A Mystery in Conformal Mappings and Division by Zero Calculus

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Abstract: We introduce a mysterious property in conformal mappings and division by zero calculus with some elementary linear mapping.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

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1 A new type example

We introduce a mysterious property in conformal mappings and division by zero calculus with some elementary linear mapping.

We consider the elementary mapping

$$W = \frac{1}{1-z} \tag{1.1}$$

on the complex plane. Then, we note that the circle $C_r : |z| = r, 0 < r < 1$ is mapped conformally to the circle with its center

$$\frac{1}{1-r^2} \tag{1.2}$$

and with its radius

$$\frac{r}{1-r^2}. \tag{1.3}$$

Its diameter is given by

$$\left[\frac{1}{1+r}, \frac{1}{1-r}\right]. \tag{1.4}$$

It is the Apollonius' circle that is given by

$$\left|\frac{w-1}{w}\right| = r. \tag{1.5}$$

Then, from the representations

$$\left(x - \frac{1}{1-r^2}\right)^2 + y^2 = \left(\frac{r}{1-r^2}\right)^2 \tag{1.6}$$

and

$$(1-r^2)x^2 - 2x + \frac{1}{1-r^2} + (1-r^2)y^2 = \frac{r^2}{1-r^2}, \tag{1.7}$$

we obtain the surprising result, by the division by zero calculus for $r = 1$

$$x = \frac{1}{2}, \tag{1.8}$$

because we will expect to obtain the usual conformal mapping to the line

$$x = 1. \tag{1.9}$$

Indeed note that by the division by zero calculus,

$$\frac{1}{1-r^2}|_{r=1} = \frac{1}{4}, \tag{1.10}$$

$$\frac{r^2}{1-r^2}|_{r=1} = -\frac{3}{4}, \tag{1.11}$$

and

$$\frac{r}{1-r^2}|_{r=1} = -\frac{1}{4}. \tag{1.12}$$

In general, we obtain

$$\frac{r^n}{1-r^2}|_{r=1} = \frac{1}{4}(1-2n). \tag{1.13}$$

The line (1.8) may be considered as the natural one from the viewpoint of the Apollonius circle (1.5). However, we see that its diameter and radius may be considered as mysterious ones.

Note that in (1.7), when we apply the division by zero calculus, we have the nonsense and mysterious result.

2 Conclusion and open questions

We discovered a new type of result in the division by zero calculus in conformal mappings.

For example, what is a general theory or a general theorem for the example?

What does the example mean?

3 Essence of division by zero calculus

We state the essence of division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{3.1}$$

we will **define**

$$f(a) = C_0. \tag{3.2}$$

For the correspondence (3.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (3.1), we **can**

define any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

However, we can consider the more general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{3.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [1, 2, 3, 4].

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

Note that

$$(x^n)' = nx^{n-1}$$

and so

$$\left(\frac{x^n}{n}\right)' = x^{n-1}.$$

Here, we obtain the right result for $n = 0$

$$(\log x)' = \frac{1}{x}$$

by the division by zero calculus.

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