega} - C\overrightarrow{M}\cdot\overrightarrow{K} + A\overrightarrow{N}\cdot\overrightarrow{K}\right)} \tag{3.28}$$

Inserting (3.27) and (3.28) in (3.26), we can factor out the exponentials to the left and right, respectively, and obtain (3.25). $\square$

Theorem 3.10. For $f \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, $\mathbf{x}, \omega \in \mathbb{R}^{3,1}$, and constant (modulation frequency) vector $\mu = \mu_t e_t + \overrightarrow{\mu} \in \mathbb{R}^{3,1}$ we have

$$\mathcal{F}\{e^{e_t t\mu_t} f(\mathbf{x}) e^{i_3 \overrightarrow{x}\cdot\overrightarrow{\mu}}\}(\omega) = e^{-e_t\left(\frac{1}{2}d(b\mu_t^2 - 2\mu_t\omega_t) + \mu_t(dm-bn)\right)} \\ \mathcal{F}\{f(\mathbf{x})\}\big(\omega - (b\mu_t e_t + B\overrightarrow{\mu})\big) e^{-i_3\left(\frac{1}{2}D(B\overrightarrow{\mu}^2 - 2\overrightarrow{\mu}\cdot\overrightarrow{\omega}) + \overrightarrow{\mu}\cdot(D\overrightarrow{M} - B\overrightarrow{N})\right)}. \quad (3.29)$$

Proof. We compute

$$\mathcal{F}\{e^{e_t t\mu_t} f(\mathbf{x}) e^{i_3 \overrightarrow{x}\cdot\overrightarrow{\mu}}\}(\omega) \\ = \int_{\mathbb{R}^{3,1}} k(t,\omega_t) e^{e_t t\mu_t} f(\mathbf{x}) e^{i_3 \overrightarrow{x}\cdot\overrightarrow{\mu}} K(\overrightarrow{x}, \overrightarrow{\omega}) d^4x. \quad (3.30)$$

First we focus on the left side factors

$$k(t,\omega_t)e^{e_t t\mu_t} = \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at^2 + 2mt + d\omega_t^2 - 2(t+dm-bn)\omega_t\right)} e^{e_t t\mu_t} \\ = \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at^2 + 2mt + d\omega_t^2 - 2(t+dm-bn)\omega_t + 2tb\mu_t\right)} \\ = \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at^2 + 2mt + d\omega_t^2 - 2db\mu_t\omega_t + db^2\mu_t^2 + 2db\mu_t\omega_t - db^2\mu_t^2\right)} \\ e^{\frac{e_t}{2b}\left(-2(t+dm-bn)(\omega_t - b\mu_t) - 2(dm-bn)b\mu_t\right)} \\ = e^{\frac{e_t}{2b}\left(2db\mu_t\omega_t - db^2\mu_t^2 - 2(dm-bn)b\mu_t\right)} \\ \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(at^2 + 2mt + d(\omega_t - b\mu_t)^2 - 2(t+dm-bn)(\omega_t - b\mu_t)\right)} \\ = e^{-e_t\left(\frac{1}{2}d(b\mu_t^2 - 2\mu_t\omega_t) + \mu_t(dm-bn)\right)} k(t, \omega_t - b\mu_t), \quad (3.31)$$

where in the last line we used the expression for the left kernel factor from (3.3). Similarly, we can treat the right factors and obtain

$$e^{i_3 \overrightarrow{x}\cdot\overrightarrow{\mu}} K(\overrightarrow{x}, \overrightarrow{\omega}) \\ = K(\overrightarrow{x}, \overrightarrow{\omega} - B\overrightarrow{\mu}) e^{-i_3\left(\frac{1}{2}D(B\overrightarrow{\mu}^2 - 2\overrightarrow{\mu}\cdot\overrightarrow{\omega}) + \overrightarrow{\mu}\cdot(D\overrightarrow{M} - B\overrightarrow{N})\right)}. \quad (3.32)$$

By inserting (3.31) and (3.32) into (3.30), we can factor out the exponential factors to the left and right, respectively, and obtain (3.29). □

Theorem 3.11. For $f, \mathcal{F}\{f\} \in L^1(\mathbb{R}^{3,1}; Cl(3,1))$, we obtain the inverse transform as

$$f(\mathbf{x}) = \int_{\mathbb{R}^{3,1}} \overline{k(\omega_t, t)}\, \mathcal{F}\{f\}(\omega)\, \overline{K(\overrightarrow{x}, \overrightarrow{\omega})}\, d^4\omega, \quad (3.33)$$

with overbar denoting the principal reverse [20], i.e. $\overline{e_t} = -e_t$, $\overline{i_3} = -i_e$.

Proof. We compute

$$\begin{aligned}
&\int_{\mathbb{R}^{3,1}} \overline{k(\omega_t,t)}\,\mathcal{F}\{f\}(\omega)\,\overline{K(\vec{x},\vec{\omega})}\,d^4\omega \\
&= \int_{\mathbb{R}^{3,1}} \frac{1}{\sqrt{2\pi b}} e^{-\frac{e_t}{2b}\left(at^2+2mt+d\omega_t^2-2(t+dm-bn)\omega_t\right)} \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)} \\
&\qquad \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t+\frac{\vec{\omega}}{B}\Big) e^{\frac{i_3}{2B}\left(D\vec{\omega}^2+2(B\vec{N}-D\vec{M})\cdot\vec{\omega}\right)} \frac{1}{(2\pi B)^{3/2}} \\
&\qquad e^{-\frac{i_3}{2B}\left(A\vec{x}^2+2\vec{M}\cdot\vec{x}+D\vec{\omega}^2-2(\vec{x}+D\vec{M}-B\vec{N})\cdot\vec{\omega}\right)} \frac{1}{(2\pi B)^{3/2}}\,d^4\omega \\
&= \int_{\mathbb{R}^{3,1}} \frac{1}{2\pi b}\frac{1}{(2\pi B)^3} e^{-\frac{e_t}{2b}\left(at^2+2mt-2t\omega_t\right)}\;\mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t+\frac{\vec{\omega}}{B}\Big) \\
&\qquad e^{-\frac{i_3}{2B}\left(A\vec{x}^2+2\vec{M}\cdot\vec{x}-2\vec{x}\cdot\vec{\omega}\right)}\,d^4\omega \\
&= \frac{1}{2\pi b}\frac{1}{(2\pi B)^3}\int_{\mathbb{R}^{3,1}} e^{-\frac{e_t}{2b}\left(at^2+2mt-2t\omega_t\right)}\;\Big\{\int_{\mathbb{R}^{3,1}} e^{-\frac{e_t}{b}\omega_t t'} e^{\frac{e_t}{2b}(at'^2+2mt')} f(\mathbf{x}') \\
&\qquad e^{\frac{i_3}{2B}(A\vec{x}'^2+2\vec{M}\cdot\vec{x}')}\,e^{-\frac{i_3}{B}\vec{\omega}\cdot\vec{x}'}\,d^4x'\Big\} e^{-\frac{i_3}{2B}\left(A\vec{x}^2+2\vec{M}\cdot\vec{x}-2\vec{x}\cdot\vec{\omega}\right)}\,d^4\omega \\
&= e^{-\frac{e_t}{2b}\left(at^2+2mt\right)} \int_{\mathbb{R}^{3,1}} \Big[\frac{1}{2\pi b}\int_{\mathbb{R}^1} e^{-\frac{e_t}{b}\omega_t t'} e^{\frac{e_t}{b}t\omega_t}\,d\omega_t\Big] e^{\frac{e_t}{2b}(at'^2+2mt')} f(\mathbf{x}') \\
&\qquad e^{\frac{i_3}{2B}(A\vec{x}'^2+2\vec{M}\cdot\vec{x}')}\Big[\frac{1}{(2\pi B)^3}\int_{\mathbb{R}^3} e^{-\frac{i_3}{B}\vec{\omega}\cdot\vec{x}'} e^{\frac{i_3}{B}\left(\vec{x}\cdot\vec{\omega}\right)}\,d^3\omega\Big]\,d^4x' \\
&\qquad e^{-\frac{i_3}{2B}\left(A\vec{x}^2+2\vec{M}\cdot\vec{x}\right)} \\
&= e^{-\frac{e_t}{2b}\left(at^2+2mt\right)} \int_{\mathbb{R}^{3,1}} \delta(t-t')\,e^{\frac{e_t}{2b}(at'^2+2mt')} f(\mathbf{x}') \\
&\qquad e^{\frac{i_3}{2B}(A\vec{x}'^2+2\vec{M}\cdot\vec{x}')}\,\delta^3(\vec{x}-\vec{x}')\,d^4x'\; e^{-\frac{i_3}{2B}\left(A\vec{x}^2+2\vec{M}\cdot\vec{x}\right)} \\
&= e^{-\frac{e_t}{2b}\left(at^2+2mt\right)}\; e^{\frac{e_t}{2b}(at^2+2mt)} f(\mathbf{x}) e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\; e^{-\frac{i_3}{2B}\left(A\vec{x}^2+2\vec{M}\cdot\vec{x}\right)} \\
&= f(\mathbf{x}).
\end{aligned} \tag{3.34}$$

For the first equality we insert the definition of the kernel factors (3.3), and Lemma 3.5 that expresses the SASFT in terms of the SFT. For the third equality we use the definitions of the SFT (2.14) and of the function g in terms of f (3.6). For the fourth equality we apply Fubini's theorem for changing the order of integration. □

For the SASFT we have the following Rayleigh (Parseval) energy theorem.

Theorem 3.12. A space-time algebra signal $f \in L^2(\mathbb{R}^{3,1}; Cl(3,1))$ and its SASFT $\mathcal{F}\{f\}$ satisfy the energy identity

$$||\mathcal{F}\{f\}||_2 = ||f||_2. \tag{3.35}$$

Proof. We compute

$$
\begin{aligned}
(\|\mathcal{F}\{f\}\|_2)^2 &= \int_{\mathbb{R}^{3,1}} |\mathcal{F}\{f\}(\omega)|^2 d^4\omega \\
&= \int_{\mathbb{R}^{3,1}} \Big| \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)} \\
&\qquad \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t + \frac{\vec{\omega}}{B}\Big) e^{\frac{i_3}{2B}\left(D\vec{\omega}^2+2(B\vec{N}-D\vec{M})\cdot\vec{\omega}\right)} \frac{1}{(2\pi B)^{3/2}}\Big|^2 d^4\omega \\
&= \int_{\mathbb{R}^{3,1}} \Big| \frac{1}{\sqrt{2\pi b}} \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t + \frac{\vec{\omega}}{B}\Big) \frac{1}{(2\pi B)^{3/2}}\Big|^2 d^4\omega \\
&= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} \Big| \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t + \frac{\vec{\omega}}{B}\Big)\Big|^2 \frac{d^4\omega}{bB^3} \\
&= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} \Big| \mathcal{F}_{SFT}\{g\}(\omega')\Big|^2 d^4\omega' \\
&= \frac{1}{(2\pi)^4}(2\pi)^4\int_{\mathbb{R}^{3,1}} |g(\mathbf{x})|^2 d^4x \\
&= \int_{\mathbb{R}^{3,1}} \Big| e^{\frac{e_t}{2b}(at^2+2mt)} f(\mathbf{x})\, e^{\frac{i_3}{2B}(A\vec{x}^2+2\vec{M}\cdot\vec{x})}\Big|^2 d^4x \\
&= \int_{\mathbb{R}^{3,1}} |f(\mathbf{x})|^2 d^4x = (\|f\|_2)^2,
\end{aligned} \tag{3.36}
$$

where we used Lemma 3.5 for the second equality, substitution $\omega' = \frac{\omega_t}{b}e_t + \frac{\vec{\omega}}{B}$, $d^4\omega' = d^4\omega/(bB^3)$, for the fifth equality, Lemma 2.6 for the sixth equality, and the definition of g by (3.6) for the seventh equality. Finally, taking the square root of both sides of $(\|\mathcal{F}\{f\}\|_2)^2 = (\|f\|_2)^2$, completes the proof. □

In analogy to Lemma 3.7 of [7] we can establish relationships between signal derivatives and multiplication with frequency components of the SA-SFT.

Lemma 3.13. *For $f, \frac{\partial}{\partial t}f, \frac{\partial}{\partial x_k}f \in L^2(\mathbb{R}^{3,1}; Cl(3,0))$, $k = 1,2,3$, provided that the derivatives exist, we obtain*

$$
\begin{aligned}
&\int_{\mathbb{R}^{3,1}} \omega_t^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega = b^2 \int_{\mathbb{R}^{3,1}} \Big| e_t\big(\frac{a}{b}t + \frac{m}{b}\big) f(\mathbf{x}) + \frac{\partial}{\partial t} f(\mathbf{x})\Big|^2 d^4x, \\
&\int_{\mathbb{R}^{3,1}} \omega_k^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega \\
&\qquad = B^2 \int_{\mathbb{R}^{3,1}} \Big| \big(\frac{A}{B}x_k + \frac{M_k}{B}\big) f(\mathbf{x}) i_3 + \frac{\partial}{\partial x_k} f(\mathbf{x})\Big|^2 d^4x.
\end{aligned} \tag{3.37}
$$

Proof. We compute

$$\begin{aligned}
&\int_{\mathbb{R}^{3,1}} \omega_t^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega = \int_{\mathbb{R}^{3,1}} |\omega_t e_t \mathcal{F}\{f\}(\omega)|^2 d^4\omega \\
&= \int_{\mathbb{R}^{3,1}} \Big| \frac{1}{\sqrt{2\pi b}} e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)} \omega_t e_t \ \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t + \frac{\overrightarrow{\omega}}{B}\Big) \\
&\qquad e^{\frac{i_3}{2B}\left(D\overrightarrow{\omega}^2+2(B\overrightarrow{N}-D\overrightarrow{M})\cdot\overrightarrow{\omega}\right)} \frac{1}{(2\pi B)^{3/2}}\Big|^2 d^4\omega \\
&= \frac{b^2}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} \Big| e^{\frac{e_t}{2b}\left(d\omega_t^2+2(bn-dm)\omega_t\right)} \frac{\omega_t}{b} e_t \ \mathcal{F}_{SFT}\{g\}\Big(\frac{\omega_t}{b}e_t + \frac{\overrightarrow{\omega}}{B}\Big) \\
&\qquad e^{\frac{i_3}{2B}\left(D\overrightarrow{\omega}^2+2(B\overrightarrow{N}-D\overrightarrow{M})\cdot\overrightarrow{\omega}\right)}\Big|^2 \frac{d^4\omega}{bB^3} \\
&= \frac{b^2}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} \Big|\mathcal{F}_{SFT}\{\frac{\partial}{\partial t}g(\mathbf{x})\}\Big(\frac{\omega_t}{b}e_t + \frac{\overrightarrow{\omega}}{B}\Big)\Big|^2 \frac{d^4\omega}{bB^3} \\
&= \frac{b^2}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} \Big|\mathcal{F}_{SFT}\{\frac{\partial}{\partial t}g(\mathbf{x})\}(\omega')\Big|^2 d^4\omega' \\
&= b^2 \int_{\mathbb{R}^{3,1}} \Big|\frac{\partial}{\partial t}g(\mathbf{x})\Big|^2 d^4x \\
&= b^2 \int_{\mathbb{R}^{3,1}} \Big| e^{\frac{e_t}{2b}(at^2+2mt)} \Big[e_t(\frac{a}{b}t + \frac{m}{b})f(\mathbf{x}) + \frac{\partial}{\partial t}f(\mathbf{x})\Big] e^{\frac{i_3}{2B}(A\overrightarrow{x}^2+2\overrightarrow{M}\cdot\overrightarrow{x})}\Big|^2 d^4x \\
&= b^2 \int_{\mathbb{R}^{3,1}} \Big|e_t(\frac{a}{b}t + \frac{m}{b})f(\mathbf{x}) + \frac{\partial}{\partial t}f(\mathbf{x})\Big|^2 d^4x, \qquad (3.38)
\end{aligned}$$

where we used Example 2.1 for the first equality, Lemma 3.5 for the second equality, Lemma 2.8 for the fourth equality, the substitution $\omega' = \frac{\omega_t}{b}e_t + \frac{\overrightarrow{\omega}}{B}$ and $d^4\omega' = d^4\omega/(bB^3)$ for the fifth equality, Lemma 2.6 for the sixth equality, and Lemma 3.6 for the seventh equality. □

3.3. Uncertainty Principle for the SASFT

We now establish the directional uncertainty principle for the SASFT. The uncertainty principle specifies how precisely a signal can be measured in space as well as in its spectral (frequency) domain, and is therefore of universal importance in quantum theory, optics and signal processing.

Theorem 3.14. For two arbitrary constant space-time vectors $\mathbf{c}, \mathbf{d} \in \mathbb{R}^{3,1}$ and $f, |\mathbf{x}|^{1/2} f \in L^2(\mathbb{R}^{3,1}; Cl(3,0))$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^{3,1}} (c_t t - \overrightarrow{c}\cdot\overrightarrow{x})^2 |f(\mathbf{x})|^2 d^4x \int_{\mathbb{R}^{3,1}} (d_t\omega_t - \overrightarrow{d}\cdot\overrightarrow{\omega})^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega \\
&\qquad \geq \frac{1}{4}\Big[(c_t d_t' - \overrightarrow{c}\cdot\overrightarrow{d}')^2 F_-^2 + (c_t d_t' + \overrightarrow{c}\cdot\overrightarrow{d}')^2 F_+^2\Big], \qquad (3.39)
\end{aligned}$$

with $\mathbf{d}' = b d_t e_t + B\overrightarrow{d}$, and energies of the left- and right traveling wavepackets

$$F_\pm = \int_{\mathbb{R}^{3,1}} |f_\pm(\mathbf{x})|^2 d^4x. \qquad (3.40)$$

Proof. We observe that according to (3.6) and Lemma 3.5

$$|\mathcal{F}\{f\}(\omega)|^2 = \frac{1}{(2\pi)^4 bB^3}|\mathcal{F}_{SFT}\{g\}(\omega')|^2, \qquad \omega' = \frac{\omega_t}{b}e_t + \frac{\vec{\omega}}{B}. \tag{3.41}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^{3,1}} (d_t\omega_t - \vec{d}\cdot\vec{\omega})^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega \\
&= \frac{1}{(2\pi)^4 bB^3}\int_{\mathbb{R}^{3,1}} (d_t\omega_t - \vec{d}\cdot\vec{\omega})^2 |\mathcal{F}_{SFT}\{g\}(\omega')|^2 d^4\omega \\
&= \frac{bB^3}{(2\pi)^4 bB^3}\int_{\mathbb{R}^{3,1}} (d_t'\omega_t' - \vec{d}'\cdot\vec{\omega}')^2 |\mathcal{F}_{SFT}\{g\}(\omega')|^2 d^4\omega'
\end{aligned} \tag{3.42}$$

with $\mathbf{d}' = bd_te_t + B\vec{d}$, such that $d_t\omega_t - \vec{d}\cdot\vec{\omega} = d_t'\omega_t' - \vec{d}'\cdot\vec{\omega}'$, and $d^4\omega = bB^3d^4\omega'$. Relabeling $\omega' \to \omega$ in the last integral of (3.42), and using $|f(\mathbf{x})| = |g(\mathbf{x})|$ of Lemma 3.7, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{3,1}} (c_tt - \vec{c}\cdot\vec{x})^2 |f(\mathbf{x})|^2 d^4x \int_{\mathbb{R}^{3,1}} (d_t\omega_t - \vec{d}\cdot\vec{\omega})^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega = \\
&\int_{\mathbb{R}^{3,1}} (c_tt - \vec{c}\cdot\vec{x})^2 |g(\mathbf{x})|^2 d^4x \frac{1}{(2\pi)^4}\int_{\mathbb{R}^{3,1}} (d_t'\omega_t - \vec{d}'\cdot\vec{\omega})^2 |\mathcal{F}_{SFT}\{g\}(\omega)|^2 d^4\omega \\
&\quad \geq \frac{1}{4}\Big[(c_td_t' - \vec{c}\cdot\vec{d}')^2F_-^2 + (c_td_t' + \vec{c}\cdot\vec{d}')^2F_+^2\Big],
\end{aligned} \tag{3.43}$$

where we applied the directional uncertainty principle of the SFT (Theorem 2.9) for the inequality, and replaced $G_\pm = \int_{\mathbb{R}^{3,1}} |g_\pm(\mathbf{x})|^2 d^4x = F_\pm$, i.e. finally applying Lemma 3.7 to the left- and right traveling energy integrals for g. □

For coordinate directions we get the following specialization.

Corollary 3.15. *For any two space-time and frequency coordinates x_λ, ω_μ, with $\lambda, \mu \in \{t, 1, 2, 3\}$, $x_t = t$, and $f, |\mathbf{x}|^{1/2}f \in L^2(\mathbb{R}^{3,1}; Cl(3,0))$ we have*

$$\int_{\mathbb{R}^{3,1}} x_\lambda^2 |f(\mathbf{x})|^2 d^4x \int_{\mathbb{R}^{3,1}} \omega_\mu^2 |\mathcal{F}\{f\}(\omega)|^2 d^4\omega \geq \frac{\delta_{\lambda,\mu}}{4}\beta^2F^2, \tag{3.44}$$

with Kronecker symbol $\delta_{\lambda,\mu}$ equal 1 if $\mu = \lambda$ and zero otherwise, $\beta = b$ for $\mu = t$ and $\beta = B$ for $\mu \in \{1, 2, 3\}$, and total signal energy (see (2.12)*)*

$$F = \int_{\mathbb{R}^{3,1}} |f(\mathbf{x})|^2 d^4x = \int_{\mathbb{R}^{3,1}} [|f_-(\mathbf{x})|^2 + |f_+(\mathbf{x})|^2] d^4x = F_- + F_+. \tag{3.45}$$

4. Conclusion

After giving some background on space-time algebra and reviewing the space-time Fourier transform (SFT) and some of its properties, we have defined the special affine space-time Fourier transform as a vast generalization of the SFT on the one hand, which now includes new transforms such as a fractional SFT and others. On the other hand this generalization means also to lift the classical special affine Fourier transforms, with their primary relevance for optics [1, 2], to the level of high dimensional hypercomplex (Clifford) integral transforms. The current work is based on [17], adding detailed proofs for important SASFT properties. We expect that a series of theoretical and applied research on this new class of hypercomplex transforms may ensue in fields like physics, electro-magnetism, optics, signal processing, GPS, space navigation and quantum computing, including quantum internet related signal processing.

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