ON THE AVERAGE NUMBER OF INTEGER POWERED DISTANCES IN $\mathbb{R}^k$

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Abstract. Using the method of compression we obtain a lower bound for the average number of d^r-unit distances that can be formed from a set of n points in the euclidean space $\mathbb{R}^k$. By letting $\mathcal{D}_{n,d^r}$ denotes the number of d^r-unit distances ($r > 1$ fixed) that can be formed from a set of n points in $\mathbb{R}^k$, then we obtain the lower bound

$$\sum_{1 \leq d \leq t} \mathcal{D}_{n,d^r} \gg n \sqrt[2r]{k} \log t.$$

for a fixed $t > 1$.

1. Introduction

The Erdős distance problem in $\mathbb{R}^k$ ($k \geq 3$) is perhaps one of the most celebrated unsolved problem in discrete geometry. The problem as is suggestive was posed by the Hungarian mathematician Paul Erdős. It has two main versions, namely the distinct distance problem and the unit distance problem, respectively. Even though both versions of the problem remains unsolved in higher dimensions, some substantial progress has been made. For the number distinct distances $\mathcal{P}_{n,k}$ that can be formed from an arrangement of n points in any euclidean space $\mathbb{R}^k$ for $k \geq 3$, Erdős obtained the upper bound

$$\mathcal{P}_{n,k} \ll n^{\frac{2}{k}}$$

and conjectured that the lower bound for the number of distinct distances that can be formed from the arrangement of n points in the space $\mathbb{R}^k$ for $k \geq 3$ is lower bounded by the same quantity. It is known that (see [3]) the number of distinct distances that can be formed a set of n points in a euclidean space $\mathbb{R}^k$ for $k \geq 3$ satisfies the lower bound

$$\mathcal{P}_{n,k} \gg n^{\frac{2}{k} - \frac{2}{k(k+2)}}.$$

The unit distances problem similarly seeks to find the number of unit distances that can be formed from a set of n points in the plane. Let $\mathcal{I}_{n,k}$ denotes the number of unit distances that can be formed from a set of n points in the euclidean space $\mathbb{R}^k$ for $k \geq 2$. Paul Erdős (see [2]) proved the lower bound for the number of unit distances that can be formed from n points in $\mathbb{R}^k$ for $k = 2$

$$\mathcal{I}_{n,k} \gg n^{1+o(1)}$$

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and conjectured that the upper bound can also be bounded by a function of this form. The best known upper is given by

$$\mathcal{I}_{n,k} \ll n^{\frac{4}{3}}$$

due to Joel Spencer, Endre Szemeredi, and William Trotter [1].
In this paper, by exploiting the method of compression of points in $\mathbb{R}^k$, we obtain a lower bound for the number of d-unit distances that can be formed from a set of n points in $\mathbb{R}^k$ for $k \geq 2$. In particular, we obtain the following lower bound

Theorem 1.1. *Let $\mathcal{D}_{n,d^r}$ denotes the number of d^r-unit distances ($d > 0$) that can be formed from a set of n points in $\mathbb{R}^k$. Then the lower bound holds for the average number of distances formed for $1 \leq d \leq t$ ($t > 1$ fixed)*

$$\gg n \sqrt[2r]{k} \log t.$$

2. Preliminaries and background

Definition 2.1. By the compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $1 \geq m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. The mass of compression.

In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i,j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i,j \leq n$.

Lemma 2.4. *The estimate holds*

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m > 0$.

Proposition 2.2. *Let* $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ *with* $x_i \neq 0$ *for each* $1 \leq i \leq n$ *and* $x_i \neq x_j$ *for* $i \neq j$*, then the estimates holds*

$$m\log\left(1-\frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \ll m\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

□

Definition 2.6. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale $m > 0$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

Definition 2.7. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $1 \leq i \leq n$. Then by the ball induced by $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of scale $m > 0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| < \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n = 2$.

Remark 2.8. In the geometry of balls under compression of scale $m > 0$, we will assume implicitly that $1 \geq m > 0$. The circle induced by points under compression is the ball induced on points when we take $n = 2$.

Proposition 2.3. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

In particular, we have the estimate

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

for $\vec{x} \in \mathbb{N}^n$, where $m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]$ is the error term in this case.

Lemma 2.9 (Compression estimate)**.** *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$ and $x_i \neq x_j$ for $i \neq j$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\mathrm{sup}(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1} - 2mn.$$

Theorem 2.10. *Let $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$. Then $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] < \mathcal{G} \circ \mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$, then it follows that $||\vec{y}|| > ||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \geq \mathcal{G} \circ \mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}|| \leq ||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] < \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}|| < ||\vec{y}||$. It follows that

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\bigg(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\bigg)\right|\right| &< \left|\left|\vec{y}-\frac{1}{2}\bigg(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\bigg)\right|\right| \\ &= \frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 2.11. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. If $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.10 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{y}] &> \mathcal{G}\circ\mathbb{V}_m[\vec{z}] \\ &\geq \mathcal{G}\circ\mathbb{V}_m[\vec{x}] \\ &> \mathcal{G}\circ\mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 2.12. Theorem 2.11 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.13. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\bigg(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\bigg)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 2.14. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball.

Theorem 2.15. *The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \neq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.10, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}] \leq \mathcal{G}\circ\mathbb{V}_m[\vec{z}] < \mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{y}|| < ||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}$ is an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The latter equality follows from assertion that two balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}] = \mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the indistinguishable balls and must satisfy the inequality

$$\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| = \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$
$$= \frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}] = \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Remark 2.16. We note that we can replace the set $\mathbb{N}^n$ used in our construction with $\mathbb{R}^n$ at the compromise of imposing the restrictions $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_i > 1$ for all $1 \leq i \leq n$ and $x_i \neq x_j$ for $i \neq j$. The following construction in our next result in the sequel employs this flexibility.

3. The lower bound

Theorem 3.1. *Let $\mathcal{D}_{n,d^r}$ denotes the number of d^r-unit distances ($d > 0$) that can be formed from a set of n points in $\mathbb{R}^k$ for a fixed $r > 1$. Then the lower bound holds*

$$\sum_{1\leq d\leq t} \mathcal{D}_{n,d^r} \gg n\sqrt[2r]{k}\log t$$

for a fixed $t > 1$.

Proof. Pick arbitrarily a point $(x_1, x_2, \ldots, x_k) = \vec{x} \in \mathbb{R}^k$ with $x_i > 1$ for $1 \leq i \leq k$ and $x_i \neq x_j$ for $i \neq j$ such that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}] = d^r$ for a fixed $d > 0$ and $r > 1$. This ensures the ball induced under compression is of radius $\frac{d^r}{2}$. Next we apply the compression of fixed scale $m \leq 1$, given by $\mathbb{V}_m[\vec{x}]$ and construct the ball induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2} = \frac{d^r}{2}$. By appealing to Theorem 2.15 admissible points $\vec{x}_l \in \mathbb{R}^k$ ($\vec{x}_l \neq \vec{x}$) of the ball of compression induced must satisfy the condition $\mathcal{G}\circ$

$\mathbb{V}_m[\vec{x}_l] = d^r$. Next we count the number of d^r-unit distances formed by a set of n points in $\mathbb{R}^k$ by counting pairs of admissible points $(\vec{x}_l, \vec{x}_h)$ on the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that $\mathbb{V}_m[\vec{x}_l] = \vec{x}_h$ so that the average number of d^r-unit distances for $1 \leq d \leq t$ with fixed $t, r > 1$ is lower bounded by

$$\sum_{1\leq d\leq t}\mathcal{D}_{n,d^r} = \sum_{1\leq d\leq t}\sum_{\substack{1\leq l\leq \frac{n}{2}\\ \vec{x}_l\in\mathbb{R}^k\\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]=d^r}} 1$$

$$= \sum_{1\leq d\leq t}\sum_{\substack{1\leq l\leq \frac{n}{2}\\ \vec{x}_l\in\mathbb{R}^k}} \frac{\sqrt[r]{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]}}{d}$$

$$\gg \sum_{1\leq d\leq t}\sum_{\substack{1\leq l\leq \frac{n}{2}\\ 1\leq i\leq k}} \frac{\sqrt[2r]{k}\,\sqrt[r]{\mathrm{Inf}(x_{l_i})}}{d}$$

$$\geq \sum_{1\leq d\leq t}\frac{\sqrt[2r]{k}}{d}\sum_{1\leq l\leq \frac{n}{2}} 1$$

$$= \sum_{1\leq d\leq t}\frac{n\sqrt[2r]{k}}{2d}$$

$$= \frac{n\sqrt[2r]{k}}{2}\sum_{1\leq d\leq t}\frac{1}{d}$$

and the lower bound follows. □

4. **Data availability statement**

The manuscript has no associated data.

5. **Conflict of interest**

The authors declare no conflict of interest regarding this manuscript. [1].

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