A PROOF OF THE KAKEYA MAXIMAL FUNCTION CONJECTURE FROM A SPECIAL CASE

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Abstract. First in this paper we will prove the Kakeya maximal function conjecture in a special case when tube intersections behave like line intersections. This paper highlights how different tube intersections can be than line intersections. However, we show that the general case can be deducted from the line like case.

Contents

1. Introduction

A line l_i is defined as

$$l_i := \{y \in \mathbf{R}^n | \exists a, x \in \mathbf{R}^n \quad \text{and} \quad t \in \mathbf{R} \quad \text{s.t} \quad y = a + xt\}$$

We define the δ-tubes as δ neighbourhoods of lines:

$$T_i^\delta := \{x \in \mathbf{R}^n | |x - y| \leq \delta, \quad y \in l_i\}.$$

The order of intersection is defined as the number of tubes intersecting in an intersection. We define $A \lesssim B$ to mean that there exists a constant C_n depending only on n such that $A \leq C_n B$. We say that tubes are δ-separated if their angles are δ-separated. Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. For each tube in $B(0,1)$ define a as it‘s center of mass. Define the Kakeya maximal function as
$f^*_\delta : S^{n-1} \to \mathbb{R}$ via

$$f^*_\delta(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{T^\delta_\omega(a) \cap B(0,1)} \int_{T^\delta_\omega(a) \cap B(0,1)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f^*_\delta||_p \leq C_\epsilon \delta^{-n/p+1-\epsilon} ||f||_p, \tag{1.1}$$

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for all $\epsilon > 0$ and some $n \leq p \leq \infty$. A very important reformulation of the problem by Tao is the following. A bound of the form (1.1) follows from a bound of the form

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon} N^{1/p'} \delta^{(n-1)/p'}, \tag{1.2}$$

for all $\epsilon > 0$, and for any set of $N \leq \delta^{1-n}$ δ-separated of δ-tubes. See for example [2] or [1]. It's enough to consider the case $p = n$ and the rest of the cases will follow via interpolation [1,2]. Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_i}(x) 1_{B(0,1)}(x) \leq 2^{k+1}\}.$$

We will prove the following theorem.

Theorem 1.1. *Let there be a $N \lesssim \delta^{1-n}$ δ-separated δ-tubes. Assume that for $k > 0$, $l \neq m$, it holds that*

$$T_l \cap T_m \cap E_{2^k} = \bigcap_{j=1}^{\sim 2^k} T_{ij}.$$

Then we have

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \leq C_n (\log(\frac{1}{\delta})^{(n-1)/n} (N\delta^{n-1})^{(n-1)/n}.$$

It is a fact that the intersection of each pair of different lines contains only one point. So this paper emphasis the difference between line and tube intersections and it can be said that we first prove the Kakeya maximal function conjecture in a line like case. However, we proof the general case also.

Corollary 1.2. Let there be a $N \lesssim \delta^{1-n}$ δ-separated δ-tubes. Then we have

$$||\sum_{\omega \in \Omega} 1_{B(0,1)} 1_{T_\omega(a_\omega)}||_{n/(n-1)} \leq C_n \log(\frac{1}{\delta})^{(n-1)/n} (N\delta^{n-1})^{(n-1)/n}.$$

One of our results is the following: a generalization of a lemma of Corbóda.

Lemma 1.3. *[A generalization of a lemma of Corbóda] For δ-separated tube intersections of order $2^k > 1$ it holds that*

$$|\bigcap_{i=1}^{2^k} T_i| \lesssim \delta^{n-1} 2^{-k/(n-1)}.$$

It's not hard to check that the above bound is essentially tight.

2. Previously known results

We will use the following bound for the pairwise intersections of δ-tubes:

Lemma 2.1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n \cap B(0,1)$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [1].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, define its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an δ-separated subset of Ω.

Lemma 2.2. *In the notation just defined*

$$N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [1].

3. A proof of the generalization of the lemma of Corbóda

Let us define

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \le \sum_{i=1}^{N} 1_{T_i}(x) 1_{B(0,1)}(x) \le 2^{k+1}\}.$$

Let us suppose that $2^k = \delta^{-\beta}, 0 < \beta \le n-1$, and let‘s suppose that tube T_j intersecting $T_i \cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim \delta^{n-1-\beta}$ on the unit sphere. Then the angle between T_j and T_i is greater than $\sim \delta^{1-\beta/(n-1)}$. Thus by lemma 1.3 the intersection

$$|\bigcap_{i=1}^{2^k} T_i| \le |T_i \cap T_j \cap E_{2^k}| \le |T_i \cap T_j| \lesssim \delta^{n-1+\beta/(n-1)} \le \delta^{n-1} 2^{-k/(n-1)}. \tag{3.1}$$

Thus, we can suppose that the directions in the intersection $E_{2^k} \cap T_i \cap T_j$ belong to a cap of size $\sim \delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2.2 we get that the cap can contain at most $\sim 2^k$ tube-directions. However, the cap contains at least 2^k tube directions. Thus, for any tube T_i in the intersection there exists a tube T_j, such that the angle between T_i and T_j is $\sim \delta^{1-\beta/(n-1)}$ and the inequality (3.1) is valid. Thus we proved the lemma 1.3.

4. The proof of the line like case

We defined

$$E_{2^k} := \{x \in \mathbf{R}^n | 2^k \le \sum_{i=1}^{N} 1_{T_i}(x) 1_{B(0,1)}(x) \le 2^{k+1}\}.$$

We have for $k > 0$ that

$$E_{2^k} = \bigcup_{i=1}^{M} \bigcap_{j=1}^{\sim 2^k} T_{ij}.$$

The number M is just the number of distinct intersections of given order. The cases $k < 2$ are trivial for our purposes and we omit them. We assume the special case that

$$E_{2^k} \cap T_l \cap T_m \subset \bigcap_{j=1}^{\sim 2^k} T_{ij}, \tag{4.1}$$

for $l \neq m$. We then say that the intersection $T_l \cap T_m$ is point like, because the above holds for tubes replaced by lines. However it’s relatively easy to construct examples of situations where (4.1) does not hold. For example, some kind of a "double hairbrush" where we would have two handles intersecting a lot with a small angle $\sim \delta$. Then we would have

$$\bigcup_{i=1}^{2} \bigcap_{j=1}^{\sim 2^k} T_{ij} \subset E_{2^k} \cap T_l \cap T_m$$

and not

$$T_l \cap T_m \cap E_{2^k} = \bigcap_{j=1}^{\sim 2^k} T_j,$$

which is implied by (4.1). Now, via standard dyadic decomposition

$$\sum_k (2^k)^{n/(n-1)} |E_{2^k}| \sim || \sum_{i=1}^{N} 1_{B(0,1)} 1_{T_i} ||_{n/(n-1)}^{n/(n-1)}$$

It suffices to proof that

$$|E_{2^k}| \lesssim 2^{-kn/(n-1)} N \delta^{n-1}. \tag{4.2}$$

We use Fubini to deduct

$$\begin{aligned} (2^k)^3 |E_{2^k}| &\sim \int_{E_{2^k}} (\sum_{i=1}^{N} 1_{T_i})^3 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} \int 1_{T_i} 1_{T_j} 1_{T_l} \\ &\sim \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} |T_i \cap T_j \cap T_l \cap E_{2^k}|. \end{aligned} \tag{4.3}$$

Next we rewrite the triple sum. The first is the diagonal term, the second contains two repeated indexes and the key triple sum will contain no repeated indexes.

$$\begin{aligned} &\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} |T_i \cap T_j \cap T_l \cap E_{2^k}| \\ &\lesssim \delta^{n-1} N + C \sum_{i=1}^{N} \sum_{j=1}^{N} |T_i \cap T_j \cap E_{2^k}| \\ &+ \sum_{l=1, l\neq i, l \neq j}^{N} \sum_{i=1, i \neq j, i \neq l}^{N} \sum_{l=1, l \neq i, l \neq j}^{N} |T_i \cap T_j \cap E_{2^k} \cap T_l| \\ &\lesssim \delta^{n-1} N + 2^k \delta^{n-1} N + \sum_{l \in N_1(j, N_0(j)), l \neq i, l \neq j} \sum_{i \in N_0(j), i \neq j} \sum_{j=1}^{N} |T_i \cap T_j \cap E_{2^k} \cap T_l|. \end{aligned} \tag{4.4}$$

First of all we have the trivial estimates for the diagonal case. Next we deal the sum in (4.4) that contains two repeated indexes. Now,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} |T_i \cap T_j \cap E_{2^k}| \sim (2^k)^2 |E_{2^k}| \lesssim 2^k \delta^{n-1} N,$$

where we used that

$$\sum_k 2^k |E_{2^k}| \sim || \sum_{i=1}^{N} 1_{T_i} ||_1 = \sum_{i=1}^{N} |T_i| \sim \delta^{n-1} N.$$

Finally, we treat the case with no repeated indexes. Now, for each two different tubes T_i and T_l there are only $\sim 2^k$ tubes such that $|B(0,1)\cap T_i\cap,\ldots,T_{2^k}\cap E_{2^k}|\neq 0$ by the condition (4.1). So we use the condition (4.1) to $N_1(j,N_0(j))\subset(1,\ldots,N)$, and deduct that

$$\#(N_1(i,N_0(i)))\leq 2^{k+1}. \tag{4.5}$$

Intuitively this follows because the condition (4.1) with two different fixed tubes, T_i and T_j, and an index l that is always different than i or j implies that we sum over just one intersection. This intuition is the foundation why we use the third power of 2^k in the beginning. In other words, with condition (4.1) and with fixed $i\neq j$, there are only $\sim 2^k$ different choices for l:

$$\sum_{l=1,l\neq i,l\neq j}^{N}|T_i\cap T_j\cap E_{2^k}\cap T_l|=\sum_{l=1,l\neq i,l\neq j}^{N_{ij}}|\bigcap_{m=1}^{\sim 2^k}T_m\cap T_i\cap T_j\cap T_l|\sim 2^k|\bigcap_{m=1}^{\sim 2^k}T_m\cap T_i\cap T_j|.$$

However, an another key problem is that in the last part we are still left with two sums over to N in (4.4). So, an another key part of our proof is the following idea - "the summing away method":

$$\begin{aligned}
&\sum_{l\in N_1(j,N_0(j)),l\neq i,l\neq j}\ \sum_{i\in N_0(j),i\neq j}\ \sum_{j=1}^{N}|T_i\cap T_j\cap E_{2^k}\cap T_l|\\
&=\sum_{l\in N_1(j,N_0(j)),l\neq i,l\neq j}\ \sum_{i\in N_0(j),i\neq j}\ \sum_{j=1}^{N}\int_{T_i\cap T_l\cap E_{2^k}}1_{T_j}\\
&\lesssim \max_{j,j\neq l,j\neq l}(\#(N_1(j,N_0(j)))\sum_{i\in N_0(j),i\neq j}\ \sum_{j=1}^{N}\int_{T_i\cap T_l\cap E_{2^k}}1_{T_j})\\
&\lesssim 2^kN\sum_{j=1}^{N}\int_{E_{2^k}}1_{T_i}1_{T_l}1_{T_j}\\
&=2^kN\int_{E_{2^k}}1_{T_i}1_{T_l}\sum_{j=1}^{N}1_{T_j}\\
&\lesssim N2^k\int_{E_{2^k}}1_{T_j}1_{T_l}2^k\\
&=N(2^k)^2|T_i\cap T_l\cap E_{2^k}|.
\end{aligned} \tag{4.6}$$

Above, the third inequality follows from (4.5) and that $\#(N(j))\leq N$. The second to last inequality follows because we are integrating over E_{2^k}, respectively. Now, it follows from the lemma 1.3 that we have

$$|T_i\cap T_l\cap E_{2^k}|\lesssim 2^{-k/(n-1)}\delta^{n-1}, \tag{4.7}$$

for $i\neq l$. Thus, the claim (4.2), follows from the equations (4.3), (4.4), (4.6) and (4.7).

5. The proof the general case

We divide each δ-tube to L parallel δ'-tubes overlapping small amount. So that we have

$$T_i^{\delta} \subset \bigcup_{j=1}^{L'} T_{ij}^{\delta'}. \tag{5.1}$$

Then we choose a maximum number of disjoint δ'-tubes. If we don't like this procedure we should define our δ-tubes in equivalent way as hyperrectangles. For completess we prove the following quite self-evident lemma.

Lemma 5.1. *Suppose that each $x \in E_{2^k}$ belongs to maximally C_n δ'-tubes. Then for maximal number of disjoint tubes $T_{ij}^{\delta'}$ we have*

$$\bigcup_{j=1}^{L} T_{ij}^{\delta'} \subset T_i^{\delta}, \tag{5.2}$$

for all different tubes we have

$$T_i^{\delta} \subset \bigcup_{j=1}^{C_n L} T_{ij}^{\delta'}, \tag{5.3}$$

and

$$|T_i^{\delta}| \leq \sum_{j=1}^{C_n L} |T_{ij}^{\delta'}|. \tag{5.4}$$

Moreover, we have

$$\{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} \sum_{j=1}^{L} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\} = \bigcup_{j=1}^{L} \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\} \tag{5.5}$$

and for some set of tubes

$$|E_{2^k}| \sim |\{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} \sum_{j=1}^{L} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\}|. \tag{5.6}$$

Proof. Now, (5.2) and (5.3) are clear. Moreover, (5.4) follows from (5.3). Clearly, because the the L tubes are disjoint, we have (5.5)

$$\bigcup_{j=1}^{L} \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\} = \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} \sum_{j=1}^{L} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\}.$$

If $x \in \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} \sum_{j=1}^{L} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\}$, then (5.2) shows that $x \in E_{2^k}$. So we can assume $x \in E_{2^k}$. So essentially we just slice each δ-tube to L parallel disjoint δ'-tubes. We can proof the equation (5.6) as follows. If $x \in E_{2^k}$ then there exists $\sim 2^k$ δ-tubes such that $x \subset T_j$ for $j \in \{1, 2, \dots, 2^{k+1}\}$. So x belongs to one or few $T_{ij}^{\delta'}$-tubes via (5.3). Thus, $x \in \bigcup_{j=1}^{C_n L} \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\}$. So we proved that

$$E_{2^k} \subset \bigcup_{j=1}^{C_n L} \{x \in \mathbf{R}^n | 2^k \leq \sum_{i=1}^{N} 1_{T_{ij}^{\delta'}} 1_{B(0,1)} \leq 2^{k+1}\},$$

and that

$$\bigcup_{j=1}^{L}\{x\in\mathbf{R}^n|2^k\le\sum_{i=1}^{N}1_{T_{ij}^{\delta'}}1_{B(0,1)}\le 2^{k+1}\}\subset E_{2^k}.$$

So, via (5.5) we only need to prove that

$$|\bigcup_{j=1}^{C_nL}\{x\in\mathbf{R}^n|2^k\le\sum_{i=1}^{N}1_{T_{ij}^{\delta'}}1_{B(0,1)}\le 2^{k+1}\}|\le C_n|\bigcup_{j=1}^{L}\{x\in\mathbf{R}^n|2^k\le\sum_{i=1}^{N}1_{T_{ij}^{\delta'}}1_{B(0,1)}\le 2^{k+1}\}|.$$

But this follows if we take

$$|\bigcup_{j=1}^{L}\{x\in\mathbf{R}^n|2^k\le\sum_{i=1}^{N}1_{T_{ij}^{\delta'}}1_{B(0,1)}\le 2^{k+1}\}|=\max_{l\in(1,\dots,C_n)}|\bigcup_{j=1}^{L_l}\{x\in\mathbf{R}^n|2^k\le\sum_{i=1}^{N}1_{T_{ij}^{\delta'}}1_{B(0,1)}\le 2^{k+1}\}|.$$

. □

Next, we define

$$E'_{j2^k}:=\{x\in\mathbf{R}^n|2^k\le\sum_{i=1}^{N}1_{T_ij^{\delta'}}1_{B(0,1)}\le 2^{k+1}\}.$$

Thus, by previous lemma 5.1, we have

$$\sum_{j=1}^{L}|E'_{j2^k}|\sim|E_{2^k}|. \tag{5.7}$$

Finally, we make δ' so small that we have point like intersections. In other words

$$E'_{j2^k}\cap T_l^{\delta'}\cap T_m^{\delta'}\subset\bigcap_{i=1}^{\sim 2^k}T_{ij}^{\delta'},$$

for all l,m and j, when $l\neq m$. This "thinning technique" is always possible when we necessary have intersections, that is when $k\neq 0$. It holds that

$$\lim_{i\to\infty}T_l^{1/i}\cap T_m^{1/i}\cap E_{j2^k}^{'1/i}\subset\{x\}, \tag{5.8}$$

for $l\neq m$. So for any l and m, $l\neq m$, we can choose $\delta'_i=1/i$ such that

$$T_l^{1/i}\cap T_m^{1/i}\cap E_{j2^k}^{1/i}=T_l,^{1/i}\cap T_m^{1/i}\cap\bigcup_{j=1}^{M}\bigcap_{k=1}^{\sim 2^k}T_{jk}^{1/i}=\bigcap_{j=1}^{\sim 2^k}T_j^{1/i},$$

for all $i>i_0$, for some i_0. This follows from (5.8) and because the intersections are disjoint. Taking the minimum of $1/i$ over all M intersections gives the desired $\delta'_j<1/i$, for E'_{j2^k}. Then we must take the minimum of δ'_j over j and we are done. In addition the conditions for our theorem 1.1 hold for any E'_{j2^k}.

Remark 5.2. With the easy thinning technique just defined we get rid of the small angle counterexamples to (4.1). Now the angles are large enough with respect to possibly really small δ'. This method solves the so called "small angle problem" for the Kakeya conjecture.

So we have
$$|E'_{j2^k}| \lesssim 2^{-kn/(n-1)} N\delta'^{(n-1)}$$
via previous discussion and theorem 1.1. And it follows from above and from (5.1) that
$$|E_{2^k}| \sim \sum_{j=1}^{L} |E'_{j2^k}| \lesssim 2^{-kn/(n-1)} NL\delta'^{(n-1)} \sim 2^{-kn/(n-1)} N\delta^{n-1},$$
which proves the corollary 1.2.

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