The Generating Function Technique and Algebraic Ordinary Differential Equations

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Abstract

In the past, theorems have shown that individuals can implement a (formal) power series method to derive solutions to algebraic ordinary differential equations, or AODEs. First, this paper will give a quick synopsis of these "bottom-up" approaches while further elaborating on a recent theorem that established the (modified) generating function technique, or [m]GFT, as a powerful method for solving differentials equations. Instead of building a (formal) power series, the latter method uses a predefined set of (truncated) Laurent series comprised of polynomial linear, exponential, hypergeometric, or hybrid rings to produce an analytic solution. Next, this study will utilize the [m]GFT to create several analytic solutions to a few example AODEs. Ultimately, one will find [m]GFT may serve as a powerful "top-down" method for solving linear and nonlinear AODEs.

1.) Introduction

AODEs use differential algebra to define differential equations with only one independent variable [1]. Numbers theory and computer-based algebra are extensively utilized and supported by this field. Finally, AODEs have many formulations, such as differential Galois theory and modules (i.e., M, D, etc.).

Most methods for solving AODEs involve a "bottom-up" approach regarding a formal power series. In other words, an individual tries to establish an analytic solution by finding a pattern within the leading coefficient of a solitary power series [2]. If the value turns out to be a combinatorial number, then the power series becomes formal [3]. This process often involves enacting many iterations, making the "bottom-up" means of acquiring an analytic solution very time-consuming [2,3].

This article will consider a new "top-down" approach for finding solutions to AODEs. It is well-known that (formal) power series can form a new analytic function [4]. For instance, GFT, which incorporates a set of Laurent series of product ring-based formal power series or generating functions, can be used to discover analytic solutions to both linear and nonlinear partial differential equations [5]. This method might be the pinnacle of power series methods to develop new functions; hence, [m]GFT is viewed as a "top-down" means for solving differential equations, such as AODEs, since it uses preformed generating functions to find analytic solutions.

There are several sections in this paper. Section two will have a more thorough discourse on methods and theorems which implement a "bottom-up" basis for deriving analytic solutions to AODEs. On the other hand, the theorem explaining why [m]GFT, an effective "top-down" instrument in solving nonlinear partial differential equations, will be further elucidated and expanded upon to show it as a method for finding solutions AODEs in section three. Section four will show the application of

[m]GFT on a few examples of AODEs. Finally, the conclusion, or section five, will gleam a terse description and beneficial facets of [m]GFT.

2.) (Formal) Power Series and their solutions to AODEs

Consider the following power series, also known as a Taylor series:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If x_0, or the center of the series, was equal to null or zero, then the above expression is considered a Maclaurin series. Some coefficients a_n of a power series incorporated a division by factorial n. It was the main purgative for an individual to define the value of this coefficient for each serial term [6]. Also, the Cauchy-Hadamard theorem stated that power series converged at specific values [7,8]: when a series was deemed convergent, it formed an analytic function [9].

There were many algorithms and methods for establishing power series solutions to various differential equations via iterative or "bottom-up" means. For instance, N. Thieu Vo and associates devised an algorithm that helped an individual iteratively access each coefficient of a prospective power series solution to an AODE [10]. Building an analytic solution via single power series is generally time-consuming, but others in the field are finding ways to accelerate the process [11].

3.) The Generating Function Technique Revisited

The central theorem that established the GFT as a method for solving (nonlinear) PDEs claimed that a (truncated) Laurent series of formal power series derived analytic solutions to many (nonlinear) differential equations [5]. The theorem suggested that formal power series within the set of (truncated) Laurent series are polynomial rings. Upon applying the polynomial rings within a differential equation, an individual would form a free ideal ring whose generators were necessary to form algebraic equations. Setting these algebraic equations to zero, then solving for as many coefficients and constants as possible would allow the individual to establish analytic solutions to many differential equations.

Definition 3.1. *The predefined set of (truncated) Laurent series of generating functions served as the general solution to an AODE of interest and was a symmetric (Lie) algebra.*

Definition 3.2. *The auxiliary function f_l, in the primary expression defining the general solution y, was a polynomial ring based upon the dependent variable, or intermediate, x. The dependent variable x was linearized, exponentiated, or hypergeometric transformed.*

Definition 3.3. *The multiplication of auxiliary functions f_l established a polynomial product ring.*

The general solution y of [m]GFT was defined as follows:

$$y(x) =$$
$$\sum_{i=1}^{2} \sum_{j=-p_{s,a}}^{p_{s,b}} \left(a\,(i,j) \left(\sum_{k=0}^{\infty} 2S_k(0)^i \left(\prod_l f_l(x)\right)^k \right)^j + b(i,j) \left(\sum_{k=0}^{\infty} 2C_k(0)^i \left(\prod_l f_l(x)\right)^k \right)^j \right),$$

where f_l is the l-th auxiliary function, $S_k$and $C_k$were the square root of the Fibonacci and Chebyshev U combinatorial numbers about zero, respectively. Note:

$$S_k\left(0\right) = sin\left(\frac{\pi k}{2}\right),$$

and

$$C_k\left(0\right) = cos\left(\frac{\pi k}{2}\right).$$

The other coefficients accompanying each formal power series or generating function were $a(i,j)$ and $b(i,j)$. The set of Laurent series, which was truncated by the specific powers $p_{s,a}$ and $p_{s,b}$, of formal power series or generating functions were used in the general solution were "complete" polynomial rings.

Note: the general solution y with a Frobenius adjustment [18] was:

$$y(x) = x^r \sum_{i=1}^{2} \sum_{j=-p_{s,a}}^{p_{s,b}} \left(a\left(i,j\right) \left(\sum_{k=0}^{\infty} 2S_k(0)^i \left(\prod_l f_l(x) \right)^k \right)^j + b(i,j) \left(\sum_{k=0}^{\infty} 2C_k(0)^i \left(\prod_l f_l(x) \right)^k \right)^j \right).$$

For this paper's purpose, we defined the auxiliary function f_l as a linearized, exponentiated, or hypergeometric transformed (truncated) Laurent polynomial ring L_l; thus, there were at least two levels of (truncated) Laurent structures in the general solution. The Laurent polynomial ring L_l was defined as:

$$L_l(x) = \alpha_l(0) + x^2\alpha_l(2) + \frac{\alpha_l(-2)}{x^2} + x\alpha_l(1) + \frac{\alpha_l(-1)}{x},$$

or

$$L_l(x) = \alpha_l(0) + x^{3/2}\alpha_l(3) + \tfrac{\alpha_l(-3)}{x^{3/2}} + x^2\alpha_l(4) + \tfrac{\alpha_l(-4)}{x^2} + x\alpha_l(2) + \sqrt{x}\alpha_l(1) + \tfrac{\alpha_l(-1)}{\sqrt{x}} + \tfrac{\alpha_l(-2)}{x},$$

where α_l was a coefficient/constant. This algebraic entity, which added to the topology of the space of future analytic solutions to an AODE that was derived using [m]GFT [12,13], were possibly truncated. (Note: the former auxiliary function L_l was used in the rest of this section and for deriving solutions to the example AODEs given in the next section of this study since it possessed a significantly lower computational cost than its counterpart.) The auxiliary function f_l of the predefined formal power series or generating function was either linearized ($l = 1$), exponentiated ($l = 2$), the hypergeometric transformed ($l = 3$), or a combination of at least two of the three last states. Thus:

$$f_1\left(x\right) = L_1\left(x\right),$$

$$f_2\left(x\right) = e^{L_2(x)},$$

and

$$f_3\left(x\right) =_1 F_1\left(\beta_1; \beta_2; L_3\left(x\right)\right).$$

If the auxiliary function f_1 were only in the predefined formal power series, then the resultant general solutions would be comprised of polynomial linear rings. If the auxiliary function f_2 were only in the predefined formal power series, then the resultant general solution would be made of polynomial exponential rings. It was important to note that the paper introducing GFT only considered (a variation of) the auxiliary function f_2 with a truncated Laurent polynomial ring $L_2(x)$ [5]. If the auxiliary function f_3 were only in the predefined formal power series, then the resultant general solution would consist of polynomial hypergeometric rings. Finally, if at least two of the three auxiliary functions f_1, f_2, and f_3 were in the predefined formal power series, then the resultant general solution would be comprised of polynomial hybrid rings. It was important to note the multiplication of at least two of the three auxiliary functions f_1, f_2, and f_3 established a product ring. Since one considered the composition of formal power series as a set of polynomial rings, (s)he could claim each completed polynomial ring was a symmetric algebra [14].

Lemma and definition 3.4. *Plugging the set of (truncated) Laurent series of polynomial rings described above into an AODE of interest and possibly considering some hypergeometric function contiguous relations established Hopf algebras.*

Lemma 3.5. *Multiplying the Hopf algebras with the multiplicative inverse of the greatest common (zero) divisor, followed by parsing the resultant principal integral/ideal domain or free ideal ring by the generators of the transformed AODE of interest, that were associated with auxiliary functions f_1, f_2, and f_3, produced algebraic equations.*

Substituting the general solution into AODEs established Hopf algebras known to be skewed polynomial rings called noncommutative principal ideal rings, or R_{PI} [15]. Sometimes the Hopf algebras did not become apparent until an individual considered hypergeometric function contiguous relations. When an individual multiplied the transformed AODEs with its common denominator/multiplicative inverse of the greatest common (zero) divisor, or $(R_{GCD})^{-1}$, (s)he obtained a particular product ring, called a principal integral/ideal domain, or R_{PID} [16]. In other words, the numerator of a principal ideal ring was a principal ideal domain.

$$R_{PI} = R_{PID} R_{GCD}$$

$$R_{PI}(R_{GCD})^{-1} = R_{PID} R_{GCD}(R_{GCD})^{-1}$$

$$R_{PI}(R_{GCD})^{-1} = R_{PID}$$

A principal integral/ideal domain was also known as a free ideal ring or *fir* [17]. Also, it was necessary to note that if one considered the auxiliary function f_3 was present in the general solution, (s)he had to include the contiguous relations of the hypergeometric function after plugging in the general solution into the AODE. For example, the derivatives of ${}_1F_1(\beta_1; \beta_2; L_3(x))$ in the transformed AODE were converted to expressions involving ${}_1F_1(\beta_1; \beta_2; L_3(x))$ and ${}_1F_1(\beta_1; 1+\beta_2; L_3(x))$ before proceeding.

A set of generators parsed the principal integral/ideal domain or free ideal ring into many symmetric algebraic equations. The gathering of linearized, exponentiated, and/or hypergeometric transformed (truncated) Laurent polynomial ring of variable/intermediate x, like $\langle x, e^{L_2(x)}, {}_1F_1(\beta_1; \beta_2; L_3(x)), {}_1F_1(\beta_1; 1+\beta_2; L_3(x))\rangle$, may serve as the generator set for the R_{PID} produced from some AODEs. (For instance, the parsing of a R_{PID} may have only used part of the entire possible generator set, like $\langle x^p, e^{L_2(x)}\rangle$.) It was important to state the generator $\langle e^{L_2(x)}\rangle$ was able to be expanded to its individual exponential terms and itself. For instance,

$$\left\langle e^{\frac{\alpha_2(-1)+x(\alpha_2(0)+\alpha_2(1)x)}{x}}\right\rangle = \left\langle e^{\frac{\alpha_2(-1)}{x}}, e^{\alpha_2(0)}, e^{\alpha_2(1)x}, e^{\frac{\alpha_2(-1)+x(\alpha_2(0)+\alpha_2(1)x)}{x}}\right\rangle.$$

The generator set was used to derive more algebraic equations by parsing the product ring R_{PID} [5]. These algebraic objects were also likely symmetric.

Corollary 3.6. *By setting the algebraic equations derived from the product ring discussed above to null or zero, an individual should be able to define at least one set of coefficients/constants properly. After plugging these coefficients/constants into the set of (truncated) Laurent series of polynomial rings described in definition 3.1, the last item may become an analytic solution to the AODE of interest.*

One exploited the algebraic equations for the values of the coefficients/constants a, b, α_l, β_1, and β_2 whenever possible after setting all the algebraic equators equal to null or zero. Using computer mathematics software, like Mathematica®, individuals derived sets of known coefficients/constants that yielded analytic solutions to AODEs.

It was important to note that when the AODE contained variable coefficients and inhomogeneous components that were either exponential or trigonometric functions, an individual had to limit the auxiliary function f_l as an exponentiated expression of the independent variable x, or let $l = 2$.

Mathematica® was used to derive solutions to the following AODEs. Thus, one was able to follow most of the work in this paper by examining the Mathematica® spreadsheet.

4.) Examples

This study section will consider and solve three examples AODEs found within the Kamke set [18]. The first equation, ODE No. 29, in the paper is as follows:

$$F\left(x, y, y'\right) = y'(x) - xy(x)^2 - 3xy(x) = 0.$$

By setting the domain of p_s of the general solution between [0,2], then plugging it into the above equation, one obtains a significant expression involving the variable x. Next, (s)he times the common denominator/multiplicative inverse of the expression's greatest common (zero) divisor with the original equation/principal ideal ring to derive a product ring or free ideal ring. The following generator set, $\left\langle \begin{matrix} x, e^{\alpha_2(0)+x(\alpha_2(1)+\alpha_2(2)x)}, \\ {}_1F_1\left(\beta_1;\beta_2;\alpha_3\left(0\right)+x\left(\alpha_3\left(1\right)+x\alpha_3\left(2\right)\right)\right), {}_1F_1\left(\beta_1;1+\beta_2;\alpha_3(0)+x\left(\alpha_3(1)+x\alpha_3(2)\right)\right) \end{matrix} \right\rangle$, creates one hundred fifty-nine algebraic equations. After setting these algebraic equations to null or zero, one derives two hundred seventy-nine sets of coefficients/constants, thus possible analytic solutions. The expression below is an example of an analytic solution derived for AODE given above:

$$y\left(x\right) = -\frac{3\alpha_1(0)e_1^{\alpha_2(0)+\frac{3x^2}{2}}F_1\left(\beta_1;\beta_2;\alpha_3(0)\right)}{\alpha_1(0)e_1^{\alpha_2(0)+\frac{3x^2}{2}}F_1\left(\beta_1;\beta_2;\alpha_3(0)\right)+i}$$

Another AODE from the Kamke set is ODE No. 1115:

$$F\left(x,y,y',y''\right)=xy''(x)-(3x-2)y'(x)+(3-2x)y(x)=0.$$

By setting the domain of p_s of the predefined set of (truncated) Laurent series consisting of formal power series also between [-1,0], then plugging it into the above equation, one obtains a significant expression involving the variable x. Again, (s)he times the common denominator/multiplicative inverse of the expression's greatest common (zero) divisor and the equation itself to derive a product ring or free ideal ring. The following generator set, $\left\langle \begin{array}{c} x, e^{\frac{\alpha_2(-1)+x(\alpha_2(0)+\alpha_2(1)x)}{x}}, \\ {}_1F_1\left(\beta_1;\beta_2;\frac{\alpha_3(-1)}{x}+\alpha_3(0)+x\alpha_3(1)\right), {}_1F_1\left(\beta_1;1+\beta_2;\frac{\alpha_3(-1)}{x}+\alpha_3(0)+x\alpha_3(1)\right) \end{array}\right\rangle$, yields thirty-six algebraic equations. After solving the coefficients/constants, one derived twenty-seven sets of possible solutions. The following is an example of an analytic solution derived by [m]GFT for the above AODE:

$$y(x)=a(1,-1)_1F_1\left(1-\frac{6}{\sqrt{17}};2;\sqrt{17}x\right)e^{\alpha_2(0)-\frac{1}{2}\left(\sqrt{17}-3\right)x}.$$

Next, the ODE No. 1700 from the Kamke set is considered:

$$F\left(x,y,y',y''\right)=y(x)y''(x)-y'(x)^2+1=0.$$

By setting the p_s domain of the general solution again between [-1,0], then plugging it into the above equation, one produces a significant expression involving the variable x. Again, (s)he multiplies the common denominator/multiplicative inverse of the greatest common (zero) divisor of the expression and the equation itself to derive a product ring or free ideal ring. The following generator set, $\left\langle x, e^{\alpha_2(1)x}, {}_1F_1\left(\beta_1;\beta_2;x\alpha_3(1)\right), {}_1F_1\left(\beta_1;1+\beta_2;x\alpha_3(1)\right)\right\rangle$, establishes one hundred thirty-seven algebraic equations. These sets of coefficients/constants generate thirty-one different results. An example of an analytic solution derived by [m]GFT for the above AODE is as follows:

$$y(x)=\frac{(a(1,-1)+a(2,-1))e^{\frac{x}{\sqrt{a(2,-1)^2-a(1,-1)^2}}}}{2\alpha_1(0)}+\frac{1}{2}\alpha_1(0)(a(1,-1)-a(2,-1))e^{-\frac{x}{\sqrt{a(2,-1)^2-a(1,-1)^2}}}.$$

The final AODE from the Kamke set is ODE No. 1483:

$$F\left(x,y,y',y'',y^{(3)}\right)=2xy^{(3)}(x)-4(v+x-1)y''(x)+(6v+2x-5)y'(x)+(1-2v)y(x)=0.$$

By setting p_s of the general solution with the Frobenius correction to negative unity or *-1*, then plugging it into the above equation, one produces a significant expression involving the variable x. Again, (s)he multiplies the common denominator/multiplicative inverse of the greatest common (zero) divisor of the expression and the equation itself to derive a product ring or free ideal ring. The following generator set, $\left\langle \begin{array}{c} x, x^r, e^{\alpha_2(1)x}, {}_1F_1\left(\beta_1;\beta_2;x\alpha_3(1)\right), \\ {}_1F_1\left(\beta_1;1+\beta_2;x\alpha_3(1)\right) \end{array}\right\rangle$, creates eighteen algebraic equations. These sets of coefficients/constants generate fifteen different results. An example of an analytic solution derived by [m]GFT for the above AODE is as follows:

$$y(x)=\alpha_1(0)a(1,-1)x^{2v}_1F_1\left(v+\tfrac{1}{2};2v+1;x\right).$$

5.) Conclusion

Unlike many "bottom-up" methods that build analytic solutions from solitary (formal) power series, [m]GFT uses a set of (truncated) Laurent series of predefined polynomial linear, exponential, hypergeometric, or hybrid rings to establish analytic solutions to differential equations. Once applied to differential equations, like AODEs, it also establishes Hopf symmetric algebras. Finally, [m]GFT is not only a powerful tool for solving (nonlinear) partial differential equations but also possesses excellent potential to find analytic solutions to many linear and nonlinear differential equations, unlike other methods.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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