Singular properties

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Abstract: The purpose of this article as a continuation of development of the multiplical topic is to find a solution for operation of differentiation and factorization of a function with points of interruption, points where function turns to zero. The solution which allows restoring the original function as result of reverse operation of integration and factorial-multiplication of previously obtained derivative and factor-derivative respectively and with an appropriate selection of an arbitrary multiplier B or addend C, respectively. As the result of the work made a number of new classes of function properties and definitions are introduced as function point properties.

Keywords: Natural singular properties, Applicable singular properties, Singular differential, Singular factorial, Symmetrical factorial, Hyperbolic singular factorial, Singular addend, Hyperbolic singular multiplier.

The preceding related article: Multiplical concept https://vixra.org/pdf/2205.0150v1.pdf

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Introduction

Function **critical points** are function singularities (function interruption point), and considering the factorials and the factorization, in addition are points where the function turns to zero.

The operations of differentiation and factorization are interrupted at function critical points, so information about function characteristics at those points are not transferred to its derivative and factor-derivative (hereinafter referred to as the first derivatives), that is, it is lost forever. As a result, with the reverse integration or factorial-multiplication, respectively, of the previously obtained first derivatives, there is no way to restore the original function.

May be we should do nothing about it, but the given circumstance violates the mutual invertibility of factor-derivative and indefinite multiplical as well as derivatives and indefinite integrals or imposes restrictive conditions on the analyzed functions, and thus indirectly violates the postulate of the mutual invertibility of factorial-multiplication and factorization as well as integration and differentiation operations, for which it is worth fighting methodologically, as I guess.

The goal is to develop a new extended logic for the above mentioned operations of mathematical analysis, in the application of which the reverse operations would lead to the effective restoration of the original function in the form of an indefinite multiplical or an indefinite integral with an appropriate selection of an arbitrary multiplier B or addend C, respectively, if not for all, then for a much larger number of cases, and which could be used in practical applications.

The new logic implies existence of new properties for functions. In this case, we are talking about the functions properties at dimensionless points, or the properties that has an effect when passing through a point in the course of analysis. The general name of the class of such properties is **singular properties** of functions at points. These can either be present at points of analytically given functions in a natural way, in which case the analysis will show them, or they can be applied to the points of functions both analytically and automatically in the course of functions factorization and differentiation when certain circumstances occur. Common to the singular properties of functions is that, when analyzed, their value does not depend in any way on the differential of argument **dx**. The first and well-known example of the singular properties is the discrete values in the definition of a function. Despite the fact that the latter does not directly participate in analytic transformations described here, nevertheless, the discrete definition falls under the definition of singular properties.

The new operation logic that takes into account the singular properties of functions finds its notational reflection, which distinguishes it from the classical logic - the one that ignores the singular properties of functions, both naturally present and applied.

Inclusive and exclusive states

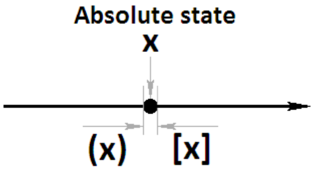
Since we are talking about such properties of functions that are effective when passing through points, it becomes necessary to determine the boundary of the analyzed segment, interval or domain of the function with an additional indication of whether or not the considered boundary point of the interval, segment, domain is included in the corresponding interval, segment, domain, which is true for both (right and left) end boundary points, except for cases when there is simply no boundary point or both at once, when the function domain has no boundary and goes to infinity. Thus, the definition of the border includes not only the coordinate of the point of this border, but also information about the state: inclusive or not inclusive (exclusive), which should be reflected in the notation when writing coordinates. Speaking about the inclusion or exclusion of points in the interval, one should not take this figure of speech literally, because it makes no sense to include or exclude a dimensionless something (points). In this case, we are talking about taking into account (inclusion) or not taking into account (exclusion) the corresponding singular property of the function at a critical point in the process of analyzing the function in one of its two directions: upward or downward of the function argument. The inclusive/exclusive state for the substituted value of the function argument also becomes decisive in the approach under consideration.

For the segment right boundary point the state included in the segment implies the position of the segment boundary when it infinitely approaches the boundary point from the right, and the state excluded in the segment implies the position of the segment boundary when it infinitely approaches the boundary point from the left, and for the left boundary point it is the other way around, included in the segment implies the position of the segment boundary when it infinitely approaches the boundary point from the left and not included in the segment implies the position of the segment boundary when it infinitely approaches the boundary point from the left. This example describes the principle of the relative state inclusive / exclusive, what about a specific segment.

In the circumstances that the analysis of functions is carried out along the entire abscissa axis, with potential division of the analyzed domain into adjacent segments located at the junction with each other, without gaps and overlaps, when the left border of one segment simultaneously means the boundary of the segment following it and adjacent to it (coincides in its definition: the coordinate of the point + inclusive/exclusive), it is necessary to switch to the principle of an absolute, non-segment-centric state of inclusion or exclusion points in intervals, segments and domains, otherwise the conflict of the latter in the “struggle for common points” is inevitable ".

The special notation for the absolute state inclusive/exclusive for the value of an argument or for the coordinate of the boundary of an interval, segment, domain is taken as follows:

- Absolutely inclusive **x**: **[x]**
- Absolutely exclusive **x**: **(x)**



Since there are discrete functions that return their values only at points, and the function analysis is carried out in the direction of growth of its argument by the default, the rule that the absolute state inclusive or exclusive for the coordinates of points, boundaries of intervals, segments, domains of functions coincides with the above-described relative state inclusive or exclusive for the right border of the segment. At the same time, as a consequence of the above reasons, the rule implies an absolute inclusion state by the default for substituted values as a function argument for specifying the coordinate of a point, or boundaries of intervals, segments, domains, including as a definition of integration and factorial-multiplication segments, if another state is not indicated specifically. Expression **f(x)** is equivalent to expression **f([x])**, where both imply inclusive **x**, which distinguishes them from expression **f((x))**, which implies exclusive **x**.

Note: The default inclusive state rule does not apply to a passed function argument at points where a discrete value definition is present. For such points, the expression **f(x)** will return a discrete value at the point **x**. In order to get the value of the continuous definition of the function when approaching the point from the right (if any), you need to give the function an explicit indication of the on state: **f([x])**.

Switching to measurement and taking into account the absolute state inclusive or exclusive for points on the abscissa axis and choosing one of the states as the default state under the conditions of analysis that takes into account the singular properties of functions, leaves valid the following equations of **Multiplical concept** article: **4**; **5**; **7.1**; **7.2,** and valid in the form in which they are presented. Similar equations for definite integrals also remain valid under the conditions described.

Taking into account the state of the point inclusive or exclusive, special binary operators for comparing two coordinate values are introduced, which define the state inclusive as greater, and the state exclusive as less, in case if both compared values have the same numerical values:

[x] **>▪** x **>▪** (x) true,

(x) **▪<** x **▪<** [x] true,

(x) **▪<** (x) false,

(x) **<** [x] false,

where **>▪** is more, taking into account the state; **▪<** is less, taking into account the state; **>** is more stateless; **<** is less stateless; **x** is absolute coordinate of the point, **[x]** is absolutely inclusive **x**; **(x)** is absolutely exclusive of **x**.

Other signs that take into account states: **≥▪** is equal and greater; **▪≤** is equal or less.

Equalities and inequalities of coordinates with states:

(x) = (x) true,

(x) **▪=▪** (x) true,

(x) = [x] true,

(x) **▪=▪** [x] false,

(x) **<▪>** [x] true,

(x) **!▪=▪** [x] true,

(x) **<>** [x] false,

(x) **!=** [x] false,

where **=** is stateless equality; **▪=▪** is stateful equality; **!▪=▪** and **<▪>** are stateful inequality; **!=** and **<>** are stateless inequality.

Arithmetic operations with coordinates ignore states inclusive or exclusive, and the return result of such operations is simply a number, and is identical to the length of the interval, or the coordinate of a point without the specified state:

[x] – (x) = 0 true.

Functions to operate with states:

$$St = get\_state(x), \quad (19.1)$$

$$x_1 = set\_state\ (x_0, St), \quad (19.2)$$

where **get_state** is a function that returns the state of the passed **x** coordinate as follows: **–1** is coordinate exclusive, **+1** is coordinate included, **0** is stateless point or just a real number, **null** is error; **set_state** is a function that returns the $\mathbf{x_1}$ coordinate with the state **St** and $\mathbf{x_0}$ position.

Natural singular properties

Singular factorial and **singular differential** are values that are present at any point of a function, including the function critical points, numerically describing the change in the function from one of its value, which is exclusive at the point, to its other value, which is inclusive at the point when passing through the points in the direction of growth of the argument and are determined as follows: the singular factorial as ratio of function value at point inclusive to function value at point exclusive, the singular differential as difference of function value at point inclusive to function value at point exclusive:

$$\mathbf{sf}f(x) = \frac{f(x)}{f\big((x)\big)}, \qquad (20.1)$$

$$\mathbf{sd}f(x) = f(x) - f\big((x)\big), \qquad (20.2)$$

where **sf*f*(x)** is the singular factorial of the function ***f(x)*** at the point **x**; **sd*f*(x)** is the singular differential of the function ***f(x)*** at the point **x**.

Through the mediation of the function ***f***, there is interdependence between the singular factorial and the singular differential, but only if the hypothetical division by zero operation (in the case of critical points) in the following equations does not prevent this:

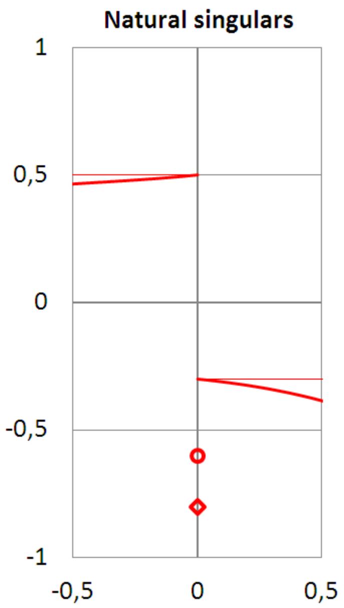
$$\mathbf{sd}f(x) = f\big((x)\big) \cdot \mathbf{sf}f(x) \ - f\big((x)\big) = f(x) - \frac{f(x)}{\mathbf{sf}f(x)}, \qquad (21.1)$$

$$\mathbf{sf}f(x) = \frac{\mathbf{sd}f(x) - f\big((x)\big)}{f\big((x)\big)} = \frac{f(x)}{f(x) - \mathbf{sd}f(x)}, \qquad (21.2)$$

The singular factorial and the singular differential can be characterized as function **singular increments** of the second and first order, respectively.

At non-critical points, the singular differential is 0, and the singular factorial is 1 by definition. This condition is necessary and sufficient. At these points, the described singular properties are not informative.

A finite value different from 1 for a singular factorial and from 0 for a singular differential takes place in function interruption points, but only if the function values at the point on both its sides are within the allowed range that is they are defined finite and for defining the singular factorial are not zero in addition. In such case the singular factorial and the singular differential are informative at the point and they numerically describe the interdependence between the two function domains separated by the interruption point, the singular differential describes the absolute difference of function values, the singular factorial describes the relative difference of function values.



The diagram shows the described case with an interruption of the function at **x=0**. Despite the fact that the singular increments are not being explicitly indicated, because there is no need for, since the graph of the function is self speaking, indirectly indicates the values of the natural singular properties, nevertheless, for clarity, the singular factorial is indicated in this diagram as circle, and the singular differential is indicated as diamante and they have visibly informative values there.

The stored value of this numerical relationship between the function domains separated by a critical point is necessary to restore the function in a reverse factorial-multiplication or reverse integration of its factor-derivative or its derivative respectively. In process of the mentioned operation the continuous definition of the function respective first derivative in the vicinity of the critical point can be excluded and replaced by the stored numerical value as a multiplier or as an addend of the intermediate result of the operation being performed. Thus, despite the obvious gap of the factor-derivative or derivative with values that are out of the allowed range at the point, the described technique allows us to effectively restore the function.

If the function value on one or both sides of the point when approaching it are out of the allowed range then the singular factorial and differential are indefinite, informative. Further analysis based on the singular differential is not possible in the case, and it is not possible to restore the function as a result of the reverse integration of the function derivative. But if the analysis is based on the singular factorial then the one can numerically relate the values of the function measured on both sides of the critical point at the same (symmetrically with respect to the point) and infinitely small distance from it. In other words, find a factorial of a function, as its change with a change in the argument from one of the measurement points to another. In view of the fact that the measurement points are located at an equal distance from the critical point **x**, this factorial can be characterized as a symmetrical factorial:

$$\mathbf{sif}f(\mathrm{x},\mathrm{dx}) = \frac{f(\mathrm{x}+\mathrm{dx})}{f(\mathrm{x}-\mathrm{dx})}, \qquad (22.1)$$

where **x** is the coordinate of the critical point; **dx** is module of an infinitesimal distance from measurement points to the critical point **x**; **si*f*** is the function symmetric factorial as a function of the coordinate x and the module of the infinitesimal distance from measurement points to the critical point **x.**

The previous ratio can be converted into the product of two multipliers: a finite non-zero proportionality factor **V** and in general an infinitesimal or infinitely large which depends on the distance from measurement points to the critical point. Both multipliers are properties of the object - **the hyperbolic singular factorial** as a natural singular property of the function at the point:

$$\mathbf{hsf}f(\mathrm{x}) = \{\,\mathrm{V},\mathrm{H}(\mathrm{dx})\,\} \qquad (22.2)$$

$$\mathbf{sif}f(\mathrm{x},\mathrm{dx}) = \mathrm{V}\cdot\mathrm{H}(\mathrm{dx}) = \mathbf{hsf}f(\mathrm{x}).\mathrm{V}\cdot\mathbf{hsf}f(x).\mathrm{H}(\mathrm{dx}), \qquad (22.3)$$

Example:

$$\mathbf{hsf}f(\mathrm{x}) = \{\,-5,\mathrm{dx}^1\,\} \qquad (22.4)$$

$$\mathbf{sif}f(\mathrm{x},\mathrm{dx}) = -5\cdot\mathrm{dx}^1 \qquad (22.5)$$

where **hsf*f*** is a hyperbolic singular factorial of the function ***f***; **V** or **hsf*f***(x).V is the value of the hyperbolic singular factorial of the function *f* at the point x as its property; **H** or **hsf*f***(x).H is the hyperbolizer as a function of the modulus of the distance **dx** of the measurement points of the function values from the point **x** and as a property of the hyperbolic singular factorial of the function ***f*** at the point **x**; **dx** is the hyperbolization arm as the hyperbolizer argument.

The value of the hyperbolic singular factorial indicates the value of the proportional relation between the domains of the function separated by a critical point.

The hyperbolizer has such a name, because with an infinitesimal value of the hyperbolization arm, it returns an infinitely large or infinitely small value, and even possibly raised to some finite power. When numerically solving definite multiplicals, the hyperbolizer returns sufficiently small or sufficiently large values, depending on the size of the hyperbolization arm and its possible exponentiation. The return value of the hyperbolizer is always positive, since the arithmetical sign of the symmetrical factorial is always bound to the value of the hyperbolic singular factorial **V**. The hyperbolizer cannot be zero or undefined, since such symmetrical singular values are assigned to the value of the hyperbolic singular factorial.

The hyperbolizer describes the order of incommensurability (not parity) of function values on both sides of a critical point when approaching it to the limit. Incommensurability is a general case of function properties at critical points. If the values of the functions on both sides of the critical point are commensurate (parity), then the returned value of the hyperbolizer is 1, in which case we can talk about a unit hyperbolizer, and in this case the symmetric factorial is equivalent to the singular factorial at the function point, and this case is special. Such cases take place for finite non-zero function values at a critical point on both sides of from it, as shown in the previous diagram, and also if on both sides of the critical point the function is represented by the same analytically given constituent functions, differing only in their multipliers.

The hyperbolizer as a multiplier of the value of the hyperbolic singular factorial, hyperbolizes it, thereby justifying the names of this class of natural singular properties. This actually explains the methodological division of the symmetrical factorial into two multipliers, one of them is a conditional constant that does not depend on the hyperbolization arm, and the other is a conditional variable that depends on the arm of the hyperbolizer only, and which has its direct decisive value when carrying out factorial-multiplication using arbitrarily chosen size of the factorial-multiplication element.

As an example, consider a hyperbolic singular factorial at the junction (at a critical point) at x = 0 of two different power polynomials as constituent functions. Based on the general formulation of the problem both functions do not necessarily have finite non-zero values at critical point. They represented by the following equation, with different values of multipliers of terms and exponents for **x** in the general case:

$$f(x) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{b_i} \cdot \mathrm{x}^{\mathrm{a_i}}, \text{for } \mathrm{a_i} \in \mathrm{Z}, \qquad (23.1)$$

$$f(x) = \mathrm{sign(x)} \cdot \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{b_i} \cdot |\mathrm{x}|^{\mathrm{a_i}}, \text{for } \mathrm{a_i} \in \mathrm{R}. \qquad (23.1)$$

where **N** is the number of power terms in the definition of the constituent function; $\mathbf{b_i}$ - **i**-th term multiplier; $\mathbf{a_i}$ is the exponent of the argument **x**; **sign** — function that returns **–1** in case if the passed argument is negative, otherwise returns **+1**.

When approaching a critical point infinitely close (x = 0), the decisive value of the function will be the term with the minimum value of $\mathbf{a_i}$ from among those with a finite non-zero value of $\mathbf{b_i}$. When **x** tends to zero, the components of the value of functions from other terms with higher $\mathbf{a_i}$ and finite nonzero **b**, regardless of the size of the latter, can be neglected, in view of the ratio of these components of the value of the function to the component of the value of the function from the decisive term tends to zero. So, if the decisive term is with negative $\mathbf{a_i}$, then the value of the function at a critical point is not defined and it tends to infinity in absolute value as **x** tends to the point, and if the decisive term is with positive $\mathbf{a_i}$, then the value of the function at a critical point is equal to zero and tends to zero as **x** tends to a point. If the decisive term is with zero $\mathbf{a_i}$, then the value of the function at a critical point is equal to **b**, a finite non-zero.

The value of the symmetrical singular factorial at a critical point at x=0:

For integer $\mathbf{a_i}$**:**

$$\mathbf{sif}f(\mathrm{x, dx}) = \frac{f(+\mathrm{dx})}{f(-\mathrm{dx})} = \frac{\mathrm{b}_k \cdot (+\mathrm{dx})^{\mathrm{a_k}}}{\mathrm{b}_j \cdot (-\mathrm{dx})^{\mathrm{a_j}}} = \mathrm{sign}\big(-|\mathrm{a_j}|\%2\big) \cdot \frac{\mathrm{b}_k}{\mathrm{b}_j} \cdot \mathrm{dx}^{(\mathrm{a_k}-\mathrm{a_j})}, \qquad (23.3)$$

For real **a_i:**

$$\mathbf{sif}f(\mathrm{x},\mathrm{dx}) = \frac{f(+\mathrm{dx})}{f(-\mathrm{dx})} = \frac{+\mathrm{b}_k \cdot \mathrm{dx}^{\mathrm{a_k}}}{-\mathrm{b}_j \cdot \mathrm{dx}^{\mathrm{a_j}}} = -\frac{\mathrm{b}_k}{\mathrm{b}_j} \cdot \mathrm{dx}^{(\mathrm{a_k}-\mathrm{a_j})}, \qquad (23.4)$$

where **j** is the index of the decisive term of the constituent function, which is before the critical point; **k** is the index of the decisive term of the constituent function, which is after the critical point; **%** is the division remainder operator.

Properties of a hyperbolic singular factorial:

$$\mathbf{hsf}f(\mathrm{x}).\mathrm{V} = \mathrm{sign}\big(-|\mathrm{a}|_\mathrm{j}\%2\big) \cdot \frac{\mathrm{b}_k}{\mathrm{b}_j} \text{ for } \mathrm{a_i} \in \mathrm{Z}, \qquad (23.5)$$

$$\mathbf{hsf}f(\mathrm{x}).\mathrm{V} = -\frac{\mathrm{b}_k}{\mathrm{b}_j} \text{ for } \mathrm{a_i} \in \mathrm{R}, \qquad (23.6)$$

$$\mathbf{hsf}f(\mathrm{x}).\mathrm{H}(\mathrm{dx}) = \mathrm{dx}^{(\mathrm{a_k}-\mathrm{a_j})}. \qquad (23.7)$$

For constituent functions represented by a power polynomial with natural **a_i** including 0, the definition of a hyperbolic singular factorial can be expressed in terms of the finite values of the constituent functions derivatives at a critical point on either side of it, both inclusive and non-inclusive:

$$\mathbf{sif}f(\mathrm{x},\mathrm{dx}) = \frac{f(\mathrm{x}+\mathrm{dx})}{f(\mathrm{x}-\mathrm{dx})} = \mathrm{sign}(-\mathrm{j}\%2) \cdot \frac{\mathrm{f}'^{\mathrm{k}}(\mathrm{x}) \cdot !\,\mathrm{j}}{\mathrm{f}'^{\mathrm{j}}\big((\mathrm{x})\big) \cdot !\,\mathrm{k}} \cdot \mathrm{dx}^{\mathrm{k}-\mathrm{j}}, \qquad (23.8)$$

$$\mathbf{hsf}f(\mathrm{x}).\mathrm{V} = \mathrm{sign}(-\mathrm{j}\%2) \cdot \frac{\mathrm{f}'^{\mathrm{k}}(\mathrm{x}) \cdot !\,\mathrm{j}}{\mathrm{f}'^{\mathrm{j}}\big((\mathrm{x})\big) \cdot !\,\mathrm{k}}, \qquad (23.9)$$

$$\mathbf{hsf}f(x).\mathrm{H}(\mathrm{dx}) = \mathrm{dx}^{\mathrm{k}-\mathrm{j}}, \qquad (23.10)$$

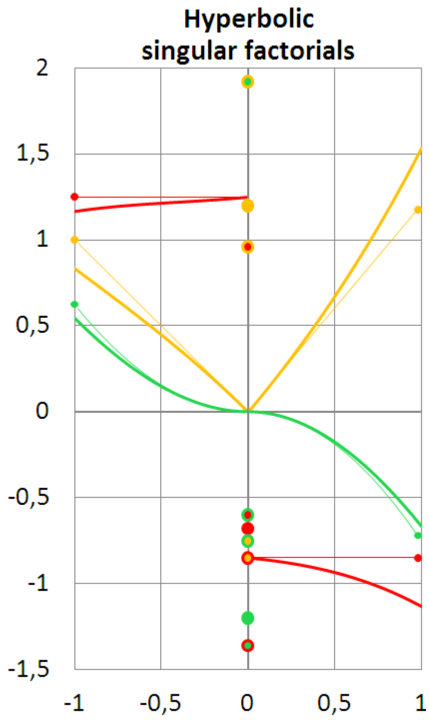
where **k** is the ordinal number of the first found derivative of a non-zero final value at the point **x** inclusive; **j** is the ordinal number of the first found derivative of a non-zero final value at the point **x** exclusive; **$f'_k(x)$** is the value of the derivative of the ordinal number **k** at the point **x** inclusive; **$f'_j((x))$** is the value of the derivative of the ordinal number **j** at the point **x** exclusive.

For the purpose of the notation universalizing the function ***f*** itself is understood as a conditional derivative of the zero ordinal number or the conditional **0^{th}** derivative and the value of the expression !0 is taken equal to 1 as the result of the product of the zero number of multipliers. The enumeration of derivatives is carried out up to the first one found with a non-zero final value in the direction of growth of their ordinal number, starting from zero. The first derivatives found with non-zero finite values, in this case under the ordinal numbers **j** and **k**, are the decisive derivatives for their respective function domains, since the increase in the derivatives of higher ordinal numbers can be neglected if **dx** tends to zero.

Since, the function value is negative to the left of the critical point for an odd exponentiation of the decisive term or for an odd ordinal number of the decisive derivative, and also since the positive symmetric factorial must describe the numerical proportional relationship of the pair of function values of the same arithmetic sign on both sides of the critical point (without passing through x-axis), and negative, on the contrary, the different arithmetic sign (with x-axis crossing), the result of the ratio is multiplied by **–1** , which is reflected in the multiplier as the returning value of the **sign** function.

If the function value on one of the sides is finite non-zero, that is in case of the polynomial at $\mathbf{a_j = 0}$ or $\mathbf{a_k = 0}$ or in case of the function's conditionally zero derivative being decisive (k=0 or j=0), and since this value, in turn, does not depend on the size of **dx**, then on the corresponding side the module of the interval between the function value measurement point and the critical point can be greater or less than the same on the opposite side, or even equal to zero. The rule of symmetry of the positions of the function values measurement points relative to the critical point can be violated in this case, and then the module of the interval on the side of the critical point, where the value of the function is determined by the derivative of the ordinal number from one and higher, is substituted as **dx**.

<u>Note</u>: It should not be assumed at all that the hyperbolizer is an exclusively power function, as in the above example with a power polynomial, since the described is a special case. So, for example, the constituent function may be logarithmic, thus the logarithm will be present in the hyperbolizer definition. Nevertheless in general a successful finding hyperbolic singular factorial via analysis of the constituent functions derivative values at the critical point, that is the confirmed fact of their finite non-zero values there, makes the hyperbolizer definition to be exactly a power function in deed.



On the diagram, colored thick lines indicate graphs of the continuous function definition of two domains on both sides of the critical point in a form of power polynomials with natural exponents. Thin lines indicate the component of the function value from the decisive term or, in other words, the function growth from the decisive derivative measured in the critical point on the corresponding side of it, with the exponent of the term or with the ordinal number of the derivative, respectively, as follows: red - zero (the function itself), orange - the first, green - the second. On both sides of the critical point at points with coordinates along the x-axis –1 and 1, components of the function values from the decisive terms were measured, or in other words, the growth of the function from the decisive derivative was measured. Small one-color bullets located at the intersection of the thin lines with the grid lines of the coordinate system at marks –1 and 1 reflect the result of the corresponding measurement. The ratio of **Y** coordinate values of all pairs of the small bullets of different colors are numerically reflected as the **Y** coordinates of large bullets, which are located in the coordinate **x** of the critical point, as a graphical reflection of the point property. For them, the outer color of the bullet is associated with the function domain to the right of the critical point, and the inner color is associated with the function domain to the left of the critical point. Thus, the numerical relation of the function domains in the vicinity of a critical point is reflected, arising from the corresponding pair of terms or from the corresponding pair of derivatives. Since the arm of the hyperbolizer **dx = 1** in the diagram, and therefore the return value of the hyperbolizer is one, these values would point to the values of the hyperbolic singular factorials if the terms or the derivatives whose graphs are displayed as thin lines of the corresponding colors would be decisive. The values of large single-color bullets do not depend on **dx**, since in this case pairs of identical functions join at a critical point and the value of the hyperbolizer for such a pair is equal to one regardless the **dx** size.

The singular factorial of a function is a special case of the hyperbolic singular factorial, and the hyperbolic singular factorial as a concept is a hyperbolic extension of the concept of a singular factorial. Probably here it is worth noting that the concept of a singular differential cannot fundamentally have its own hyperbolic expansion since this does not make sense. An evidence of this is also the fact that the concept of zero-order derivatives is principally absent, in contrast to how the existence of the concept of first-order derivatives (one order lower than the order of a singular factorial) gives sense to the concept of a hyperbolic singular factorial as it is described above. Concepts and entities of higher orders are based on concepts and entities of lower orders, and we can only recognize the higher and lower orders in entities and concepts. This is the philosophy of mathematics, which once again brings us back to its source - to the concept of a hyper-operator.

Singular factorial and singular differential, as well as the hyperbolic singular factorial, are inherent natural singular properties of functions at points revealed by analysis, they cannot be removed from the function definition and cannot be added to it as they are not part of it, but they depend on it.

Applicable singular properties and logic of factorization and differentiation taking into account applicable singular properties

According to the updated logic of the operations of factorization and differentiation of functions, when these operations pass through critical points, the hyperbolic singular factorial or singular differential existing at such points and belonging to them, respectively, is automatically copied into the **applicable singular properties** of the functions, respectively: **hyperbolic singular multiplier** or **singular addend**, generally named as **applicable singulars**, and are automatically applied to the generated, respectively factor-derivative or derivative at the appropriate points and by this becoming **applied singulars**. Thus, the operation of factorization or differentiation of a function is not interrupted at critical points of the respectively modulated or differentiable function:

$$\mathbf{hsm}f^{\bullet}(\mathrm{x}) = \mathbf{hsf}f(\mathrm{x}), \qquad (24.1)$$

$$\mathbf{hsm}f(\mathrm{x}) = \mathbf{hsf}F^{\bullet}(\mathrm{x}), \qquad (24.2)$$

$$\mathbf{sa}f'(\mathrm{x}) = \mathbf{sd}f(\mathrm{x}), \qquad (24.3)$$

$$\mathbf{sa}f(\mathrm{x}) = \mathbf{sd}F(\mathrm{x}), \qquad (24.4)$$

where **hsm*f*(**x) is the hyperbolic singular multiplier of the function ***f*** at the point **x**; **sa*f***(x) is the singular addend of the function ***f*** at the point **x**; ***f*•** - factor-derivative of the function ***f***; ***f'*** - derivative of the function ***f***; ***F*•** factor-anti-derivative of the function ***f***; ***F*** is the anti-derivative of the function ***f***.

It seems important not to confuse natural singular properties with applicable ones, despite the identity of the content of one and the other, since they exhibit their properties in different ways, nevertheless determined by the common content. So, unlike natural singular properties, applicable singulars are part of the function definition, they can be added (to be applied) or removed from the function definition, changing the latter in this way. Adding or removing an applicable singular changes the first or second order anti-derivatives and derivatives according to the type of applied singular.

It is proposed to graphically display the applied singulars on the diagrams as part of the function definition as follows: the **x** coordinate is the position of the singular; the **y** coordinate is the numerical value of the singular. Singular sign in the diagram: singular addends - "**+**", hyperbolic singular multipliers - "⨯”. The crosshairs of the signs must accurately indicate the position of the singular on the diagram (**x**-coordinate and numerical value in **y**).

For a hyperbolic singular multiplier the definition of a hyperbolizer is indicated on the right and above its sign, but only if the hyperbolizer is not unit. If the hyperbolizer is a power function, then it is allowed to indicate only the exponentiation at **dx**, while omitting **dx** itself. Examples:

$$\times^{\ln \mathrm{dx}}, \qquad (25.1)$$

$$\times^{\mathrm{dx}^{-2}} \Leftrightarrow \times^{-2}, \qquad (25.2)$$

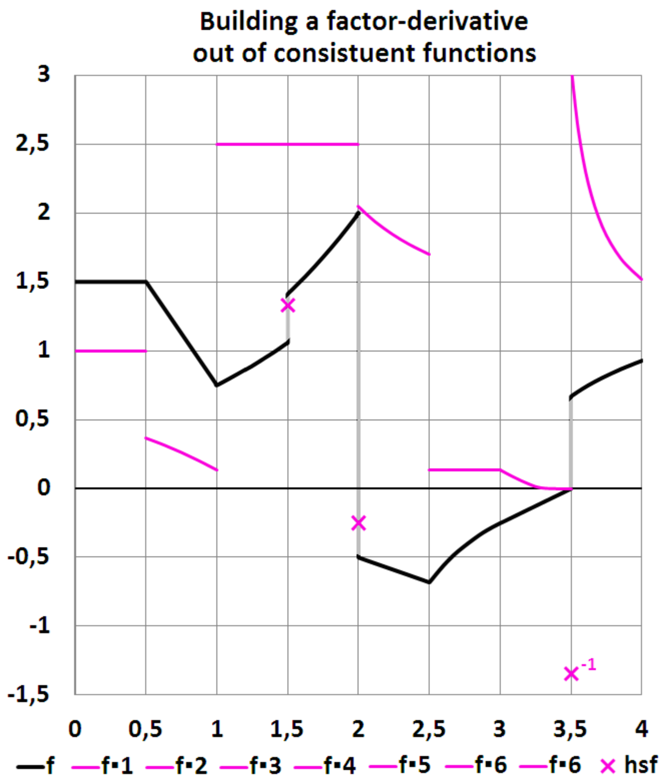
$$\times^{h}, \qquad (25.3)$$

where ***h*** is a function – hyperbolizer which definition can be given additionally, for example in cases where the latter is simply bulky.

When the continuous function definition and the applied singulars describe different physical quantities and/or have different physical dimensions, for example, the Power (Watts) and the Energy or Work (Joules), respectively, if the Time (seconds) is an argument, or there is a different numerical order of values of the continuous function definition and singular values, then the diagrams are supposed to use two mutually independent Y scales, separately for the function continuous definition and for the applied singulars.

Similar to the factorial-multiplication, when factorizing, the function domain can be divided into several constituent domains with analytically given functions and their separate factorization can be performed.

In the example, the construction of the factor-derivative (magenta) for a function (black) is divided into domains by break points and three critical points of the function. At the break points of the function, as expected, its factor-derivative is interrupted. Also, as expected, the graphs of the factor-derivatives of the domains are located above the x-axis, which allows



reverse factorial-multiplication of the factor-derivative to restore the function. In this regard, the function interruption at **x=1.5** and **x=2** and function critical point at **x=3.5** where the function crosses the x-axis, both cause interrupting the factorization process with the loss of information about the change in the function at the those critical points.

At the points **x=1.5** and **x=2** , the function has singular factorials as ratio of the finite non-zero value of the function to the right of the respective point to the finite non-zero value of the function to the left of the respective point:

$$\mathbf{sf}f(1.5) = \frac{f([1.5])}{f((1.5))} = \frac{1.41421}{1.06066} = 1⅓\,, \quad (42.1)$$

$$\mathbf{sf}f(2) = \frac{f([2])}{f((2))} = \frac{-1}{4} = -¼\,. \quad (42.2)$$

Specially the point **x=1.5** represents a good example of how the presence of a singular multiplier at function (the factor-derivative in our case) point causes an interruption of its multiplical (the analyzed function in our case) at the same point even though the function itself has no interruption in this point.

At the point **x=3.5**, the function has a hyperbolic singular factorial that is determined by finite non-zero values of the function derivatives, including the function itself, at the point **x=3.5** on both sides of it:

$$\mathbf{hsf}f(3.5) = \left\{\operatorname{sign}(-1\%2) \cdot \frac{\mathrm{f}([3.5]) \cdot !\,1}{\mathrm{f}'((3.5)) \cdot !\,0}; \mathrm{dx}^{0-1}\right\}$$

$$= \left\{-1 \cdot \frac{0.66943 \cdot 1}{0.5 \cdot 1}; \mathrm{dx}^{-1}\right\} = \left\{-1.33887; \mathrm{dx}^{-1}\right\}. \quad (42.3)$$

Factorizing the function domain at critical points, taking into account the singular properties of the functions, applies corresponding hyperbolic singular multipliers to the factor-derivative. In the process of the reverse factorial-multiplication of the factor-derivative at infinitely close approximation to the point **x=2**, the finite non-zero intermediate result of the factorial-multiplication is multiplied by the finite value of the singular at this point, and at infinitely close approximation to the point **x=3.5**, the infinitely small intermediate result will be multiplied by product of the singular finite value and the hyperbolizer return value, which is, in this case, equals to the size of extremely small interval to the interruption point (hyperbolizer arm), raised to the power of –1, that is, the intermediate result is multiplied by an infinitely large number, and as a result become non-zero and finite. Also, the proportion of the relative distance to the x-axis for the locus of points of the indefinite multiplical of factor-derivative and the same of the initial function is preserved, which makes the indefinite multiplical of the factor-derivative be identical to the original function when choosing and coordinating arbitrary factors **B**.

Working with applicable singulars using helper functions

In addition to the automatic generation and application of applicable singulars to functions in the process of factorization and differentiation, these singulars can also be applied to functions in an analytical way, using the universal “apply” method, which accepts a singular directly or generates it according to data submitted to the method: singular type, its numerical value and hyperbolizer (the hyperbolizer is only for the hyperbolic singular multiplier, which is assumed to be y=1 by the default, hereinafter information about the hyperbolizer, which is given in these brackets is not mentioned, but is meant). The application of singulars is possible both ways by one singular per call of the “**apply**” method, and by an array, where the coordinates **x** of the points of application of singulars are the array keys:

apply(f, x, s), (26.1)

apply(f, x, V, T, [H]), (26.2)

apply(f, array), (26.3)

where ***f*** is the identifier of the function as the target of the object's application; **x** is the value of the function argument - point as the target of the object’s application; **V** is a real number as the numerical value of the singular to be applied; **s** is the applicable object; **T** is the type of the applicable object with the following possible constants: "**D**" - discrete value for the function point; "**SA**" - singular addend, "**HSM**" - hyperbolic singular multiplier; **H** - hyperbolizer; **array** - an array of objects to be applied at the given coordinates **x** of their application in the function.

The “**apply**” method works on the principle of replacing an existing object with a newly applied object in cases where the latter has the same setting criteria as those already existing in the function definition: the **x** coordinate (the new one is placed at the existing one location point) and the object type (the new object matches the existing one type). Two or more objects or singulars of the same type cannot exist at the same point of the function.

Separately, it is worth clarifying that placing a discrete value in the definition of a function at a point will override the continuous definition of the function for this point, if it existed there, but will not lead to an effective interruption of the function when doing differentiation or factorization and integrating or factorial-multiplicating, as it is shown by example of factorial-multiplication of $\mathbf{y = |bn| \cdot |x|^n}$ at the zero point, but will cause the function to return this discrete value when it is called, submitting the coordinate of the requested point without explicitly specifying the inclusive or exclusive state.

The given singular can be removed from the function definition using the universal function "**remove**" according to the submitted singular setting criteria to the method:

$$s = remove(f,[x_0,[x_1]],[T]), \quad (27)$$

where $\mathbf{x_0}$ is the coordinate of the object for a single removal, or the first of the coordinates of the removal interval; $\mathbf{x_1}$ is the second coordinate of the removal interval; **T** - type of function definition - the object of removal: "**V**" - continuous function definition, coupled with discrete function definition; "**C**" - continuous function definition; "**D**" - discrete function definition; "**SA**" - singular addend, "**HSM**" - hyperbolic singular multiplier; **s** is entity to be removed, as object or as function.

If one does not submit the type of the object to the "**remove**" function, then the entire existing definition is being removed from the function. If an interval is passed to the "**remove**" function, then the specified function definition object type is being removed within the specified interval, if neither the object coordinate nor the interval is passed, then the specified function definition object type is being removed in the entire interval from $-\infty$ to $+\infty$. If one coordinate is passed to the "**remove**" function, then an object of the specified type will be removed at the specified coordinate: singular or discrete function definition. The "**remove**" function returns the removed object, or null, if the latter is not found in the function definition according to the passed setting criteria. In case of multiple removal of objects of the specified function definition object type by the specified interval, or the same, but without specifying the interval, then the "remove" function returns a function created with all objects of the specified function definition object type that are removed: continuous definition and/or, discrete values in points, all singulars of the specified type in relation to their original coordinates.

The "**create_singular**" function transforms a passed real number or a discrete values of a function definition, which are passed as an argument, respectively, into the value of the applicable singular created and returned or into the values of one or more applicable singulars in the function created and returned in relation to original discrete values coordinates, and all having passed singular type and hyperbolizer:

$$a = create\_singular(V, T, [H]), \quad (28.1)$$

$$f_1 = create\_singular(f_0, T, [H]), \quad (28.2)$$

where **V** is a real number as the numerical value of the returned singular; $\mathbf{f_0}$ is a discretely defined function; $\mathbf{f_1}$ is a function containing singulars.

The "**merge**" function merges several functions into one and returns the result of the merge made:

$$f = merge(f_1, [f_2, [f_3, .. [f_n]]]), \quad (29)$$

where **f** – result function; $\mathbf{f_1}$, $\mathbf{f_2}$, $\mathbf{f_3}$ .. $\mathbf{f_n}$ – functions to merge.

The "**merge**" function works on the principle of replacing the function definition to the right with the function definition to the left in the list of passed arguments, acting in the direction from left to right along the list. Discrete values, singulars with matching setting criteria, and overlapping domains of continuous function definition are subject of substitution.

The "**extract**" function extracts a part of the definition of a function of the specified function definition object type at the specified point or specified interval, and returns the result of the extraction as a function:

$$s = \text{extract}(f,[x_0,[x_1]],[T]). \quad (30)$$

The returned object, the set of arguments and the logic of their processing of the "**extract**" function completely coincides with that of the "**remove**" function. The only difference between the "**extract**" function and the "**remove**" function is that "**extract**" does not modify the function passed as an argument.

In order to find the first singular and its coordinate in the function definition, the following functions are called:

$$s = \text{find\_one}\ (f,\ [x_0,[x_1],T), \quad (31.1)$$

$$x = \text{find\_one\_coordinate}\ (f,\ [x_0,[x_1],T), \quad (31.2)$$

where **s** is an object (singular); **x** - coordinate of the object (singular).

The functions "**find_one**" and "**find_one_coordinate**" search in the direction from $\mathbf{x_0}$ to $\mathbf{x_1}$ until an object of the specified type is encountered. The set of arguments and the logic of their processing of functions completely coincides with that of the "**extract**" function. The only difference between these two functions and "extract" function is that they do not return a function under any circumstances.

All of the listed helper functions that accept intervals support the submitting of interval boundaries with the specified absolute state inclusive or exclusive and with the absolute state inclusive by the default.

The hyperbolic singular multiplier and singular addend applied to the functions do not exhibit themselves in any way during an ordinary function call with the specified argument value corresponding to the position of these features, they are in a hidden state. In order to explicitly get their numerical value, type and hyperbolizer, one needs to call the following functions, respectively, passing them a singular as an argument:

$$V = \text{singular\_value}(s), \quad (32.1)$$

$$t = \text{singular\_type}(s), \quad (32.2)$$

$$H = \text{singular\_hyperbolizer}(s), \quad (32.3)$$

where **s** is an object (singular); **T** - type of object (singular); **V** is the numerical value of the object (singular), **H** is the singular hyperbolizer.

If undefined is passed to the "**singular_type**", "**singular_value**", and "**singular_hyperbolizer**" functions as an argument, they also return undefined. The "**singular_hyperbolizer**" function returns undefined in case if a singular addend is passed as a singular. Also one can get the value of a singular or its hyperbolizer by directly referring to its property: **V = s.V**; **H = s.H**.

Algorithm for calculating defined multiplical and defined integral taking into account applied singulars

Operation constants:

Common for factorial-multiplication and integration:

d_0x – positive infinitely small or small enough when applying numerical method;

Only for factorial-multiplication: **T = "HSM", N = 1**.

Only for integration: **T = "SA", N = 0**.

Input variables:

Common for factorial-multiplication and integration: ***f_1, f_2, X_0, X_1***.

Setting the initial state of the variables: $x_0 := X_0$, $g := X_1$ ▪< X_0? –1:+1, R := N.

Executing following actions of the present paragraph in the given order in a loop till x0 <▪> X1 and R is finite and not equal to ln(N):

Common for factorial-multiplication and integration:

$$x_1' := x_0 = X_1 \ ?\ \text{set\_state}(X_1, g) : \text{set\_state}(g \cdot x_0 + d_0x > g \cdot X_1 ?\ X_1 : x_0 + g \cdot d_0x, -g) \quad (33.1),$$

$$x_c := x_0 = x_1' \ ?\ \text{undefined} :$$

$$\left(\text{find\_one\_coordinate}(f_1, x_0, x_1', T) \ || \ \text{find\_one\_interruption\_coordinate}(f_2, x_0, x_1')\right), \quad (33.2)$$

Only for factorial-multiplication:

$$i := |x_1 - x_0| \ / \ 2 \geq |x_c - x_0|, \quad (33.3.1)$$

$$x_1 := x_c \ ?\ \Big(x_c = x_0 \ ?\ \text{set\_state}(2 \cdot x_0 - x_1', -g) :$$

$$\Big(x_c = x_1' ?\ \text{set\_state}(x_1', g) : \text{set\_state}\big(2 \cdot x_c - (i \ ?\ x_0 : x_1), -g\big)\Big)\Big) : x_1', \quad (33.4.1)$$

Only for integration:

$$i := x_c = x_0, \quad (33.3.2)$$

$$x_1 := x_c \ ?\ \text{set\_state}\big(x_c, i \ ?\ g : -g\big) : x_1', \quad (33.4.2)$$

Common for factorial-multiplication and integration:

$$dx := x_1 - x_0, \quad (33.5)$$

$$s := i \ ?\ \text{find\_one}(f_1, x_0, x_1, t) : \text{undefined}, \quad (33.6)$$

$$V := \text{singular\_value}(s), \quad (33.7)$$

$$x := i \ ?\ x_c : (x_1 + x_0) / 2, \quad (33.8)$$

Only for factorial-multiplication:

$$H := \text{singular\_hyperbolizer}(s), \quad (33.9)$$

$$d_h x := \mathrm{sign}\left(\ln\left(\mathrm{H}(d_0 x)\right)\right) \cdot g \ ?\ x_1 - x_c : x_c - x_0, \quad (33.10)$$

$$\mathrm{tsf} := \left(\mathrm{V} \cdot \mathrm{H}(d_h x)\right)^{g}, \quad (33.11.1)$$

$$\mathrm{tf} := f_2(x)\ !\ ||\ f_2(\mathrm{x})^{\mathrm{dx}/2} \cdot f_2(\mathrm{x}_1)^{\mathrm{dx}/2}, \quad (33.12.1)$$

$$\mathrm{R} := \mathrm{R} \cdot (\mathrm{tsf}\ ||\ \mathrm{tf}), \quad (33.13.1)$$

Only for integration:

$$\mathrm{tsd(x)} := \mathrm{V} \cdot \mathrm{g}, \quad (33.11.2)$$

$$\mathrm{td} := f_2(x)\ !\ ||\ f(\mathrm{x}_0)\,\mathrm{dx}/2 + f(\mathrm{x}_1)\,\mathrm{dx}/2, \quad (33.12.2)$$

$$\mathrm{R} := \mathrm{R} + (\mathrm{tsd}\ ||\ \mathrm{td}), \quad (33.13.2)$$

Common for factorial-multiplication and integration: $x_0 := x_1$. (33.14)

Returning the result: R.

where **d_0x** is the nominal element of factorial-multiplication or integration, sets the length of the corresponding value; **T** is the type of the applicable singular; it depends on the operation being carried out: factorial-multiplication or integration; **N** is a neutral element which depends on the operation being carried out; ***f_1*** is a multiplicand or integrand function containing hyperbolic singular multipliers or singular addends, respectively; ***f_2*** is a multiplicand or integrand function containing a continuous function definition; **X_0** is the absolute beginning boundary of the factorial-multiplication or integration segment; **X_1** is the absolute ending boundary of the factorial-multiplication or integration segment; **:=** - assignment sign, value to the right of itself is set to variable to the left of itself; **x_0** is the absolute beginning boundary of the factorial-multiplication or integration element; **{..} ? {..} : {..}** is a conditional operator that works according to the scheme: {questioned statement} ? {expression in case of truth} : {expression in case if false}; **g** - sign of the element of factorial-multiplication or integration, at the same time it is an indicator of the direction of factorial-multiplication or integration, respectively: +1 in the direction of the growth of the argument, –1 in the opposite direction of the direction of the growth of the argument; **R** is the result of factorial-multiplication or integration, and also the intermediate result of factorial-multiplication or integration, respectively; **x_1'** is the absolute preliminary ending boundary of the factorial-multiplication or integration element; **||** - sign of the enumeration operator, returns the value to the right of itself, if the value to the left of it is not defined, otherwise it returns the value to the left of it; **find_one_interruption_coordinate(*f*, x_0, x_1)** - a function that returns the absolute coordinate of the first interruption of f_2 within the boundaries of the specified interval from x_0 to x_1, searching in the direction from x_0 to x_1; x_c is the absolute coordinate of the applied singular in f_1 or the f_2 interruption; **i** – indicator of singular inclusion or function interruption in the operation element; x_1 is the absolute ending boundary of the factorial-multiplication or integration element; **s** is the applied singular; **V** is the singular numerical value; **x** is the absolute coordinate of the conditional middle of the factorial-multiplication or integration element; **dx** is the effective element of factorial-multiplication or integration; $d_h x$ - part of the factorial-multiplication element - the hyperbolization arm; **H** is the hyperbolizer; **tsf** – technical singular factorial of the factor-anti-derivative; **!||** - sign of the negative enumeration operator, returns the value to the left of itself if it is not defined, otherwise it returns value to the right of itself; **tf** – technical continuous factorial of the factor-anti-derivative; **tsd** – technical singular differential of the anti-derivative; **td** is the technical continuous differential of the anti-derivative.

Note: functions f_1, f_2 can be represented by one function. In present case, the explicit separation to two functions demonstrates the fundamental possibility of calculating over two separate functions.

Explanation of the algorithm for calculating a definite multiplical and integral taking into account applied singulars

Factorial-multiplication or integration (hereinafter referred to as the operation) begins with setting the initial state: the beginning boundary of the first element of the operation x_0 is set to the beginning of the segment of the operation X_0. The intermediate result **R** is set to a neutral number **N**. The direction of operation relative to the direction of the x-axis is determined by means of determining the indicator of the direction of operation **g**.

The iterations of the loop are repeated till the intermediate result of the operation does not go beyond the allowable values and till the initial boundary of the operation element is less than the final boundary of the operation segment, taking into account the absolute state of the boundaries (inclusive or exclusive). Further explanation is given on the example of the forward direction of the operation, which is in the direction of growth of the argument. And here it is necessary to clarify that if considering the operation in the opposite direction then the described absolute states of the boundaries change to the opposite: inclusive to exclusive, and exclusive to inclusive, which **g** is responsible for.

At the beginning of each cycle, based on the nominal size of the operation element, the preliminary ending boundary of the operation element x_1' is determined. The ending boundary of the segment of the operation is taken into account here. By default, the ending boundary of the operation element is defined as exclusive, except for the case when the beginning boundary of the operation element with the state exclusive coincides in coordinate value with the ending boundary of the segment of the operation with the state inclusive, by other words, when the last element of the operation contains only a transition through point of the ending boundary of the segment of the operation.

After determining the preliminary ending boundary of the operation element, a check is made for the presence of applied singulars of $\boldsymbol{f_1}$ inside this element, and if they are not found, then a check for the presence of $\boldsymbol{f_2}$ interruptions inside this element, and in that order, and the coordinate of such a point $\mathbf{x_c}$ is extracted, if it is found. If the values of the coordinates of the beginning and preliminary ending of the operation element coincide (the operation element is only a transition through a point), then it makes no sense to look for the coordinate of the applied singular or interruption, if one of them exists, then it is at the point.

Once $\mathbf{x_0}$, $\mathbf{x_1'}$ and $\mathbf{x_c}$ are known the final coordinate and state of the ending boundary of the operation element $\mathbf{x_1}$ can be determined, which will finally determine the position and size of the operation element. The method of obtaining this value is different for factorial-multiplication and integration.

The peculiarity of determining the operation element for factorial-multiplication is that, the position of the applied singular found, if possible, should fall in the middle of the operation element with the equality of the lengths of the resulting parts of the operation element on both sides of the applied singular in order to comply with the above described symmetry. The exceptions are cases where symmetry cannot be achieved in principle potentially for the first and last element of the operation, if the corresponding boundaries of the operation segment are specified with the state exclusive for the beginning or inclusive for the ending boundary, and same time singulars are applied at these boundary points.

The ending boundary of the operation segment approaches the beginning one in order to comply with the symmetry condition indicated above, or in order not to include the found singular at all to the current iteration, but to include it in the next one, because otherwise the length of the operation segment potentially will exceed the nominal length, which may be critically important when applying the numerical method for calculating the multiplical or integral.

The specific feature of defining an operation element in integration is the separation of singulars or interruptions of a function from domains of continuous function definition and their allocation among different operation elements. Thus, if the applied singular or interruption is found at the beginning of the operation element, then the operation element includes only the transition through this point, otherwise the operation element ends before the point, without passing through it and excluding the point in itself.

The indicator **i** indicates the inclusion of the found singular or interruption in the current iteration.

After determining the ending boundary of the operation element, it is possible to determine the length of the operation element **dx**, which can be zero if the operation element is only a transition through a point, and also it is possible to find the applied singular **s** in the operation element if it is certain that it is included in the current element of the operation, for which **i** is responsible. If the singular is found then it is possible to read its numerical value **V**, and for the hyperbolic singular multiplier, also its hyperbolizer **H**. If the inclusion of the applicable singular or function interruption in the current element of the operation is confirmed, then the coordinate of the detected one of them is taken as the midpoint **x** of the operation element, thereby recreating the conditions of the singular creation from a symmetric factorial of the multiplicand function anti-derivative (if it is created during factorization). If the inclusion of the applicable singular or function interruption in the current operation element is not confirmed, then the midpoint of the operation element is taken as the default midpoint. One of the two halves of the factorial-multiplication element. formed by its division by the midpoint, is designated as the hyperbolization arm $\mathbf{d_h x}$.

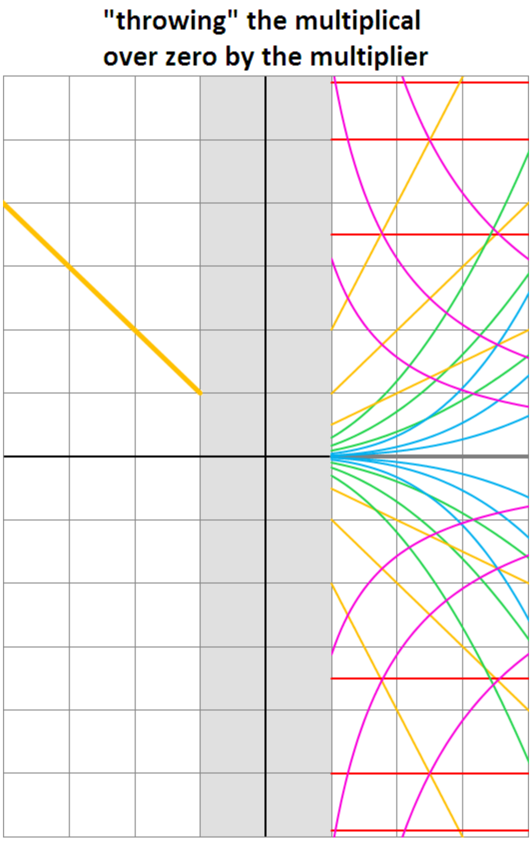
After the preparatory steps are completed, it is possible to begin to calculate the elementary increment of the operation result, for factorial-multiplication it is an elementary multiplier, for integration it is an elementary addend. Each of them is determined by the values of its two parts: the first and prior, responsible for the increment associated with the presence of the applicable singular, and the second and secondary, responsible for the increment associated with the continuous definition of the function. So, if the first part has certainty (the applicable singular is present in the operation element), then the second part is not taken into account. As a rule, the interruption of the function is covered by the presence of the applicable singular, which, as it were, puts a patch on this section of the function, which, in particular, justifies the priority of the first part over the second. Also, the reason for the priority of the first part is the immeasurability of the parts as multipliers or addends. The second part influence on the result of the operation шы tending to nothing, compared to the influence of the first part, when the size of the operation element approaches zero.

The calculation of the first part of **tsd** for integration is quite trivial and is the result of multiplying the value of the singular addend by the integration direction indicator **g**, depending on the integration direction, either leaving the singular addend as an addend (for direct) or effectively turning it into a subtrahend (when reversed). The calculation of the first part **tsf** for factorial-multiplication is somewhat similar, but just as complicated. Instead of multiplying the hyperbolic singular multiplier is raised to power of the direction indicator **g**, depending on the direction of factorial-multiplication, either leaving the hyperbolic singular multiplier as a multiplier (when forward) or effectively turning it into divider (when reverse). Also, the hyperbolic singular multiplier itself has a positive hyperbolizer return value **H** as its own multiplier, which for obvious reasons is not present with the singular addend in any form.

As it is already clear, the value of the hyperbolic singular multiplier can be negative. The negative arithmetic sign allows, during the factorial-multiplication, to translate the intermediate result of the factorial-multiplication, and hence the multiplical function from the positive zone to the negative one, and vice versa, an unlimited number of times, which is one of the guarantees of restoring the function from its derivative filled with hyperbolic singular multipliers. If we are talking about an indefinite multiplical, then the freedom to choose an arbitrary multiplier **B** saves us from the need to choose the arithmetic sign of multiplical domains, which are separated by the transition of the multiplical through zero, directly in the process of factorial-multiplication. In this case, the determining factor is the position of the zero crossing points and their number within the factorial-multiplication segment or within the entire domain of the indefinite multiplical definition.

Speaking about the number of transitions of the multiplical from the positive to the negative zone and vice versa, and therefore about the number of negative multipliers encountered in the process of factorial-multiplication, we can refer to the fact that hyperbolic singular multipliers in general, being discrete quantities, assume the finiteness of their number relative to the finite interval of the multiplicand function, which obviously casts to the finiteness of the number of negative multipliers of the intermediate factorial-multiplication result in the finite factorial-multiplication segment, and thus fundamentally removes the prohibition on the negative value of the hyperbolic singular multiplier, which is fundamentally imposed on the negativity of the multiplicand function continuous definition the precisely because of the appearance in this case of a sequential series of negative multipliers of infinite quantity.

tsf or **tsd**, having priority over **tf** or **td**, respectively, effectively “cuts out” this function continuous definition in the interval of the operation element and replaces it with itself. From a technical point of view, the intermediate result corresponding to the beginning point of the operation element is multiplied by **tsf** or added by **td**, respectively, at the entrance of the iteration and passed to the iteration exit, to the point corresponding to the ending of the operation element.



Thus, the intermediate result of the factorial-multiplication does not reach extreme values: uncertainty or zero. Suppose it approaches zero at an infinitely small distance from it and then either “throws” over the x-axis by the negative value of the hyperbolic singular multiplier, or “reflects” from it if the latter is positive. With or without “refraction”, with the equal angle of reflection to the angle of incidence or not, with “jump” in the value of the function or its derivatives or not, all this depends on the value of the modulus of the value of the singular hyperbolic multiplier, whether it differs from 1, and for how much.

The diagram shows an illustration of the "throwing" and "reflection" varieties in the vicinity of the critical point (gray zone) depending on the value of the hyperbolic singular multiplier and the definition of its hyperbolizer.

Strictly speaking, the multiplical obtained as a result of the reverse factorial-multiplication of the factor-derivative with the applied singulars is not an exact copy of the original function due to the presence of “throws” through zero or “reflections”, which we know about, but nevertheless is effectively identical to it which, in particular, is necessary for the practical use. Despite the presence of such an infinitely small "patches", the new function has hyperbolic singular factorials indistinguishable from those of the original function, which makes it possible to repeat the factorization with it again and obtain an identical factor-derivative.

In the vicinity of the interruption point of the integrand or multiplicand function (hereinafter the sub-operational function) we can calculate respectively arithmetical mean or geometrical mean out of two values of the sub-operational function taken on both sides of the interruption point and at the same infinitesimal distance from it. We call this the value as the conditional average at the interruption point.

So the finite value of this quantity indicates that the influence of the sub-operational function values before the interruption point on the intermediate result of the operation (on the integral or multiplical, respectively) is compensated by its values after the interruption point. We can say that the values of the function in the vicinity of the interruption point on both sides of it are mutually compensating. In this case a singular addend or a hyperbolic singular multiplier with unit hyperbolizer can be placed at this point as a "patch", which generates the finite size values of **tsd** and **tsf** respectively, establishing the absolute difference or the relative (proportional) difference between the two domains of the integral or multiplical respectively.

During the factorial-multiplication, if the conditional average value at the critical point is infinitely small or infinitely large in modulus, then compensation is performed by the return value of the hyperbolizer, by the hyperbolizer definition which is corresponding to the case, and which is different from unity. The hyperbolizer is something what brings the value of the hyperbolic singular multiplier into line with the continuous definition of the multiplicand function on either side of the interruption point. An infinitely large conditional average is compensated by an infinitely small hyperbolizer return value, an infinitely small conditional average is compensated by an infinitely large hyperbolizer return value.

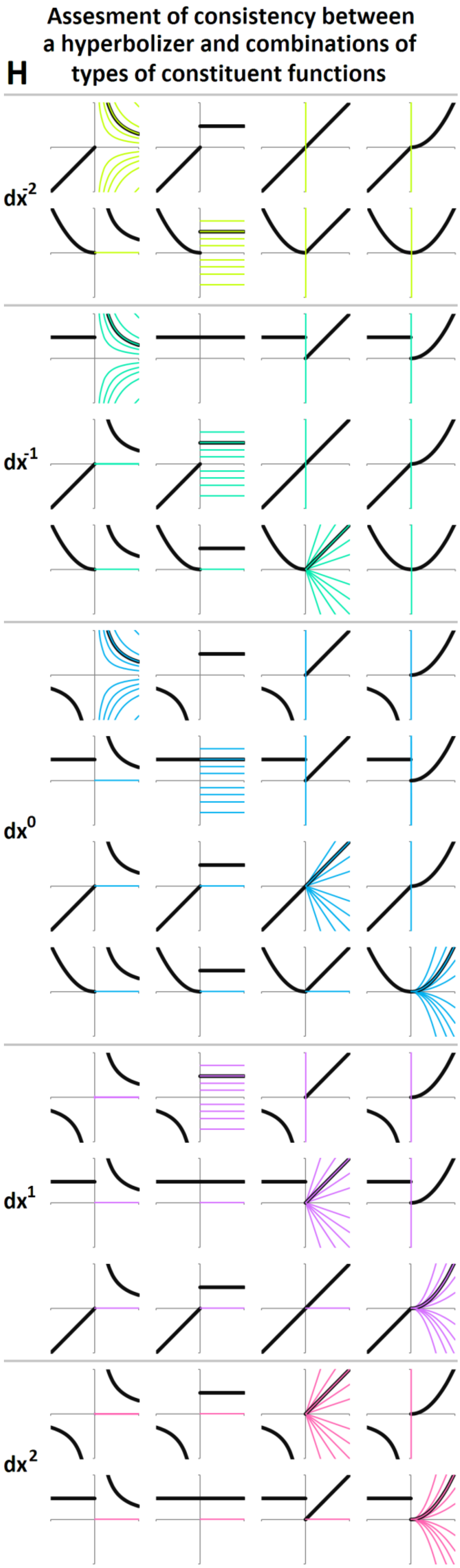
For this reason, a definite multiplical for a segment consisting of only one point with a singular with a non-unit hyperbolizer will not give a finite result. Also, the result of factorial-multiplicating the segment, the boundaries of which include points with hyperbolic singular multipliers with a non-unit hyperbolizer, may not give the finite result. For the integration operation, the conditional average value, which is infinitely large in modulus, at the critical point, unfortunately, cannot be compensated.

If the hyperbolizer definition at a point does not correspond to the conditional average value at a point, and not necessarily at the interruption point of the multiplicand function, then the factorial-multiplication can be interrupted by turning the multiplical to zero or to infinity modulo. In such cases, the hyperbolizer is insufficient or excessive. The automatic generation of the applicable singulars during the factorization of functions by definition cannot lead to a mismatch of the hyperbolizer definition. Here deliberately applied singulars, function modification are meant.

The diagram shows combinations of the continuous definition of the multiplical on both sides of the critical point (black graphs) given by the following power equations: **$y=C/x$; $y=C$; $y=C\cdot x$; $y=C\cdot x^2$**. The colored graphs show how the multiplical possibly could develop to the right of the critical point when a hyperbolic singular multiplier is placed at the critical point, depending on the definition of the hyperbolizer, namely on the value of exponentiation, as follows (from right to left): light green: –2, aquamarine: –1, blue: 0, lilac: 1, magenta: 2. A fan of color plots of non-zero finite value indicates a matched hyperbolizer definition of a combination of indefinite multiplicals on either side of a critical point. Zero or undefined values of these graphs indicate inconsistency: insufficiency or excessivity of the hyperbolizer, respectively.

A hyperbolizer return value depends on the factorial-multiplication element passed as the hyperbolization arm which is infinitesimal or small enough respectively for the analytical or numerical solution of a definite multiplical. Metaphorically speaking, the hyperbolic singular multiplier is a “seed” containing quantitative (finite value of the singular) and qualitative (hyperbolizer definition) information about its potential growth, the factorial-multiplication element is a growth resource, “fertile soil”, and the result of “powering” the hyperbolic singular multiplier by the factorial-multiplication element generates a "fruit" ready for use as a multiplier or divisor of the intermediate result of the factorial-multiplication.

The question of why the singular contains qualitative information, why the nature of hyperbolization is not determined "automatically", "in place" in the process of factorial-multiplication, disappears for two reasons. Firstly, an applicable singular is a property of a function at a point, which in



principle cannot depend on the analysis being carried out, on its logic and surrounding circumstances, in other words, the analyst must have the right and the technical ability to purposely apply a singular at a point with a hyperbolizer definition that does not correspond to this point. Secondly, the factorial-multiplication process is sequential, it does not "run ahead" in order to possibly determine the nature of the development of the factorial-multiplication after the interruption point, and then choose the appropriate definition of the hyperbolizer for the interruption point. The operation logic of factorial-multiplication or integration is built in such a way that it “works blindly”, as it should be in principle, it should not care about the input value and state of the intermediate result at the entrance to the iteration, whether it is finite, infinitesimal or infinitely large (the arbitrary multiplier B is purely responsible for the input value), so it does not care about the output value, whether it is finite, infinitesimal, or infinitely large.

Analysis of the continuous definition of a sub-operational function solely (function derivative or function factor-derivative, respectively) without taking into account the applied singulars does not allow us to guarantee the restoration of the function itself, if we admit the possibility of interruptions of the function or interruption of its derivatives at critical points (for a multiplical at zero points), that is, due to for the presence of the problem of coordinating arbitrary addends or arbitrary multipliers. This circumstance once again indicates the need to introduce the class of applicable singulars and their application to sub-operational functions in the circumstances of the analysis, including the critical points themselves and the properties of the functions at these points.

If a singular is not found in the operation element, then the process is effectively executed by the continuous definition of the sub-operational function. A preliminary check is made of the definiteness of the sub-operational function at the midpoint. For example, the zero point passes such a check in the absence of an applicable singular when factorial-multiplicating the modulus of the function that intersects the x-axis. To ensure significantly greater practical accuracy in the numerical solution of a definite multiplical or a definite integral, the multiplier or addend of the intermediate result, respectively, of factorial-multiplication or integration is calculated as the geometric mean and arithmetic mean, respectively, out of the values of the sub-operational function at the beginning boundary and at the ending boundary of the operation element.

Recording the multiplical and integral, taking into account applied singulars

Since the singulars applied to functions do not manifest themselves in any way when the latter is usually called, the formal approach requires modifying the notation of the multiplical and integral, which implies the operation that takes into an account applicable singulars in the function definition:

For multiplical:

$$\bullet \int \overset{|}{f_1}(\mathrm{x}, \mathrm{dx}) \;||\; f_2(\mathrm{x})^{\mathrm{dx}} \;||\; 1, \quad (34.1.1)$$

$$\bullet \int \overset{|}{f}(\mathrm{x}, \mathrm{dx}) \;||\; 1, \quad (34.2.1)$$

$$\bullet \int \overset{+}{f}(\mathrm{x}, \mathrm{dx})^{\mathrm{dx}} \;||\; 1, \quad (34.3.1)$$

$$\bullet \int f(\mathrm{x})^{\mathrm{dx}} \;||\; 1, \quad (34.4.1)$$

For integral:

$$\int \overset{|}{f_1}(\mathrm{x}, \mathrm{dx}) \;||\; f_2(\mathrm{x})\mathrm{dx} \;||\; 0, \quad (34.1.2)$$

$$\int \overset{|}{f}(\mathrm{x}, \mathrm{dx}) \;||\; 0, \quad (34.2.2)$$

$$\int \overset{+}{f}(\mathrm{x}, \mathrm{dx})\mathrm{dx} \;||\; 0, \quad (34.3.2)$$

$$\int f(\mathrm{x})\mathrm{dx} \;||\; 0, \quad (34.4.2)$$

where "**|**" - the function accent, forcing the function to return only multipliers or only addends for the intermediate result of factorial-multiplication or integration, respectively, which are due to the applied singulars that fall into the operation elements, otherwise it returns uncertainty; "**+**" the function accent, forcing the function to return multipliers or addends for the intermediate result of factorial-multiplication or integration, respectively, which is due both to the singulars that fall into the operation elements and to the continuous definition of the function.

Accenting gives the function the logic described above, essentially defining a new function, which is based on the function under the accent. **dx** is passed to this new function as an additional argument. This is necessary so that, acting in the new logic, the function could search for a singular inside the operation element and generate the hyperbolizer return value. At the same time, the reduced notation, the one with an accent "**+**", formally implies the return of infinite values by the function under the accent: **tsd / dx** when integrating, $\sqrt[\mathbf{dx}]{\mathbf{tsf}}$ and when factorial-multiplicating. It is assumed that before extracting the root from **tsf**, the latter decomposes into two multipliers: its module **|tsf|** and its arithmetic sign **sign(tsf)** as follows: $\sqrt[\mathbf{dx}]{\mathbf{sign(tsf)}} \cdot \sqrt[\mathbf{dx}]{|\mathbf{tsf}|}$, which is necessary for the possibility of transferring the sign of the multiplier through the operation of raising to an infinitesimal power **dx**. In this case, whatever the expression $\sqrt[dx]{-1}$ is, but subsequently raised to the power of **dx**, must result **—1** .

The expanded notation, the one with the accent "**|**", allows us to perform operations using two separate functions: one is for function definition with singulars and another for continuous function definition.

Equations **34.4.1** and **34.4.2** reflect the classical conduct of operations without taking into account the applicable singulars, even though they are present in the definition of the sub-operational function.

The optional additions **||1** and **||0** can be used to prevent the operation from being aborted if there is no continuous function definition.

Solely for the purpose of demonstrating the possibilities of factorial-multiplication taking into account the singular properties of functions, it is possible to define **x!** using a definite multiplical. To do this, we need to introduce the **getZ** function, which generates a discrete function from a continuous one passed as an argument by removing the definition of the function from its entire domain except for the points of the integer value of the argument:

$$\mathrm{X(x) = x}, \quad (35.4)$$

$$\mathrm{x!} = \bullet\int_0^{\mathrm{x}} \overset{|}{\text{create\_singular}}(\text{getZ}(\mathrm{X}), \text{"HSM"})(\mathrm{x}', \mathrm{dx}') \mid\mid 1. \quad (35.5)$$

The rule for matching an arbitrary multiplier "B" and an arbitrary addend "C" taking into account applied singulars

The coordinating equations for arbitrary multipliers and arbitrary addends for adjacent domains of functions in the analytical construction of indefinite multiplicals and indefinite integrals, taking into account the presence of hyperbolic singular multipliers and singular addends, take the following form:

$$F_0\big((\mathrm{x})\big) + \mathrm{C}_0 + \mathbf{sa}f(\mathrm{x}) = F_1(\mathrm{x}) + \mathrm{C}_1, \quad (36.1)$$

$$\lim_{\mathrm{dx}\to 0} F^{\bullet}{}_0(\mathrm{x}-\mathrm{dx}) \cdot \mathrm{B}_0 \cdot \mathbf{hsm}f(\mathrm{x}).\mathrm{V} \cdot \mathbf{hsm}f(\mathrm{x}).\mathrm{H(dx)} = \lim_{\mathrm{dx}\to 0} F^{\bullet}{}_1(\mathrm{x}+\mathrm{dx}) \cdot \mathrm{B}_1, \quad (36.2)$$

$$\mathrm{p_x} = \lim_{\mathrm{dx}\to 0} \log_{dx} \mathbf{hsm}f(\mathrm{x}).\mathrm{H(dx)}, \quad (36.3)$$

$$(F^{\bullet}{}_0)'^{\mathrm{j_x}}\big((\mathrm{x})\big) \cdot !\,\mathrm{k_x} \cdot \mathrm{B}_0 \cdot \mathbf{hsm}f(\mathrm{x}).\mathrm{V} \cdot 0^{(\mathrm{p_x}-\mathrm{k_x}+\mathrm{j_x})} = (F^{\bullet}{}_1)'^{\mathrm{k_x}}(\mathrm{x}) \cdot !\,\mathrm{j_x} \cdot \mathrm{B}_1, \quad (36.4)$$

where $\mathbf{F_0}$ is the anti-derivative of the function to the left of the point **x** with a singular addend; $\mathbf{F_1}$ - anti-derivative of the function to the right of the point with the singular addend; $\mathbf{C_0}$ is an arbitrary addend of the indefinite integral of the function to the left of the point **x**; $\mathbf{C_1}$ is an arbitrary addend of the indefinite integral of the function to the right of the point **x**; **sa*f*(x)** is the value of the singular addend at the point **x**; **p** – exponentiation at **dx** in the hyperbolizer definition; $\mathbf{j_x}$ is the ordinal number of the first found derivative of the anti-derivative with a non-zero finite value at the point **x** exclusive, starting from the 0^{th} ordinal number, meaning the anti-derivative itself; $\mathbf{k_x}$ is the ordinal number of the first found derivative of the anti-derivative with a non-zero finite value at the point **x** inclusive, starting from the 0^{th} ordinal number, meaning the anti-derivative itself; $\mathbf{(F^{\bullet}_0)'^{j_x}}$ – $\mathbf{j}^{th}$ derivative of the factorial-anti-derivative function to the left of the point with a singular **x** exclusive; $\mathbf{(F^{\bullet}_1)'^{k_x}}$ – $\mathbf{k}^{th}$ derivative of the factorial-anti-derivative function to the right of the point with a singular **x** inclusive; $\mathbf{B_0}$ is an arbitrary multiplier of an indefinite function multiplical to the left of the point **x**; $\mathbf{B_1}$ is an arbitrary multiplier of the indefinite multiplical of the function to the right of the point **x**; **hsm*f*(x).V** is the value of the hyperbolic singular multiplier at point **x**; **hsm*f*(x).H** is the hyperbolizer of the hyperbolic singular multiplier at point **x**.

The coordinating equation for arbitrary addends of indefinite integral is a coordinating equation for the singular differential of the anti-derivative with the singular addend of the integrand function (**36.1**). For an indefinite multiplical, the analogy is expressed in coordinating the symmetric factorial of the anti-derivative with the hyperbolic singular multiplier of the multiplicand function (**36.2**), and for this reason the equation is solved through the limit as **dx** tends to zero.

If it is possible to find a solution for a hyperbolic singular factorial of indefinite multiplical at the point **x** in terms of non-zero finite value derivatives of its adjacent constituent functions and if the hyperbolizer of the hyperbolic singular multiplier at the point **x** is represented by a power function with natural, including zero, exponentiation at **dx**, the coordinating equation for arbitrary multipliers for these constituent functions can be expressed in terms of their derivatives (**36.4**). The size of **dx** in this case is not critical, it is necessary only to extract the integer exponentiation at **dx** from the hyperbolizer definition (**36.3**). As it is seen in **36.4** a successful arbitrary multipliers coordination is only possible in case if the hyperbolizer compensates a potential immeasurability of the indefinite multiplical constituent functions values on both sides of point **x**. The following equation has to be valid: $\mathbf{p_x - k_x + j_x = 0}$, implying $\mathbf{0^0 = 1}$, as checking for satisfaction of the compensation.

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