On the Integral's Substitution Rule

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Abstract

In this article, I discuss the proof in [5] in more detail and add some details. The first part of the article mainly gives some concepts and lemmas, such as the definition of some function spaces [6], Radon integral and its basic properties [3], L^1-seminorm [3], the definition of Lebesgue integral [3, 8] and so on. These basic concepts and lemmas are very helpful in proving the main theorem later. The proof of the main theorem is divided into two parts. The main idea is to locally approximate the transformation $\boldsymbol{\phi} : U \to V$ around a point $\mathbf{a} \in U$ by an affine map. In order to be able to use this local approximation meaningfully, we decompose the given function $\psi \in C_c(V)$ into a sum of functions with very small support, so that the approximation of $\boldsymbol{\phi} : U \to V$ by the local affine approximation is very good. This is done with the help of the $\zeta-$function introduced below, which provides a practical partition of the one on $\mathbb{R}^n$ [5]. In the second part, the proof will be performed on suitable approximations of any integrable function [5]. Lemma (3.1) shows that f is $I_R\big|_U-$integrable iff the trivial continuation $f_U : X \to \mathbb{C}$ is I_R-integrable. It is also very important for the proof of the main theorem.

1 Some basic concepts and lemmata

Notation 1.1:

Let X be a locally compact topological space. Let $f : X \to \mathbb{C}$ be a function. We call:

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}} \tag{1.1}$$

the ***support*** *of f . We denote the set of all continuous functions $f : X \to \mathbb{C}$ with $C(X)$,thus:*

$$C(X) := \{f : X \to \mathbb{C} : f \text{ continuous}\}. \tag{1.2}$$

In this papar, we will make frequent use of the following function space:

$$C_c(X) := \{f \in C(X) : \operatorname{supp}(f) \text{ is compact}\}. \tag{1.3}$$

We denote $C^{\mathbb{R}}(X)$ and $C_c^{\mathbb{R}}(X)$ for the analogously defined spaces of real-valued functions.

Lemma 1.1 (Urysohn):

Let X be a normal topological space and $A, B \subseteq X$ closed and disjoint. Then there is a continuous function $f : X \to [0, 1]$ with $f\big|_A = 0$ and $f\big|_B = 1$.

For the proof of Urysohn's lemma please refer to [2]. A normal space is a topological space in which any two disjoint closed sets can be separated by neighbourhoods. Urysohn's lemma states that a topological space is normal if and only if any two disjoint closed sets can be separated by a continuous function.

Definition 1.1 (Radon−Integral):

Let X be a locally compact space. A ***Radon-Integral*** *[3] on X is a positive linear functional:*

$$I_R : C_c(X) \to \mathbb{C}, \quad f \mapsto I_R(f). \tag{1.4}$$

Which means that I_R is linear and $I_R \geq 0$ if is $f \geq 0$.

Lemma (1.2) shows some elementary properties of Radon−integral.

Lemma 1.2:

Let $I_R : C_c(X) \to \mathbb{C}$ be a Radon-integral. Then:

(i). *For all* $f \in C_c^{\mathbb{R}}(X)$ *holds* $I_R(f) \in \mathbb{R}$.

(ii). *If* $f_1,\, f_2 \in C_c^{\mathbb{R}}(X)$ *with* $f_1 \leq f_2$ *then* $I_R(f_1) \leq I_R(f_2)$.

(iii). $|I_R(f)| \leq I_R(|f|)$ *for all* $f \in C_c(X)$.

Proof:

Let $I_R : C_c(X) \to \mathbb{C}$ be a Radon-integral.

(i). We break the function $f \in C_c(X)$ into positive part $f^+ := \max\{f, 0\} \geq 0$ and negative part $f^- := \min\{-f, 0\} \geq 0$. Then we can write f as the difference of the positive and negative part $f = f^+ - f^-$. Because I_R is linear thus:

$$I_R(f) = I_R\left(1f^+ + (-1)\,f^-\right) = 1I_R\left(f^+\right) + (-1)\,I_R\left(f^-\right) = I_R\left(f^+\right) - I_R\left(f^-\right) \in \mathbb{R}. \tag{1.5}$$

Because $I_R(f^+),\, I_R(f^-) \geq 0$.

(ii). Actually because I_R is a linear functional, then I_R is linear and $I_R(f) \geq 0$ when $f \geq 0$. For any $f_1,\, f_2 \in C_c^{\mathbb{R}}(X) := \{f \in C_c(X) : f \text{ is real}\}$ with $f_1 \leq f_2$ holds $f_2 - f_1 \geq 0$ and hence $I_R(f_2 - f_1) = I_R(f_2) - I_R(f_1) \geq 0$.

(iii). We choose $\theta \in [0, 2\pi)$ and write f as $|f|\, e^{i\theta}$. Then $I_R(f) = |I_R(f)|\, e^{i\beta}$. Obviously holds $I_R\left(fe^{-i\theta}\right) = e^{-i\theta} I_R(f) = |I_R(f)| \geq 0$. Thus without loss of generality we can assume $I_R\left(fe^{-i\theta}\right) := I_R(f') \geq 0$. We write a $f \in C_c(X)$ as $f := f_r + if_i$ and we know from the elementary property of complex number that $|f| \geq f_r,\, f_i$. Thus we get:

$$|I_R(f)| = I_R(f) = I_R(f_r) + iI_R(f_i) = I_R(f_r) \leq I_R(|f|)\,. \tag{1.6}$$

□

Definition 1.2 (Lower semicontinuous):

Let X *be a locally compact space. A function* $f : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$ *is* ***lower semicontinuous****, if for all* $a \in (-\infty, +\infty]$ *holds:*

$$f^{-1}\left((a, +\infty]\right) \text{ open in } X. \tag{1.7}$$

We denote the set of all lower semicontinuous function $f : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$ *as* $C_u(X)$.

In the next, we will prove some properties of Radon−integral.

Lemma 1.3:

For every $f : X \to \mathbb{R}_0^+ := [0, +\infty]$ *are equivalent:*

(i). $f \in C_u^+(X) := \{f \in C_u(X) : f \geq 0\}$.

(ii). $f(x) = \sup\{\varphi(x) : \varphi \in C_c^+(X),\ \varphi \leq f\},\ \forall x \in X$. *Here* $C_c^+(X) := \{f \in C_c(X) : f \geq 0\}$.

Definition 1.3:

For $f \in C_u^+(X)$ *we define:*

$$I_R^*(f) := \sup\left\{I_R(\varphi) : \varphi \in C_c^+(X),\ \varphi \leq f\right\}. \tag{1.8}$$

Lemma 1.4:

$C_c^+(X) \subseteq C_u^+(X)$ *and* $I_R^*(\varphi) = I_R(\varphi),\ \forall\, \varphi \in C_c^+(X)$.

Proof:

Let $f \in C_c^+(X)$. Then f is a continuous function on X , thus $f \in C_c^+(X)$ is lower semicontinuous on X . Because f is a continuous on X iff it is lower semicontinuous and upper semicontinuous on X . Thus $f \in C_u^+(X)$. Which implies $C_c^+(X) \subseteq C_u^+(X)$. From definition (1.3) we have:

$$I_R^*(\varphi) := \sup\left\{I_R(\varphi) : \varphi \in C_c^+(X),\ \varphi \leq \varphi\right\} = I_R(\varphi)\,. \tag{1.9}$$

□

Lemma 1.5:

Is $\psi \in C_c^+(X)$ *and* $\varepsilon > 0$. *So exists a* $\widetilde{\psi} \in C_c^+(X)$ *with* $\widetilde{\psi} \leq \psi,\ I_R\left(\psi - \widetilde{\psi}\right) < \varepsilon$ *and* $\widetilde{\psi}(x) < \psi(x),\ \forall x \in \operatorname{supp}\left(\widetilde{\psi}\right)$.

Proof:

For $\delta > 0$ let $\psi_\delta := (\psi - \delta) \vee 0 := \max\{\psi - \delta, 0\} \in C_c^+(X)$. Obviously hold $\psi_\delta \leq \psi$ and $\psi_\delta(x) = \psi(x) - \delta,\ \forall x \in \operatorname{supp}(\psi_\delta)$. According to Urysohn's lemma exists a $\chi \in C_c^+(X)$ with $\chi \equiv 1$ on $\operatorname{supp}(\psi)$. Thus $0 \leq \psi - \psi_\delta \leq \delta\chi$ and $I_R(\psi - \psi_\delta) \leq \delta I_R(\chi)$. Now we can choose δ such that $\delta I_R(\chi) < \varepsilon$.

□

Lemma 1.6:

Are $g,\ g_1,\ g_2 \in C_u^+(X)$ and $\lambda \geq 0$. Then λg and $g_1 + g_2$ are also in $C_u^+(X)$ and hold:

$$I_R^*(g_1 + g_2) = I_R^*(g_1) + I_R^*(g_2), \quad I_R^*(\lambda g) = \lambda I_R^*(g). \tag{1.10}$$

Proof:

Here we only prove first identity in equation (1.10). Are $g,\ g_1,\ g_2 \in C_u^+(X)$. For any given $\varepsilon > 0$ .Then from lemma (1.3) exist $\varphi_1,\ \varphi_2 \in C_c^+(X)$ such that $\varphi_1 \leq g_1$ and $\varphi_2 \leq g_2$. Because $I_R^*(g_j),\ j = 1,2$ are supremum, thus holds $I_R^*(g_j) - \varepsilon/2 < I_R(\varphi_j),\ j = 1,2$. Then because I_R is linear, thus:

$$I_R^*(g_1 + g_2) \geq I(\varphi_1 + \varphi_2) = I(\varphi_1) + I(\varphi_2) > I_R^*(g_1) + I_R^*(g_2) - \varepsilon. \tag{1.11}$$

Because $\varepsilon > 0$ random, then $I_R^*(g_1 + g_2) \geq I_R^*(g_1) + I_R^*(g_2)$.
Given a $\varepsilon > 0$ again. Let $\varphi \in C_c^+(X)$ with $\varphi \leq g_1 + g_2$ and $I_R^*(g_1 + g_2) - \varepsilon < I_R(\varphi)$. From lemma (1.5) we know that exists $\psi \in C_c^+(X)$ with $\psi \leq \varphi,\ \psi(x) < \varphi(x),\ \forall x \in \operatorname{supp}(\psi)$ and $I_R(\varphi - \psi) < \varepsilon$. Thus $\psi(x) < (g_1 + g_2)(x),\ \forall x \in \operatorname{supp}(\psi)$ and $I_R(\psi) > I_R^*(g_1 + g_2) - 2\varepsilon$.
Let $x \in \operatorname{supp}(\psi)$ fixed and define a function $\delta_x := (g_1 + g_2)(x) - \psi(x) > 0$. We consider the set:

$$V_x := \left\{ z \in X : g_j(z) > g_j(x) - \frac{\delta_x}{3},\ j = 1,2 \right\}. \tag{1.12}$$

Obviously elements in V_x satisfy the condition $z \in g_j^{-1}(g_j(x) - \delta_x/3, +\infty],\ j = 1,2$. Because $g_j,\ j = 1,2$ are lower semicontinuous, thus $V_x = \bigcap_{j=1}^{2} g_j^{-1}(g_j(x) - \delta_x/3, +\infty]$ is an open set. Furthermore $x \in V_x \subseteq V_x$, thus V_x is a open neighbourhood of x . Then form Urysohn's lemma (1.1) exist $\varphi_1^x,\ \varphi_2^x \in C_c^+(X)$ with $\operatorname{supp}\left(\varphi_j^x\right) \subseteq V_x,\ 0 \leq \varphi_j^x \leq g_j - \delta_x/3$ and $\varphi_j^x(x) = g_j(x) - \delta_x/3,\ j = 1,2$. Thus $\varphi_j^x \leq g_j,\ j = 1,2$ and $(\varphi_1^x + \varphi_2^x)(x) > \psi(x)$. Let $W_x := \{z : (\varphi_1^x + \varphi_2^x)(z) > \psi(x)\}$. Then W_x is also a open neighbourhood of x . Because $K := \operatorname{supp}(\psi)$ is compact thus exist $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} W_{x_j}$. Then define $\varphi_j := \max_{1 \leq i \leq n} \varphi_j^{x_i},\ j = 1,2$ then it follows $\varphi_j \leq g_j,\ j = 1,2$ and $\psi \leq \varphi_1 + \varphi_2$. Thus:

$$I_R^*(g_1) + I_R^*(g_2) \geq I_R(\varphi_1) + I_R(\varphi_2) = I_R(\varphi_1 + \varphi_2) \geq I_R(\psi) > I_R^*(g_1 + g_2) - 2\varepsilon. \tag{1.13}$$

Because $\varepsilon > 0$ random, then $I_R^*(g_1) + I_R^*(g_2) \geq I_R^*(g_1 + g_2)$.

□

Definition 1.4 (L^1−seminorm):

Let $f : X \to \mathbb{C}$ (or $f : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$) be any function. Then we define:

$$\|f\|_{L^1,\text{semi}} := \inf\left\{ I_R^*(g) : g \in C_c^+(X),\ |f| \leq g \right\} \in [0, +\infty]. \tag{1.14}$$

*We call $\|f\|_{L^1,\text{semi}}$ the L^1−**seminorm** of f .*

Lemma 1.7:

For functions $f,\ f_1,\ f_2 : X \to \mathbb{C}$ and $\lambda \in \mathbb{C}$ hold:

(i). $\|f_1 + f_2\|_{L^1,\text{semi}} \leq \|f_1\|_{L^1,\text{semi}} + \|f_2\|_{L^1,\text{semi}}$.

(ii). $\|\lambda f\|_{L^1,\text{semi}} = \lambda \|f\|_{L^1,\text{semi}}$.

(iii). $\||f|\|_{L^1,\text{semi}} = \|f\|_{L^1,\text{semi}}$.

(iv). $(|f_1| \leq |f_2|) \Rightarrow \left(\|f_1\|_{L^1,\text{semi}} \leq \|f_2\|_{L^1,\text{semi}} \right)$.

(v). If $f \in C_u^+(X)$ then holds $\|f\|_{L^1,\text{semi}} = I_R^*(f)$.

(vi). If $\varphi \in C_c(X)$ then holds $\|\varphi\|_{L^1,\text{semi}} = I(|\varphi|)$.

Proof:

Here we only prove (i). Are $f_1,\ f_2 : X \to \mathbb{C}$ .We can without loss of generality assume $\|f_j\|_{L^1,\text{semi}} < +\infty,\ j=1,2$. Then exist functions $g_1,\ g_2 \in C_c^+(X)$ with $|f_j| \le g_j,\ j=1,2$. Because $\|f_j\|_{L^1,\text{semi}}$ are infimum, thus for any given $\varepsilon > 0$ we have $\|f_j\|_{L^1,\text{semi}} + \varepsilon/2 > I_R^*(g_j),\ j=1,2$. Then it follows $|f_1+f_2| \le |f_1| + |f_2| \le g_1 + g_2$ and:

$$\|f_1+f_2\|_{L^1,\text{semi}} \le I_R^*(g_1+g_2) = I_R^*(g_1) + I_R^*(g_2) < \|f_1\|_{L^1,\text{semi}} + \|f_2\|_{L^1,\text{semi}} - \varepsilon. \tag{1.15}$$

Because $\varepsilon > 0$ is random then we get finally $\|f_1+f_2\|_{L^1,\text{semi}} \le \|f_1\|_{L^1,\text{semi}} + \|f_2\|_{L^1,\text{semi}}$.

□

Lemma 1.8:

Let $f : X \to \mathbb{C}$ (or $f : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$) be any function, such that exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ with $\|f-\varphi_n\|_{L^1,\text{semi}} \to 0$. Then $(I_R(\varphi_n))_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathbb{C}$. Is $(\psi_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ another sequence in $C_c(X)$ with $\|f-\varphi_n\|_{L^1,\text{semi}} \to 0$,then:

$$\lim_{n\to\infty} I_R(\varphi_n) = \lim_{n\to\infty} I_R(\psi_n). \tag{1.16}$$

Proof:

Let $f : X \to \mathbb{C}$ (or $f : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$) be any function, such that exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ with $\|f-\varphi_n\|_{L^1,\text{semi}} \to 0$. Thus exists an $N \in \mathbb{N}$ such that $\|f-\varphi_n\|_{L^1,\text{semi}} \le \varepsilon/2$ for all $n \ge N$ and any given $\varepsilon > 0$. Then for all $n,\ m \ge N$ with lemma (1.2) (iii) we have:

$$|I_R(\varphi_n) - I_R(\varphi_m)| = |I_R(\varphi_n - \varphi_m)| \le I_R(|\varphi_n - \varphi_m|). \tag{1.17}$$

Because $|\varphi_n - \varphi_m| \in C_c^+(X)$ then from lemma (1.4) we know that $I_R^*(|\varphi_n - \varphi_m|) = I_R(|\varphi_n - \varphi_m|)$ and:

$$\begin{aligned} & \|\varphi_n - \varphi_m\|_{L^1,\text{semi}} \\ = \quad & \inf\left\{I_R^*(|\varphi_n - \varphi_m|) : |\varphi_n - \varphi_m| \in C_c^+(X),\ |\varphi_n - \varphi_m| \le |\varphi_n - \varphi_m|\right\} \\ = \quad & \inf\left\{I_R(|\varphi_n - \varphi_m|) : |\varphi_n - \varphi_m| \in C_c^+(X),\ |\varphi_n - \varphi_m| \le |\varphi_n - \varphi_m|\right\} \\ = \quad & I_R(|\varphi_n - \varphi_m|). \end{aligned} \tag{1.18}$$

Hence with lemma (1.7) (i) we have:

$$I_R(|\varphi_n - \varphi_m|) = \|\varphi_n - \varphi_m\|_{L^1,\text{semi}} \le \|\varphi_n - f\|_{L^1,\text{semi}} + \|f - \varphi_m\|_{L^1,\text{semi}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{1.19}$$

Thus $(I_R(\varphi_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.
Let now $(\psi_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ be an another with $\|f-\varphi_n\|_{L^1,\text{semi}} \to 0$,then:

$$\begin{aligned} & |I_R(\varphi_n) - I_R(\psi_n)| = |I_R(\varphi_n - \psi_n)| \le I_R(|\varphi_n - \psi_n|) = \|\varphi_n - \psi_n\|_{L^1,\text{semi}} \\ \le \quad & \|\varphi_n - f\|_{L^1,\text{semi}} + \|f - \psi_n\|_{L^1,\text{semi}} \to 0, \quad (n\to\infty). \end{aligned} \tag{1.20}$$

□

Definition 1.5 (I_R-integrable):

*Let $I_R : C_c(X) \to \mathbb{C}$ be a Radon-integral. Then a function $f : X \to \mathbb{C}$ (or $f : X \to \overline{\mathbb{R}} \setminus \{-\infty\}$) is called I_R-**integrable**, if a sequence $(\varphi_n)_{n\in\mathbb{R}} \subseteq C_c(X)$ with $\|f-\varphi_n\|_{L^1,\text{semi}} \to 0$ and then call:*

$$\int_X f(x)\ \mathrm{d}_{I_R}x := \lim_{n\to\infty} I_R(\varphi_n) \tag{1.21}$$

*the I_R-**integral** of f . Obviously from lemma (1.8) we know that I_R-integral is well-defined.*

Let:

$$\lambda : C_c(\mathbb{R}) \to \mathbb{C}, \quad f \mapsto \lambda(f) := \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x. \tag{1.22}$$

The $\lambda-$integrable function $f : \mathbb{R} \to \mathbb{C}$ (or $f : \mathbb{R} \to \overline{\mathbb{R}} \setminus \{-\infty\}$) is called ***Lebesgue integrable function*** on $\mathbb{R}$ and:

$$\int_{\mathbb{R}} f(x)\ \mathrm{d}_\lambda x \tag{1.23}$$

is called the ***Lebesgue integral*** of f on $\mathbb{R}$. Analogous is $f \in C_c(\mathbb{R}^n)$, thus:

$$\boldsymbol{\lambda}_n(f) := \int_{-\infty}^{+\infty} \left(\dots \int_{-\infty}^{+\infty} f(x_1, \dots, x_n) \, \mathrm{d}x_n \dots \right) \mathrm{d}x_1 \tag{1.24}$$

define a Radon integral on $\mathbb{R}^n$. The $\boldsymbol{\lambda}_n-$integrable functions are called ***Lebesgue integrable function*** on $\mathbb{R}^n$ and if a function $f : \mathbb{R} \to \mathbb{C}$ (or $f : \mathbb{R} \to \overline{\mathbb{R}} \setminus \{-\infty\}$) is Lebesgue integrable then we call:

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \, \mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x} \tag{1.25}$$

the ***Lebesgue integral*** of f on $\mathbb{R}^n$. Lemma (1.9) [8] tell us:

Lemma 1.9:

Let $f : (a, b) \to \mathbb{R}$ be a positive measurable (relating to the Borel $\sigma-$algebra) function such that for every interval $[c, d] \subseteq (a, b)$ the function $f\big|_{[c,d]}$ is Riemann–integrable. Then the following statements are equivalent:

(i). f is improperly Riemann–integrable.

(ii). f is $\lambda-$integrable.

In this case:

$$\int_a^b f(x) \, \mathrm{d}x = \int_{(a,b)} f(x) \, \mathrm{d}_\lambda x. \tag{1.26}$$

Where λ is the $1-$dimensional Lebesgue's measure.

Because every (relating to the natural topologies (inductive by any norm) on $\mathbb{R}$ and $\mathbb{C}$) continuous function $f : \mathbb{R} \to \mathbb{C}$ is measurable (relating to the Borel $\sigma-$algebras on $\mathbb{R}$ and $\mathbb{C}$) [8]. Then for $f \in C_c(\mathbb{R}) \subseteq C(\mathbb{R})$ we have:

$$\int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x = \lambda(f) = \int_{\mathbb{R}} f(x) \, \mathrm{d}_\lambda x. \tag{1.27}$$

Analogous for $n-$dimensional case:

$$\int_{-\infty}^{+\infty} \left(\dots \int_{-\infty}^{+\infty} f(x_1, \dots, x_n) \, \mathrm{d}x_n \dots \right) \mathrm{d}x_1 = \boldsymbol{\lambda}_n(f) = \int_{\mathbb{R}^n} f(\mathbf{x}) \, \mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x}. \tag{1.28}$$

Where $\boldsymbol{\lambda}_n$ is the $n-$dimensional Lebesgue's measure. Lemma (1.10) [8] tell us the Lebesgue measure is invariant under affine transformations:

Lemma 1.10:

Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation with the form:

$$\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{T}(\mathbf{x}) := \mathbf{A}\mathbf{x} + \mathbf{b}. \tag{1.29}$$

Where $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) := \{A \in M(n \times n, \mathbb{R}) : A \text{ is invertible}\}$. Hence:

$$\boldsymbol{\lambda}_{n,\mathbf{T}} = \frac{1}{|\det \mathbf{A}|} \boldsymbol{\lambda}_n. \tag{1.30}$$

Is $\mathbf{A} \in \mathbb{O}(n, \mathbb{R}) := \{A \in M(n \times n, \mathbb{R}) : A \text{ is orthogonal}\}$ then we have:

$$\boldsymbol{\lambda}_{n,\mathbf{T}} = \boldsymbol{\lambda}_n. \tag{1.31}$$

The second conclusion in lemma (1.10) is trivial, because the determinant of an orthogonal matrix is ± 1 . Now we can use the lemma (1.10) on the $\boldsymbol{\lambda}_n-$integrable function f and hence:

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \, \mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x} = |\det \mathbf{A}| \int_{\mathbb{R}^n} f(\mathbf{A}\mathbf{x} + \mathbf{b}) \, \mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x}. \tag{1.32}$$

Where $\boldsymbol{\lambda}_{n,\mathbf{T}}(f) = \boldsymbol{\lambda}_n(f \circ \mathbf{T})$. Thus with the equation (1.28) we get:

$$\boldsymbol{\lambda}_n(f) = |\det \mathbf{A}| \, \boldsymbol{\lambda}_n(f \circ \mathbf{T}). \tag{1.33}$$

The equation (1.32) is also established for any $f \in C_c(\mathbb{R}^n)$ [4].

Lemma 1.11:

Let $I_R : C_c(X) \to \mathbb{C}$ be the Radon-integral on X . We consider the set:

$$\mathcal{L}^1(X, I_R) := \{f : C_c(X) \to \mathbb{C} : f \text{ is } I_R \text{ integrable}\}. \tag{1.34}$$

Then $\mathcal{L}^1(X, I_R)$ is a $\mathbb{C}-$vector space. And the map:

$$\int_X : \mathcal{L}^1(X, I_R) \to \mathbb{C}, \quad f \mapsto \int_X f(x)\, \mathrm{d}_{I_R} x \tag{1.35}$$

*is linear. Furthermore for every I_R-**integrable** function f is $|f|$ also integrable, and:*

$$\left| \int_X f(x)\, \mathrm{d}_{I_R} x \right| \le \int_X |f(x)|\, \mathrm{d}_{I_R} x = \|f\|_{L^1,\text{semi}}. \tag{1.36}$$

Proof:

Are $f,\, g \in \mathcal{L}^1(X, I_R)$ and $(\varphi_n)_{n\in\mathbb{N}},\, (\psi_n)_{n\in\mathbb{N}} \subseteq C_c(X)$ with $\|f - \varphi_n\|_{L^1,\text{semi}} \to 0$ and $\|g - \psi_n\|_{L^1,\text{semi}} \to 0$. Thus:

$$\|f + g - (\varphi_n + \psi_n)\|_{L^1,\text{semi}} \le \|f - \varphi_n\|_{L^1,\text{semi}} + \|g - \psi_n\|_{L^1,\text{semi}} \to 0. \tag{1.37}$$

Which $f + g \in \mathcal{L}^1(X, I_R)$ and from definition (1.5) we get:

$$\begin{aligned} \int_X (f+g)\, \mathrm{d}_{I_R} x &= \lim_{n\to\infty} I_R(\varphi_n + \psi_n) = \lim_{n\to\infty} (I_R(\varphi_n) + I_R(\psi_n)) \\ &= \lim_{n\to\infty} I_R(\varphi_n) + \lim_{n\to\infty} I_R(\psi_n) = \int_X f\, \mathrm{d}_{I_R} x + \int_X g\, \mathrm{d}_{I_R} x. \end{aligned} \tag{1.38}$$

Is $\lambda \in \mathbb{C}$. Then:

$$\|\lambda f - \lambda\varphi_n\|_{L^1,\text{semi}} = \lambda \|f - \varphi_n\|_{L^1,\text{semi}} \to 0 \tag{1.39}$$

hence:

$$\begin{aligned} \int_X (\lambda f)\, \mathrm{d}_{I_R} x &= \lim_{n\to\infty} I_R(\lambda\varphi_n) = \lim_{n\to\infty} (\lambda I_R(\varphi_n)) \\ &= \lambda \lim_{n\to\infty} I_R(\varphi_n) = \lambda \int_X f\, \mathrm{d}_{I_R} x. \end{aligned} \tag{1.40}$$

We still show the last statement. Because of the triangle inequality we know that $||f(x)| - |\varphi_n(x)|| \le |f(x) - \varphi_n(x)|,\ \forall x \in X$. Thus with (iii) in lemma (1.7) we have $\||f| - |\varphi_n|\|_{L^1,\text{semi}} \le \|f - \varphi_n\|_{L^1,\text{semi}} \to 0$. Thus $|f|$ is I_R-integrable and hence form (iii) in lemma (1.2):

$$\begin{aligned} \int_X |f|\, \mathrm{d}_{I_R} x &= \lim_{n\to\infty} I_R(|\varphi_n|) \ge \lim_{n\to\infty} |I_R(\varphi_n)| \\ &= \left| \lim_{n\to\infty} I_R(\varphi_n) \right| = \left| \int_X f\, \mathrm{d}_{I_R} x \right|. \end{aligned} \tag{1.41}$$

Furthermore because of the triangle inequality we have:

$$\begin{aligned} & \|f\|_{L^1,\text{semi}} - \|f - \varphi_n\|_{L^1,\text{semi}} \le \|f - (f - \varphi_n)\|_{L^1,\text{semi}} = \|\varphi_n\|_{L^1,\text{semi}} \\ = \quad & \|f - (f - \varphi_n)\|_{L^1,\text{semi}} \le \|f\|_{L^1,\text{semi}} + \|f - \varphi_n\|_{L^1,\text{semi}}. \end{aligned} \tag{1.42}$$

Then for $n \to \infty$:

$$\|f\|_{L^1,\text{semi}} = \lim_{n\to\infty} \|\varphi_n\|_{L^1,\text{semi}} \overset{\text{eq.(1.18)}}{=} \lim_{n\to\infty} I_R(|\varphi_n|) \overset{\text{de.(1.5)}}{=} \int_X |f|\, \mathrm{d}_{I_R} x. \tag{1.43}$$

□

Definition 1.6:

*Let $I_R : C_c(X) \to \mathbb{C}$ be a Radon integral and $U \subseteq X$. Is $f : U \to \mathbb{C}$ a function. Then we call $f : U \to \mathbb{C}$ is I_R-**integrable on** U , if the trivial continuation of $f : U \to \mathbb{C}$:*

$$f_U : U \subseteq X \to \mathbb{C} := \begin{cases} f(x), & x \in U, \\ 0, & x \in (X \setminus U) \end{cases} \tag{1.44}$$

is I_R-integrable on X and hence we define:

$$\int_U f\, \mathrm{d}_{I_R} x := \int_X f_U\, \mathrm{d}_{I_R} x. \tag{1.45}$$

2 Part 1 of the proof

First of all, we need some definitions and lemmata. The main lemma in this part is lemma (2.6), which plays an important role in the proof of theorem (3.1).

Definition 2.1 ($\zeta-$function):
we define a function from $\mathbb{R}$ *to* $[0,1]$ *by:*

$$\zeta : \mathbb{R} \to [0,1], \quad x \mapsto \zeta(x) := \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| \geq 1, \end{cases} \tag{2.1}$$

and call it $\zeta-$***function****. We define the scaling of the* $\zeta-$*function on the domain by* $\zeta_\varepsilon(x) := \zeta(x/\varepsilon)$ *. Similarly define the scaled* $\zeta-$*function in* $\mathbb{R}^n$ *by:*

$$\Pi^n \zeta_\varepsilon : \mathbb{R}^n \to [0,1], \quad \mathbf{x} \mapsto \Pi^n \zeta_\varepsilon(\mathbf{x}) := \prod_{j=1}^{n} \zeta_\varepsilon(x_j). \tag{2.2}$$

Definition 2.2 (Translation in $\mathbb{R}^n$):
We define the ***translation in*** $\mathbb{R}$ *by a operator* $\tau_\cdot$ *:*

$$\tau_\cdot f(\mathbf{x}) := f(\mathbf{x} - \cdot). \tag{2.3}$$

Lemma (2.1) states that the support of the $z-$translated and $\varepsilon-$scaled $\zeta-$function in $\mathbb{R}_n$ is the closed ball in with z as the center and ε as the radius with respect to the maximum norm.

Lemma 2.1 (Support of $\zeta-$function):

$$\operatorname{supp}(\tau_{\mathbf{z}} \Pi^n \zeta_\varepsilon) = \overline{B_{\|\cdot\|_\infty}}(\mathbf{z}, \varepsilon) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{z}\|_\infty \leq \varepsilon\}, \quad \mathbf{z} \in \mathbb{R}^n, \ \varepsilon > 0. \tag{2.4}$$

with the maximum norm $\|\mathbf{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$ *and* $\operatorname{supp}(f)$ *the support of function* f *.*

Proof:
Let $\mathbf{z} \in \mathbb{R}^n$ and $\varepsilon > 0$. By the definition of $\zeta-$function in $\mathbb{R}^n$:

$$\begin{aligned} \tau_{\mathbf{z}} \Pi^n \zeta_\varepsilon(\mathbf{x}) &= \Pi^n \zeta_\varepsilon(\mathbf{x} - \mathbf{z}) = \prod_{j=1}^{n} \zeta_\varepsilon(x_j - z_j) \\ = \ & \prod_{j=1}^{n} \zeta\left(\frac{x_j - z_j}{\varepsilon}\right) = \prod_{j=1}^{n} \left(1 - \frac{x_j - z_j}{\varepsilon}\right), \quad \left|\frac{x_j - z_j}{\varepsilon}\right| \leq 1, \ \forall j = 1, \ldots, n. \end{aligned} \tag{2.5}$$

Obviously, iff:

$$(|x_j - z_j| \leq \varepsilon, \ \forall j = 1, \ldots, n) \Rightarrow (\max\{|x_1 - z_1|, \ldots, |x_n - z_n|\} \leq \varepsilon) \Leftrightarrow (\|\mathbf{x} - \mathbf{z}\|_\infty \leq \varepsilon) \tag{2.6}$$

$\tau_{\mathbf{z}} \Pi^n \zeta_\varepsilon(\mathbf{x})$ is not equal to 0 . This indicates:

$$\operatorname{supp}(\tau_{\mathbf{z}} \Pi^n \zeta_\varepsilon) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{z}\|_\infty \leq \varepsilon\}. \tag{2.7}$$

□

Lemma 2.2:

$$\sum_{\mathbf{j} \in \mathbb{Z}^n} \tau_{\mathbf{j}\varepsilon} \Pi^n \zeta_\varepsilon(\mathbf{x}) = 1. \tag{2.8}$$

where in this sum only finitely many summands are not equal to 0 *.*

Proof:
We first prove the lemma for the case $n = 1$ and $\varepsilon = 1$. Is $x \in \mathbb{R}$ random. Then $\tau_j \zeta(x) = \zeta(x - j) \neq 0$ iff $|x - j| \leq 1$ for $j \in \mathbb{Z}$. Then in case $x = j \in \mathbb{Z}$ we have $\tau_j \zeta(x) = \zeta(x - j) = \zeta(0) = 1 - |0| = 1$ and $\tau_i \zeta(x) \neq 0$ for all $i \neq j \in \mathbb{Z}$. Is $x \notin \mathbb{Z}$ then $j := \lfloor x \rfloor < x$ .Thus:

$$\begin{aligned} &\tau_{j<x} \zeta(x) = \zeta(x - j) = 1 - |x - j| = 1 - x + j, \\ &\tau_{j+1>x} \zeta(x) = \zeta(x - j - 1) = 1 - |x - j - 1| = 1 - j - 1 + x = x - j. \end{aligned} \tag{2.9}$$

and $\tau_i\zeta(x) = \zeta(x-i) = 0$ for all $i \in \mathbb{Z} \setminus \{j, j+1\}$. For example is the distance from x to $j-1$ is the distance from x to j plus the distance of j and $j-1$, thus the total distance from x to $j-1$ is $1 \leq |x-j| + |j-j+1| = |x-j|+1 = |(x-j)+1| = |x-(j-1)|$ and hence $\tau_{j-1}\zeta(x) = \zeta(x-j+1) = 0$. Then we have:

$$\sum_{j\in\mathbb{Z}} \tau_j\zeta(x) = \sum_{j\in\{m,m+1\}} \tau_j\zeta(x) + \sum_{j\in(\mathbb{Z}\setminus\{m,m+1\})} \tau_j\zeta(x) = 1 - x + j + x - j + \sum 0 = 1. \tag{2.10}$$

Now let any $\varepsilon > 0$. Then $\tau_{j\varepsilon}\zeta_\varepsilon(x) = \zeta((x-j\varepsilon)/\varepsilon) = \zeta(x/\varepsilon - j) = \tau_j\zeta(x/\varepsilon)$. Then we get from above conclusion:

$$\sum_{j\in\mathbb{Z}} \tau_{j\varepsilon}\zeta_\varepsilon(x) = \sum_{j\in\mathbb{Z}} \tau_j\zeta\left(\frac{x}{\varepsilon}\right) := \sum_{j\in\mathbb{Z}} \tau_j\zeta(\widetilde{x} \in \mathbb{R}) = 1. \tag{2.11}$$

We now show by induction on n that the claim also holds for all $n \in \mathbb{N}$. Obviously:

$$\tau_{(\mathbf{j},j)\varepsilon}\Pi^{n+1}\zeta_\varepsilon(\mathbf{x}, x) := \Pi^{n+1}\zeta_\varepsilon(\mathbf{x} - \mathbf{j}\varepsilon, x - j\varepsilon) = \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\,\tau_{j\varepsilon}\zeta_\varepsilon(x) \tag{2.12}$$

for all $(\mathbf{x}, x) \in \mathbb{R}^{n+1}$ and $(\mathbf{j}, j) \in \mathbb{Z}^{n+1}$. For a fixed $j \in \mathbb{Z}$ we get the subsum:

$$\sum_{\mathbf{j}\in\mathbb{Z}^n} \tau_{(\mathbf{j},j)\varepsilon}\Pi^{n+1}\zeta_\varepsilon(\mathbf{x}, x) = \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\right)\tau_{j\varepsilon}\zeta_\varepsilon(x) = 1 \cdot \tau_{j\varepsilon}\zeta_\varepsilon(x) = \tau_{j\varepsilon}\zeta_\varepsilon(x). \tag{2.13}$$

Where the last equation follows from the induction hypothesis. Finally:

$$\sum_{(\mathbf{j},j)\in\mathbb{Z}^{n+1}} \tau_{(\mathbf{j},j)\varepsilon}\Pi^{n+1}\zeta_\varepsilon(\mathbf{x}, x) = \sum_{j\in\mathbb{Z}}\left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \tau_{(\mathbf{j},j)\varepsilon}\Pi^{n+1}\zeta_\varepsilon(\mathbf{x}, x)\right) = \sum_{j\in\mathbb{Z}} \tau_{j\varepsilon}\zeta_\varepsilon(x) = 1. \tag{2.14}$$

It is evident from the above proof that in this sum only finitely many summands are not equal to 0 .

□

For every choice of coefficients $a_{\mathbf{j}} \in \mathbb{C}$ and $\mathbf{j} \in \mathbb{Z}^n$ we get well-defined continuous functions:

$$g : \mathbb{R}^n \to \mathbb{C}, \quad \mathbf{x} \mapsto g(\mathbf{x}) := \sum_{\mathbf{j}\in\mathbb{Z}^n} a_{\mathbf{j}}\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x}). \tag{2.15}$$

We use this construction in the following lemma with $a_{\mathbf{j}} := \psi(\mathbf{j}\varepsilon)$.

Lemma 2.3:

Let $\psi \in C_c(\mathbb{R}^n)$. Then for every $r > 0$ there exists an $\varepsilon_0 > 0$ with:

$$\left\|\psi - \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\,\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon\right\|_{\mathbb{R}^n} < r, \quad \forall\, \varepsilon \leq \varepsilon_0. \tag{2.16}$$

Here the set $C_c(\mathbb{R}^n)$ represents the set of all continuous functions with a compact support.

Proof:

Let $\psi \in C_c(\mathbb{R}^n)$. Let $r > 0$ and $\mathbf{x} \in \mathbb{R}^n$ fixed. Then:

$$\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x}) = \prod_{k=1}^n \zeta\left(\frac{x_k - j_k\varepsilon}{\varepsilon}\right) = \prod_{k=1}^n \left(1 - \left|\frac{x_k - j_k\varepsilon}{\varepsilon}\right|\right), \quad \left|\frac{x_k - j_k\varepsilon}{\varepsilon}\right| \leq 1,\ \forall\, j = 1, \ldots, n. \tag{2.17}$$

Thus iff $|(x_k - j_k\varepsilon)/\varepsilon| \leq 1,\ \forall\, j = 1, \ldots, n$ is $\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x}) \neq 0$. This is equivalent to:

$$(|x_k - j_k\varepsilon| \leq \varepsilon,\ \forall\, j = 1, \ldots, n) \Rightarrow (\max\{|x_1 - j_1\varepsilon|, \ldots, |x_n - j_n\varepsilon|\} \leq \varepsilon) \Leftrightarrow (\|\mathbf{x} - \mathbf{j}\varepsilon\|_\infty \leq \varepsilon) \tag{2.18}$$

This condition is only fulfilled for finitely many $\mathbf{j} \in \mathbb{Z}^n$. Because ψ is continuous and has a compact support, it is uniformly continuous. Now let us choose a $\varepsilon_0 > 0$ such that for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ with $\|\mathbf{x} - \mathbf{y}\|_\infty < \varepsilon$ it follows that $|\psi(\mathbf{x})-\psi(\mathbf{y})|<r$, then for all $0<\varepsilon<\varepsilon_0$ and for all $\mathbf{x}\in\mathbb{R}^n$:

$$\begin{aligned}\left|\psi(\mathbf{x})-\sum_{\mathbf{j}\in\mathbb{Z}^n}\psi(\mathbf{j}\varepsilon)\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\right| &= \left|\psi(\mathbf{x})\left(\sum_{\mathbf{j}\in\mathbb{Z}^n}\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\right)-\sum_{\mathbf{j}\in\mathbb{Z}^n}\psi(\mathbf{j}\varepsilon)\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\right| \\ &= \left|\sum_{\mathbf{j}\in\mathbb{Z}^n}\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\left(\psi(\mathbf{x})-\psi(\mathbf{j}\varepsilon)\right)\right| \\ &\le \sum_{\mathbf{j}\in\mathbb{Z}^n}\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})\left|\psi(\mathbf{x})-\psi(\mathbf{j}\varepsilon)\right| \\ &< \underbrace{\sum_{\mathbf{j}\in\mathbb{Z}^n}\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x})}_{=1} r = r.\end{aligned} \tag{2.19}$$

□

Lemma 2.4:

For all $\mathbf{z}\in\mathbb{R}^n,\ \varepsilon>0$ *and* $\mathbf{x},\,\mathbf{y}\in\mathbb{R}^n$ *:*

$$|\tau_{\mathbf{z}}\Pi^n\zeta_\varepsilon(\mathbf{x})-\tau_{\mathbf{z}}\Pi^n\zeta_\varepsilon(\mathbf{y})|\le\frac{n}{\varepsilon}\|\mathbf{x}-\mathbf{y}\|_\infty. \tag{2.20}$$

Proof:

Without loss of generality let $\mathbf{z}=\mathbf{0}$ and $\varepsilon=1$. Because:

$$|\Pi^n\zeta_\varepsilon(\mathbf{x})-\Pi^n\zeta_\varepsilon(\mathbf{y})|=\left|\Pi^n\zeta\left(\frac{\mathbf{x}}{\varepsilon}\right)-\Pi^n\zeta\left(\frac{\mathbf{y}}{\varepsilon}\right)\right| \tag{2.21}$$

we must prove:

$$|\Pi^n\zeta(\mathbf{x})-\Pi^n\zeta(\mathbf{y})|\le n\|\mathbf{x}-\mathbf{y}\|_\infty,\quad\forall\mathbf{x},\,\mathbf{y}\in\mathbb{R}^n. \tag{2.22}$$

We use complete induction to prove.

(i). In case $n=1$ the $\zeta-$function is:

$$\Pi^1\zeta(x)=\zeta(x)=\begin{cases}1-|x|, & |x|\le1,\\ 0, & |x|\ge1,\end{cases} \tag{2.23}$$

(1). For $|x|,\,|y|\le1$:

$$|\zeta(x)-\zeta(y)|=|1-|x|-1+|y||=||y|-|x||\le|x-y|. \tag{2.24}$$

(2). For $|x|,\,|y|\le1$:

$$|\zeta(x)-\zeta(y)|=|0-0|=0\le|x-y|. \tag{2.25}$$

(3). For $(|x|\le1)\wedge(|y|\ge1)$:

$$|\zeta(x)-\zeta(y)|=|1-|x|-0|=1-|x|\le|y|-|x|\le|y-x|. \tag{2.26}$$

(4). For $(|y|\le1)\wedge(|x|\ge1)$:

$$|\zeta(y)-\zeta(x)|=|1-|y|-0|=1-|y|\le|x|-|y|\le|x-y|. \tag{2.27}$$

(ii). For $(n+1)\in\mathbb{N}$ let $(\mathbf{x},x)$ and $(\mathbf{y},y)$ in $\mathbb{R}^{n+1}$ .Then first:

$$\Pi^{n+1}\zeta(\mathbf{x},x)=\Pi^n\zeta(\mathbf{x})\zeta(x),\quad\Pi^{n+1}\zeta(\mathbf{y},y)=\Pi^n\zeta(\mathbf{y})\zeta(y). \tag{2.28}$$

Thus:

$$\begin{aligned} & \left|\Pi^{n+1}\zeta(\mathbf{x},x)-\Pi^{n+1}\zeta(\mathbf{y},y)\right|=|\Pi^n\zeta(\mathbf{x})\zeta(x)-\Pi^n\zeta(\mathbf{y})\zeta(y)| \\ \le\ & |\Pi^n\zeta(\mathbf{x})\zeta(x)-\Pi^n\zeta(\mathbf{y})\zeta(x)|+|\Pi^n\zeta(\mathbf{y})\zeta(x)-\Pi^n\zeta(\mathbf{y})\zeta(y)| \\ =\ & \zeta(x)|\Pi^n\zeta(\mathbf{x})-\Pi^n\zeta(\mathbf{y})|+\Pi^n\zeta(\mathbf{y})|\zeta(x)-\zeta(y)|,\quad(0\le\zeta(x),\Pi^n\zeta(\mathbf{y})\le1) \\ \le\ & |\Pi^n\zeta(\mathbf{x})-\Pi^n\zeta(\mathbf{y})|+|\zeta(x)-\zeta(y)| \\ \le\ & n\|\mathbf{x}-\mathbf{y}\|_\infty+|x-y|\le(n+1)\|(\mathbf{x},x)-(\mathbf{y},y)\|_\infty.\end{aligned} \tag{2.29}$$

Thus we close the complete induction.

□

Lemma 2.5:

Let $K \subseteq \mathbb{R}^m$ compact. Let $\mathbf{f} : K \to \mathbb{R}^n$ continuous. Then there is a monotonically increasing function $\eta : [0, +\infty) := \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{t\to 0} \eta(t) = 0$:

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq \eta\left(\|\mathbf{x} - \mathbf{y}\|'\right), \quad \forall \mathbf{x}, \mathbf{y} \in K. \tag{2.30}$$

Here $\|\cdot\|$ and $\|\cdot\|'$ are any norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Proof:

For $t \geq 0$ we define a function η by:

$$\eta(t) := \sup\left\{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| : \mathbf{x}, \mathbf{y} \in K,\ \|\mathbf{x} - \mathbf{y}\|' \leq t\right\}. \tag{2.31}$$

(i). Obviously the function η is defined on $\mathbb{R}_0^+$ and the codomain is also $\mathbb{R}_0^+$ bescause of the positive semi-definiteness of norms $\|\cdot\|$ and $\|\cdot\|'$.

(ii). Because $K \subseteq \mathbb{R}^n$ is compact and $\mathbf{f} : K \to \mathbb{R}^n$ continuous, thus $\mathbf{f} : K \to \mathbb{R}^n$ is uniformly continuous [7], then:

$$\forall \varepsilon > 0 \,\forall \mathbf{x}, \mathbf{y} \in K \,\exists \delta > 0\ \|\mathbf{x} - \mathbf{y}\|' < \varepsilon : \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \delta. \tag{2.32}$$

This implies $\eta(0) = 0$.

(iii). For $t := \|\mathbf{x} - \mathbf{y}\|'$:

$$\begin{aligned} &\eta\left(\|\mathbf{x} - \mathbf{y}\|'\right) := \sup\left\{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| : \mathbf{x}, \mathbf{y} \in K,\ \|\mathbf{x} - \mathbf{y}\|' \leq \|\mathbf{x} - \mathbf{y}\|'\right\} \\ = \quad &\sup\left\{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| : \mathbf{x}, \mathbf{y} \in K\right\} \geq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|. \end{aligned} \tag{2.33}$$

(iv). Because $\mathbf{f} : K \to \mathbb{R}^n$ is uniformly continuous, exist $2\delta > 0$ for all $2\varepsilon > 0$ and $\mathbf{x}, \mathbf{y} \in K$ such that:

$$\|\mathbf{x} - \mathbf{y}\|' < 2\varepsilon : \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < 2\delta. \tag{2.34}$$

This implies that the function $\eta : [0, +\infty) := \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is monotonically increasing.

(v). Because $\mathbf{f} : K \to \mathbb{R}^n$ is uniformly continuous, therefore for all $\varepsilon > 0$ exist a $\delta > 0$, such that for all $\mathbf{x}, \mathbf{y} \in K$:

$$\eta(\varepsilon) := \sup\left\{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| : \mathbf{x}, \mathbf{y} \in K,\ \|\mathbf{x} - \mathbf{y}\|' \leq \varepsilon\right\} < \delta. \tag{2.35}$$

Thus $0 \leq \eta(t) \leq \eta(\varepsilon) < \delta$ for all $0 \leq t \leq \varepsilon$ is established. Thus $\lim_{t\to 0} \eta(t) = 0$.

In summary, the function $\eta : [0, +\infty) := \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is well-defined.

□

Lemma 2.6:

Let $U, V \subseteq \mathbb{R}^n$ open and let $\boldsymbol{\phi} : U \to V$ a C^1-diffeomorphism. Then:

$$\int_V \psi(\mathbf{y})\, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y} = \int_U \psi(\boldsymbol{\phi}(\mathbf{x}))\, |\det D\boldsymbol{\phi}(x)|\, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x}, \quad \forall \psi \in C_c(V). \tag{2.36}$$

with $C_c(V) := \{\psi \in C_c(\mathbb{R}^n) : \mathrm{supp}(\psi) \subseteq V\}$.

Proof:

First we define a new function $\widehat{\psi}(\mathbf{x}) := \psi(\boldsymbol{\phi}(\mathbf{x}))\, |\det \boldsymbol{\phi}(\mathbf{x})|$. Then $\widehat{\psi} \in C_c(\mathbb{R}^n)$ with $\mathrm{supp}\left(\widehat{\psi}\right) \subseteq U$. Because $\mathrm{supp}(\psi) \subseteq V$ and $\mathrm{supp}\left(\widehat{\psi}\right) \subseteq U$ are compact, exist a $\varepsilon_1 > 0$,such that for all $\mathbf{y} \in \mathrm{supp}(\psi)$ and for all $\mathbf{x} \in \mathrm{supp}\left(\widehat{\psi}\right)$:

$$\overline{B_{\|\cdot\|_\infty}}(\mathbf{y}, \varepsilon_1) \subseteq V, \quad \overline{B_{\|\cdot\|_\infty}}(\mathbf{x}, \varepsilon_1) \subseteq U. \tag{2.37}$$

Because $\overline{B_{\|\cdot\|_\infty}}(\mathbf{y}, \varepsilon_1)$ and $\overline{B_{\|\cdot\|_\infty}}(\mathbf{x}, \varepsilon_1)$ are two closed and bounded sets in $\mathbb{R}^n$,thus they are also compact. Now we consider a continuous map $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{F}(\mathbf{x}, \mathbf{y}) := \mathbf{x} + \mathbf{y}$. This map maps compact sets $\mathrm{supp}(\psi) \times \overline{B_{\|\cdot\|_\infty}}(\mathbf{0}, \varepsilon_1)$ and $\mathrm{supp}\left(\widehat{\psi}\right) \times \overline{B_{\|\cdot\|_\infty}}(\mathbf{0}, \varepsilon_1)$ to compact sets $S_\psi := \mathrm{supp}(\psi) + \overline{B_{\|\cdot\|_\infty}}(\mathbf{0}, \varepsilon_1)$ and $S_{\widehat{\psi}} := \mathrm{supp}\left(\widehat{\psi}\right) + \overline{B_{\|\cdot\|_\infty}}(\mathbf{0}, \varepsilon_1)$ respectively [1]. For example means $\mathrm{supp}(\psi) + \overline{B_{\|\cdot\|_\infty}}(\mathbf{0}, \varepsilon_1)$ that each point $\mathbf{x}$

in $\text{supp}(\psi)$ is assigned a closed ball with radius ε_1 ,whose center is $\mathbf{x}$.
For $A \in M(n \times n, K)$ wir define the matrix norm $\|A\| := \sup\{\|A\mathbf{x}\|_\infty : \|\mathbf{x}\|_\infty \leq 1\}$. Because $S_\psi,\ S_{\widehat{\psi}}$ are compact and $\mathbf{x} \mapsto D\boldsymbol{\phi}(\mathbf{x}),\ \mathbf{y} \mapsto D\boldsymbol{\phi}^{-1}(\mathbf{y})$ are continuous then there is a $C > 0$ with:

$$\|D\boldsymbol{\phi}(\mathbf{x})\|,\ \left\|D\boldsymbol{\phi}^{-1}(\mathbf{y})\right\| \leq C, \quad \forall\, \mathbf{x} \in S_{\widehat{\psi}},\ \mathbf{y} \in S_\psi. \tag{2.38}$$

With the mean value theorem we have:

$$\begin{aligned} \|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\phi}(\mathbf{a})\|_\infty &= \left\| \int_{t=0}^{t=1} D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})\ \mathrm{d}t \right\|_\infty \\ &\leq \int_{t=0}^{t=1} \|D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})\|_\infty\ \mathrm{d}t \\ &= \int_{t=0}^{t=1} \underbrace{\|D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))\|_\infty}_{\leq C} \|(\mathbf{x}-\mathbf{a})\|_\infty\ \mathrm{d}t \\ &\leq \int_{t=0}^{t=1} C\,\|\mathbf{x}-\mathbf{a}\|_\infty\ \mathrm{d}t = C\,\|\mathbf{x}-\mathbf{a}\|_\infty \end{aligned} \tag{2.39}$$

for all $\mathbf{a} \in \text{supp}\left(\widehat{\psi}\right)$ and $\mathbf{x} \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1)$. similarly, we have:

$$\begin{aligned} \left\|\boldsymbol{\phi}^{-1}(\mathbf{y}) - \boldsymbol{\phi}^{-1}(\mathbf{b})\right\|_\infty &= \left\| \int_{t=0}^{t=1} D\boldsymbol{\phi}^{-1}(\mathbf{b} + t(\mathbf{y}-\mathbf{b}))(\mathbf{y}-\mathbf{b})\ \mathrm{d}t \right\|_\infty \\ &\leq \int_{t=0}^{t=1} \left\|D\boldsymbol{\phi}^{-1}(\mathbf{b} + t(\mathbf{y}-\mathbf{b}))(\mathbf{y}-\mathbf{b})\right\|_\infty\ \mathrm{d}t \\ &= \int_{t=0}^{t=1} \underbrace{\left\|D\boldsymbol{\phi}^{-1}(\mathbf{b} + t(\mathbf{y}-\mathbf{b}))\right\|_\infty}_{\leq C} \|(\mathbf{y}-\mathbf{b})\|_\infty\ \mathrm{d}t \\ &\leq \int_{t=0}^{t=1} C\,\|\mathbf{y}-\mathbf{b}\|_\infty\ \mathrm{d}t = C\,\|\mathbf{y}-\mathbf{b}\|_\infty \end{aligned} \tag{2.40}$$

for all $\mathbf{b} \in \text{supp}(\psi)$ and $\mathbf{y} \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon_1)$. Then follows from this:

$$\boldsymbol{\phi}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon)\right) \subseteq \overline{B_{\|\cdot\|_\infty}}(\boldsymbol{\phi}(\mathbf{a}), C\varepsilon), \quad \boldsymbol{\phi}^{-1}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon)\right) \subseteq \overline{B_{\|\cdot\|_\infty}}\left(\boldsymbol{\phi}^{-1}(\mathbf{b}), C\varepsilon\right) \tag{2.41}$$

for all $0 < \varepsilon \leq \varepsilon_1$. In fact:

$$\begin{aligned} \boldsymbol{\phi}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon \leq \varepsilon_1)\right) &= \boldsymbol{\phi}\left(\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}-\mathbf{a}\|_\infty < \varepsilon \leq \varepsilon_1\}\right) \\ &:= \{\boldsymbol{\phi}(\mathbf{x}) \in \mathbb{R}^n : \|\mathbf{x}-\mathbf{a}\|_\infty < \varepsilon \leq \varepsilon_1\}. \end{aligned} \tag{2.42}$$

Thus if $\boldsymbol{\phi}(\mathbf{x}) \in \boldsymbol{\phi}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon \leq \varepsilon_1)\right)$ then can be inferred from equation (2.39) that:

$$\|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\phi}(\mathbf{a})\|_\infty \leq C\,\|\mathbf{x}-\mathbf{a}\|_\infty \overset{\text{eq.}(2.42)}{\leq} C\varepsilon \leq C\varepsilon_1 \tag{2.43}$$

which implies $\boldsymbol{\phi}(\mathbf{x}) \in \overline{B_{\|\cdot\|_\infty}}(\boldsymbol{\phi}(\mathbf{a}), C\varepsilon)$. Thus $\boldsymbol{\phi}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon)\right) \subseteq \overline{B_{\|\cdot\|_\infty}}(\boldsymbol{\phi}(\mathbf{a}), C\varepsilon)$. Similar reason for $\boldsymbol{\phi}^{-1}(\mathbf{y}) \in \boldsymbol{\phi}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon \leq \varepsilon_1)\right)$.
Let now $\varepsilon_2 := \varepsilon_1 / C > 0$ .For every $0 < \varepsilon \leq \varepsilon_2$ define the map ψ_ε by:

$$\psi_\varepsilon := \sum_{\mathbf{j} \in \mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\, \tau_{\mathbf{j}\varepsilon} \Pi^n \zeta_\varepsilon. \tag{2.44}$$

Then from lemma (2.3) it can be deduced that:

$$\|\psi - \psi_\varepsilon\|_{\mathbb{R}^n} = \left\| \psi - \sum_{\mathbf{j} \in \mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\, \tau_{\mathbf{j}\varepsilon} \Pi^n \zeta_\varepsilon \right\|_{\mathbb{R}^n} \to 0, \quad (\varepsilon \to 0). \tag{2.45}$$

Then:

$$\text{supp}(\psi_\varepsilon) := \bigcup \left\{ \overline{B_{\|\cdot\|_\infty}}(\mathbf{j}\varepsilon, \varepsilon) : \mathbf{j} \in \mathbb{Z}^n,\ \mathbf{j}\varepsilon \in \text{supp}(\psi) \right\} \subseteq S_\psi. \tag{2.46}$$

In fact:

$$\psi_\varepsilon(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\, \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \prod_{k=1}^{n} \zeta\left(\frac{x_k - j_k\varepsilon}{\varepsilon}\right). \tag{2.47}$$

Thus:

$$\begin{aligned} (\psi_\varepsilon(\mathbf{x}) \neq 0) &\Leftrightarrow \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \prod_{k=1}^{n} \zeta\left(\frac{x_k - j_k\varepsilon}{\varepsilon}\right) \neq 0\right) \\ \Leftrightarrow\quad & \underbrace{(\psi(\mathbf{j}\varepsilon)\neq 0)}_{\Leftrightarrow(\mathbf{j}\varepsilon\in\mathrm{supp}(\psi))} \wedge \underbrace{\left(\prod_{k=1}^{n} \zeta\left(\frac{x_k - j_k\varepsilon}{\varepsilon}\right) \neq 0\right)}_{\Leftrightarrow(|(x_k - j_k\varepsilon)/\varepsilon|\leq 1,\ \forall k=1,\ldots,n)} \\ \Leftrightarrow\quad & (\mathbf{j}\in\mathrm{supp}(\psi)) \wedge (\max\{|x_1 - j_1\varepsilon|,\ldots,|x_n - j_n\varepsilon|\} = \|\mathbf{x}-\mathbf{j}\varepsilon\|_\infty \leq \varepsilon) \\ \Leftrightarrow\quad & (\mathbf{j}\varepsilon\in\mathrm{supp}(\psi)) \wedge \left(\mathbf{x}\in\overline{B_{\|\cdot\|_\infty}}(\mathbf{j}\varepsilon, \varepsilon\leq\varepsilon_1)\right). \end{aligned} \tag{2.48}$$

This implies that equation (2.46) is the support set of ψ_ε . Thus ψ_ε converges to ψ even in the inductive limit-topology, and it follows:

$$\lim_{\varepsilon\to 0}\int_V \psi_\varepsilon(\mathbf{y})\, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y} = \int_V \psi(\mathbf{y})\, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y}. \tag{2.49}$$

We define now $\widehat{\psi}_\varepsilon(\mathbf{x}) := \psi_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})|$ and $\widetilde{C} := \max_{\mathbf{x}\in S_{\widehat{\psi}}} |\det D\boldsymbol{\phi}(\mathbf{x})|$. Thus with equation (2.44) the function $\widehat{\psi}_\varepsilon$ can be written as:

$$\begin{aligned} \widehat{\psi}_\varepsilon(\mathbf{x}) &= \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\, \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})| \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\, \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})| \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \prod_{k=1}^{n} \zeta\left(\frac{\phi_k(\mathbf{x}) - j_k\varepsilon}{\varepsilon}\right) |\det D\boldsymbol{\phi}(\mathbf{x})|. \end{aligned} \tag{2.50}$$

Now we want to find the support of $\widehat{\psi}_\varepsilon$. Let $\widehat{\psi}_\varepsilon(\mathbf{x}) \neq 0$ then we have:

$$\sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \prod_{k=1}^{n} \zeta\left(\frac{\phi_k(\mathbf{x}) - j_k\varepsilon}{\varepsilon}\right) |\det D\boldsymbol{\phi}(\mathbf{x})| \neq 0, \tag{2.51}$$

which means $\mathbf{j}\varepsilon\in\mathrm{supp}(\psi)$ and $\boldsymbol{\phi}(\mathbf{x})\in\overline{B_{\|\cdot\|_\infty}}(\mathbf{j}\varepsilon,\varepsilon)$ .Thus with equation (2.41) we get:

$$\mathbf{x}\in\boldsymbol{\phi}^{-1}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{j}\varepsilon,\varepsilon)\right) \subseteq \overline{B_{\|\cdot\|_\infty}}\left(\boldsymbol{\phi}^{-1}(\mathbf{j}\varepsilon), C\varepsilon\leq\varepsilon_1\right). \tag{2.52}$$

Totally we get the support of the function $\widehat{\psi}_\varepsilon$:

$$\begin{aligned} \mathrm{supp}\left(\widehat{\psi}_\varepsilon\right) &= \bigcup\left\{\boldsymbol{\phi}^{-1}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{j}\varepsilon,\varepsilon)\right) : \mathbf{j}\in\mathbb{Z}^n,\ \mathbf{j}\varepsilon\in\mathrm{supp}(\psi)\right\} \\ \subseteq\quad & \bigcup\left\{\overline{B_{\|\cdot\|_\infty}}\left(\boldsymbol{\phi}^{-1}(\mathbf{j}\varepsilon), C\varepsilon\leq\varepsilon_1\right) : \mathbf{j}\in\mathbb{Z}^n,\ \mathbf{j}\varepsilon\in\mathrm{supp}(\psi)\right\} \subseteq S_{\widehat{\psi}}. \end{aligned} \tag{2.53}$$

And:

$$\left\|\widehat{\psi}_\varepsilon - \widehat{\psi}\right\|_{\mathbb{R}^n} \to 0, \quad (\varepsilon\to 0). \tag{2.54}$$

In fact for all $\mathbf{x} \in S_{\widehat{\psi}}$ we can estimate the value of $\left\|\widehat{\psi}_\varepsilon - \widehat{\psi}\right\|_{\mathbb{R}^n}$ with lemma (2.3):

$$\begin{aligned}
\left|\widehat{\psi}_\varepsilon(\mathbf{x}) - \widehat{\psi}(\mathbf{x})\right| &= \left|\sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\,\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})| - \widehat{\psi}(\mathbf{x})\right| \\
&= \left|\sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon)\,\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})| - \left(\sum_{\mathbf{j}\in\mathbb{Z}^n} \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\right)\widehat{\psi}(\mathbf{x})\right| \\
&= \left|\sum_{\mathbf{j}\in\mathbb{Z}^n} \left(\psi(\mathbf{j}\varepsilon)\,|\det D\boldsymbol{\phi}(\mathbf{x})| - \widehat{\psi}(\mathbf{x})\right)\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\right| \\
&\leq \sum_{\mathbf{j}\in\mathbb{Z}^n} \left|\psi(\mathbf{j}\varepsilon)\,|\det D\boldsymbol{\phi}(\mathbf{x})| - \widehat{\psi}(\mathbf{x})\right|\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^n} \left|\psi(\mathbf{j}\varepsilon)\,|\det D\boldsymbol{\phi}(\mathbf{x})| - \psi(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})|\right|\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^n} |\det D\boldsymbol{\phi}(\mathbf{x})|\,|\psi(\mathbf{j}\varepsilon) - \psi(\boldsymbol{\phi}(\mathbf{x}))|\,\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \\
&\leq \max_{\mathbf{x}\in S_{\widehat{\psi}}} |\det D\boldsymbol{\phi}(\mathbf{x})| \sum_{\mathbf{j}\in\mathbb{Z}^n} |\psi(\mathbf{j}\varepsilon) - \psi(\boldsymbol{\phi}(\mathbf{x}))|\,\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \\
&= \widetilde{C}\sum_{\mathbf{j}\in\mathbb{Z}^n} |\psi(\mathbf{j}\varepsilon) - \psi(\boldsymbol{\phi}(\mathbf{x}))|\,\tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \to 0, \quad (\varepsilon \to 0).
\end{aligned} \tag{2.55}$$

Similar to equation (2.49) we get:

$$\lim_{\varepsilon\to 0}\int_U \psi_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})|\ \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} = \int_U \psi(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})|\ \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x}. \tag{2.56}$$

Now it only remains to show, that:

$$\lim_{\varepsilon\to 0}\left|\int_U \psi_\varepsilon(\boldsymbol{\phi}(\mathbf{x}))\,|\det D\boldsymbol{\phi}(\mathbf{x})|\ \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} - \int_V \psi_\varepsilon(\mathbf{y})\ \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y}\right| = 0 \tag{2.57}$$

holds. As already indicated, we approximate the C^1-diffeomorphism $\boldsymbol{\phi}$ locally around $\mathbf{a} \in U$ by the affine map:

$$\boldsymbol{\lambda}_\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \boldsymbol{\lambda}_\mathbf{a}(\mathbf{x}) := \boldsymbol{\phi}(\mathbf{a}) + D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a}). \tag{2.58}$$

Claim 1:

There is a monotonically increasing function $\eta : [0, \varepsilon_1] \to \mathbb{R}_0^+$ with $\lim_{t\to 0}\eta(t) = 0$ such that for all $\mathbf{a} \in \operatorname{supp}\left(\widehat{\psi}\right) \subseteq U$ and for all $\mathbf{x} \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1)$:

$$\|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\lambda}_\mathbf{a}(\mathbf{x})\|_\infty \leq \eta\left(\|\mathbf{x} - \mathbf{a}\|_\infty\right)\|\mathbf{x} - \mathbf{a}\|_\infty \tag{2.59}$$

Proof:

With the mean value theorem we have for all $\mathbf{x} \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1) \subseteq U$:

$$\|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\lambda}_\mathbf{a}(\mathbf{x})\|_\infty \leq \left(\int_{t=0}^{t=1}\|D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) - D\boldsymbol{\phi}(\mathbf{a})\|\ \mathrm{d}t\right)\|\mathbf{x} - \mathbf{a}\|_\infty. \tag{2.60}$$

In fact:

$$\begin{aligned}
\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\lambda}_\mathbf{a}(\mathbf{x}) &= \boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\phi}(\mathbf{a}) - D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a}) \\
&= \int_{t=0}^{t=1} D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a})\ \mathrm{d}t - D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a}) \\
&= \int_{t=0}^{t=1}\left(D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a}) - D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a})\right)\mathrm{d}t.
\end{aligned} \tag{2.61}$$

Because $D\boldsymbol{\phi} : S_{\widehat{\psi}} \to M(n \times n, \mathbb{R})$ is continuous, we use lemma (2.5) on $D\boldsymbol{\phi}$ in terms of the matrix norm $\|\cdot\|$ and the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$, it follows for the there given function $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$:

$$\begin{aligned}
&\int_{t=0}^{t=1}\|D\boldsymbol{\phi}(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) - D\boldsymbol{\phi}(\mathbf{a})\|\ \mathrm{d}t \leq \int_{t=0}^{t=1}\eta\left(\|t(\mathbf{x} - \mathbf{a})\|_\infty\right)\mathrm{d}t \\
= &\int_{t=0}^{t=1}\eta\left(|t|\,\|\mathbf{x} - \mathbf{a}\|_\infty\right)\mathrm{d}t \leq \int_{t=0}^{t=1}\eta\left(\|\mathbf{x} - \mathbf{a}\|_\infty\right)\mathrm{d}t = \eta\left(\|\mathbf{x} - \mathbf{a}\|_\infty\right).
\end{aligned} \tag{2.62}$$

□ (Claim 1)

Claim 2:

Let ε_1, $C \geq 0$ and $\varepsilon_2 := \varepsilon_1/C$ as above. Then there is a function $\xi : [0, \varepsilon_2] \to \mathbb{R}_0^+$ with $\xi(0) = 0$ and ξ continuous in 0 , such that:

$$\left| \int_U \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| \mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x} - \int_V \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\mathbf{y}) \, \mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{y} \right| \leq \xi(\varepsilon) \varepsilon^n \tag{2.63}$$

for all $\mathbf{b} \in \operatorname{supp}(\psi)$ and $0 < \varepsilon \leq \varepsilon_2$.

Proof:

Let $\mathbf{a} := \boldsymbol{\phi}^{-1}(\mathbf{b}) \in \operatorname{supp}\left(\widehat{\psi}\right)$ and $\eta : [0, \varepsilon_1] \to \mathbb{R}_0^+$ as in claim 1. Then it follows with claim 1 and lemma (2.4):

$$\begin{aligned} \left| \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) - \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \left(\boldsymbol{\lambda}_{\boldsymbol{\phi}^{-1}(\mathbf{b})}(\mathbf{x}) \right) \right| &\leq \frac{n}{\varepsilon} \left\| \boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\lambda}_{\boldsymbol{\phi}^{-1}(\mathbf{b})}(\mathbf{x}) \right\|_\infty \\ &\leq \frac{n}{\varepsilon} \eta \left(\left\| \mathbf{x} - \boldsymbol{\phi}^{-1}(\mathbf{b}) \right\|_\infty \right) \left\| \mathbf{x} - \boldsymbol{\phi}^{-1}(\mathbf{b}) \right\|_\infty \end{aligned} \tag{2.64}$$

for all $\mathbf{x} \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1)$. With equation (2.41) if $\boldsymbol{\phi}(\mathbf{x}) \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon)$ then it follows:

$$\mathbf{x} \in \boldsymbol{\phi}^{-1}\left(\overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon) \right) \subseteq \overline{B_{\|\cdot\|_\infty}}\left(\boldsymbol{\phi}^{-1}(\mathbf{b}), C\varepsilon \right) = \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, C\varepsilon). \tag{2.65}$$

We get an analogous statement for the map $\mathbf{x} \mapsto \boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x}) = D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \boldsymbol{\phi}(\mathbf{a})$. From lemma 1 we know that $\operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon) = \overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon)$, thus it follows for all $\varepsilon \leq \varepsilon_2 = \varepsilon_1/C$, that:

$$\begin{aligned} \operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\phi}) \subseteq \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, C\varepsilon) \subseteq \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1) \subseteq S_{\widehat{\psi}}, \\ \operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\lambda}_{\mathbf{a}}) \subseteq \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, C\varepsilon) \subseteq \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1) \subseteq S_{\widehat{\psi}}. \end{aligned} \tag{2.66}$$

In fact:

$$(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\phi})(\mathbf{x}) = \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) = \prod_{k=1}^n \zeta \left(\frac{\phi_k(\mathbf{x}) - b_k}{\varepsilon} \right), \tag{2.67}$$

so $\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\phi} \neq 0$ iff $|(\phi_k(\mathbf{x}) - b_k)/\varepsilon| \leq 1$ for all $k = 1, \ldots, n$,which equivalent to $\boldsymbol{\phi}(\mathbf{x}) \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{b}, \varepsilon)$. Thus with equation (2.65) we know that $\mathbf{x} \in \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, C\varepsilon \leq \varepsilon_1) \subseteq \overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, \varepsilon_1)$, which implies the first inclusion relation in equation (2.66). Similar reason to another inclusion relation in equation (2.66). Together with equation (2.64) it now follows:

$$\left| \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) - \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \left(\boldsymbol{\lambda}_{\boldsymbol{\phi}^{-1}(\mathbf{b})}(\mathbf{x}) \right) \right| \leq \frac{n}{\varepsilon} \eta(C\varepsilon) C\varepsilon, \quad \forall \mathbf{x} \in U. \tag{2.68}$$

Obviously for all $\mathbf{x} \in U \backslash \operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\phi})$ and $\mathbf{x} \in U \backslash \operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\lambda}_{\mathbf{a}})$ holds $\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) = \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) = 0 \leq nC\eta(C\varepsilon)$.

Because the map $|\det D\boldsymbol{\phi}| : S_{\widehat{\psi}} \to \mathbb{R}_0^+$ is uniformly continuous, thus lemma (2.5) can be used, thus we can find a monotonically increasing function $\eta' : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{t \to 0} \eta'(t) = 0 = \eta'(0)$ such that:

$$\left| \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| - \left| \det D\boldsymbol{\phi}(\mathbf{a}) \right| \right| \leq \eta'\left(\|\mathbf{x} - \mathbf{a}\|_\infty \right), \quad \forall \mathbf{a}, \mathbf{x} \in S_{\widehat{\psi}}. \tag{2.69}$$

With $\widetilde{C} := \sup_{\mathbf{x} \in S_{\widehat{\psi}}} |\det D\boldsymbol{\phi}(\mathbf{x})|$ we have for all $0 < \varepsilon \leq \varepsilon_2$ and for all $\mathbf{x} \in U$:

$$\begin{aligned} & \left| \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| - \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{a}) \right| \right| \\ \leq \; & \left| \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| - \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| \right| \\ & + \left| \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| - \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left| \det D\boldsymbol{\phi}(\mathbf{a}) \right| \right| \\ \leq \; & \widetilde{C} \left| \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\phi}(\mathbf{x})) - \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \right| + \tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left| \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| - \left| \det D\boldsymbol{\phi}(\mathbf{a}) \right| \right| \\ \leq \; & \widetilde{C} \frac{n}{\varepsilon} \eta(C\varepsilon) C\varepsilon + \eta'(C\varepsilon) := \frac{1}{(2C)^n} \xi(\varepsilon). \end{aligned} \tag{2.70}$$

In fact, when $\mathbf{x} \in U \setminus \operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\lambda}_{\mathbf{a}})$ we get $\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) = 0$. For all $\mathbf{x} \in \operatorname{supp}(\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon \circ \boldsymbol{\lambda}_{\mathbf{a}})$ is $\tau_{\mathbf{b}} \Pi^n \zeta_\varepsilon (\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) = 1$ and because the function η' monotonically increasing, thus:

$$\left| \left| \det D\boldsymbol{\phi}(\mathbf{x}) \right| - \left| \det D\boldsymbol{\phi}(\mathbf{a}) \right| \right| \leq \eta'\left(\|\mathbf{x} - \mathbf{a}\|_\infty \right) \leq \eta'(C\varepsilon). \tag{2.71}$$

for all $\mathbf{x} \in \operatorname{supp}(\tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon \circ \boldsymbol{\lambda}_{\mathbf{a}})$. This is why the last inequality of equation (2.70) holds. The function $\xi : [0, \varepsilon_2] \to \mathbb{R}_0^+$ defined in this way naturally also satisfies the condition $\lim_{\varepsilon\to 0} \xi(\varepsilon) = 0$. Together with equation (2.66) we have:

$$\begin{aligned} & \left| \int_U \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{x})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} - \int_U \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon(\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{a})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} \right| \\ \leq \; & \int_U \left| \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{x})\right| - \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon(\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{a})\right| \right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} \\ \leq \; & \int_{\mathbf{x}\in\overline{B_{\|\cdot\|_\infty}}(\mathbf{a},C\varepsilon)} \frac{1}{(2C)^n}\xi(\varepsilon) \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} = \xi(\varepsilon)\varepsilon^n. \end{aligned} \tag{2.72}$$

Here the volume of the closed ball $\overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, C\varepsilon)$ in $\mathbb{R}^n$ is $(2\varepsilon C)^n$. Because the closed "ball" $\overline{B_{\|\cdot\|_\infty}}(\mathbf{a}, C\varepsilon)$ is an $n-$dimensional cube with edge length $2\varepsilon C$. On the other hand, the formula holds for the affine map $\mathbf{x} \mapsto \boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{a}) + D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a})$ with lemma (1.10). We define:

$$\mathbf{T}(\mathbf{x}) := \boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{a}) + D\boldsymbol{\phi}(\mathbf{a})(\mathbf{x} - \mathbf{a}) := \mathbf{A}\mathbf{x} + \mathbf{c}. \tag{2.73}$$

Then $\mathbf{A} = D\boldsymbol{\phi}(\mathbf{a})$ is invertible and $f := \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon \in C_c(\mathbb{R}^n)$ thus with equation (1.32) we get finally:

$$\int_U \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon(\boldsymbol{\lambda}_{\mathbf{a}}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{a})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} = \int_V \tau_{\mathbf{b}}\Pi^n\zeta_\varepsilon(\mathbf{y}) \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y}. \tag{2.74}$$

☐ (Claim 2)

Now we define:

$$\psi_\varepsilon := \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \, \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon \tag{2.75}$$

as above. We use claim 2 for all $\mathbf{j} \in \mathbb{Z}^n$ and $\mathbf{j}\varepsilon \in \operatorname{supp}(\psi)$ thus:

$$\begin{aligned} & \left| \int_V \psi_\varepsilon(\mathbf{y}) \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y} - \int_U \psi_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{x})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} \right| \\ = \; & \left| \int_V \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \, \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{y}) \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y} - \int_U \sum_{\mathbf{j}\in\mathbb{Z}^n} \psi(\mathbf{j}\varepsilon) \, \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{x})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} \right| \\ \leq \; & \sum_{\mathbf{j}\in\mathbb{Z}^n} \left|\psi(\mathbf{j}\varepsilon)\right| \left| \int_V \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\mathbf{y}) \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y} - \int_U \tau_{\mathbf{j}\varepsilon}\Pi^n\zeta_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{x})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} \right| \\ \overset{\text{eq.(2.72)}}{\leq} \; & \sum_{\mathbf{j}\in\mathbb{Z}^n} \left|\psi(\mathbf{j}\varepsilon)\right| \xi(\varepsilon)\varepsilon^n. \end{aligned} \tag{2.76}$$

If $N \in \mathbb{N}$ with $\operatorname{supp}(\psi) \subseteq [-N, N]^n$, then (at least for $0 < \varepsilon \leq 1$) there exist at most:

$$\left(\left\lfloor \frac{N}{\varepsilon} \right\rfloor - \left\lceil -\frac{N}{\varepsilon} \right\rceil + 1 \right)^n \leq \left(\frac{2N}{\varepsilon} + 1 \right)^n \overset{0<\varepsilon\leq 1}{\leq} \left(\frac{2N+1}{\varepsilon} \right)^n \tag{2.77}$$

elements $\mathbf{j} \in \mathbb{Z}^n$ with $\mathbf{j}\varepsilon \in \operatorname{supp}(\psi) \subseteq [-N, N]^n$. So that it follows:

$$\begin{aligned} & \left| \int_V \psi_\varepsilon(\mathbf{y}) \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y} - \int_U \psi_\varepsilon(\boldsymbol{\phi}(\mathbf{x})) \left|\det D\boldsymbol{\phi}(\mathbf{x})\right| \, \mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x} \right| \leq \sum_{\mathbf{j}\in\mathbb{Z}^n} \left|\psi(\mathbf{j}\varepsilon)\right| \xi(\varepsilon)\varepsilon^n \\ \leq \; & \left(\frac{2N+1}{\varepsilon} \right)^n \|\psi\|_{\mathbb{R}^n} \xi(\varepsilon)\varepsilon^n = (2N+1)^n \|\psi\|_{\mathbb{R}^n} \xi(\varepsilon) \to 0, \quad (\varepsilon \to 0). \end{aligned} \tag{2.78}$$

☐ (Lemma 2.6)

3 Part 2 of the proof

If X is a locally compact space and $\emptyset \neq U \subseteq X$ is open, then U is also locally compact and the following holds: If $\psi : U \to \mathbb{C}$ is continuous with compact support $\operatorname{supp}(\psi)$ in U , then the trivial continuation:

$$\psi_U := U \subseteq X \to \mathbb{C}, \quad x \mapsto \psi_U(x) := \begin{cases} \psi(x), & x \in U, \\ 0, & x \in (X \setminus U) \end{cases} \tag{3.1}$$

is also continuous on X with $\operatorname{supp}(\psi_U) = \operatorname{supp}(\psi)$ compact in X .We know that the compactness of a set $K \subseteq X$ depends only on the topology induced on K , but not on the larger space X . Actually, we can let $(X, \mathcal{T}_X)$ be a topological space and $K \subseteq X$. Then:

$$\mathcal{T}_X\big|_K := \left\{K \cap U : U \in \mathcal{T}_X \subseteq 2^X\right\} \subseteq 2^K \tag{3.2}$$

is the topology induced by $\mathcal{T}_X$. It is also a topology on K . Now we let $K \subseteq X$ compact. Then for each open cover $\{O_i : i \in I\}$ of K holds $K \subseteq \bigcup_{i\in I} O_i$. Now we assume $O_i \in \mathcal{T}_X\big|_K$ for each $i \in I$. Then for each $i \in I$ exists a $U_i \in \mathcal{T}_X$ such that $O_i = K \cap U_i$. Thus:

$$K \subseteq \bigcup_{i\in I} O_i = \bigcup_{i\in I} K \cap U_i = K \cap \bigcup_{i\in I} U_i \subseteq \bigcup_{i\in I} U_i. \tag{3.3}$$

Thus $\{U_i \in \mathcal{T}_X : i \in I\}$ is a open cover of K . Because of the compactness of K in X , exists $i_1, \dots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$ and $O_{i_j} = K \cap U_{i_j}$. Finally we get:

$$K \subseteq K = K \cap \bigcup_{j=1}^{n} U_{i_j} = \bigcup_{j=1}^{n} K \cap U_{i_j} = \bigcup_{j=1}^{n} O_{i_j}. \tag{3.4}$$

If we then use ψ with the trivial continuation identify ψ_U , we get the identification:

$$C_c(U) := \{\psi_U = \psi \in C_c(X) : \operatorname{supp}(\psi_U) = \operatorname{supp}(\psi) \subseteq U\}. \tag{3.5}$$

Now we let $I_R : C_c(X) \to \mathbb{C}$ a Radon−integral, then we can define the restriction $I_R\big|_U : C_c(U) \to \mathbb{C}$ by $I_R\big|_U(\psi) := I_R(\psi)$.

Lemma 3.1:

Let X be a locally compact space and $\emptyset \neq U \subseteq X$ be open. Let $I_R : C_c(X) \to \mathbb{C}$ be a Radon−integral, we can define the restriction $I_R\big|_U : C_c(U) \to \mathbb{C}$ by $I_R\big|_U(\psi) := I_R(\psi)$. Let $f : U \to \mathbb{C}$ be any function. Then f is $I_R\big|_U$−integrable iff the trivial continuation $f_U : X \to \mathbb{C}$ is I_R−integrable, and then holds:

$$\int_X f_U(x)\, \mathrm{d}_{I_R} x = \int_U f(x)\, \mathrm{d}_{I_R|_U} x. \tag{3.6}$$

Proof:

First, a function $g : U \to [0, +\infty]$ is lower semicontinuous iff the trivial continuation:

$$g_U : U \subseteq X \to [0, +\infty] := \begin{cases} g(x), & x \in U, \\ 0, & x \in (X \setminus U) \end{cases} \tag{3.7}$$

on X is lower semicontinuous. In fact, according to the definition of lower semicontinuous and equation (3.7), for any $a > 0$ holds:

$$g_U^{-1}((a, +\infty]) = g^{-1}((a, +\infty]) \subseteq U \tag{3.8}$$

and hence $g_U^{-1}((a, +\infty]) \subseteq X$ is open iff $g^{-1}((a, +\infty]) \subseteq U$ is open. In fact if $g^{-1}((a, +\infty]) \subseteq U$ is open, then $g^{-1}((a, +\infty]) \in \mathcal{T}_X\big|_U$, thus there is a $O \in \mathcal{T}_X$ such that $g^{-1}((a, +\infty]) = g_U^{-1}((a, +\infty]) = U \cap O$. Because U, $O \subseteq X$ are open, thus $g_U^{-1}((a, +\infty])$ is open in X . Necessity is similarly. Next we prove:

$$I_R^*(g_U) = I_R^*\big|_U(g), \quad \forall\, g \in C_u^+(U) := \{f \in C_u(U) : f \geq 0\}. \tag{3.9}$$

Here $C_u(U)$ is the set of all lower semicontinuous $f : U \to \overline{\mathbb{R}} \setminus \{-\infty\}$. Now we consider the set $S := \{\psi \in C_c^+(U) : \psi \leq g\} \subseteq C_c^+(X)$ and obviously the set S meets the following properties.

(i). $\psi \le g$ for all $\psi \in S$. This term is trivial.

(ii). $(\psi_1, \psi_2 \in F) \Rightarrow (\max(\psi_1, \psi_2) \in S)$. In fact, without loss of generality, it can be assumed that $\psi_1 \le \psi_2$. Then $\psi_1, \psi_2 \in F$ implies that $\psi_1 \le \psi_2 \le g$ hence $\psi_2 = \max(\psi_1, \psi_2) \in S$.

(iii). $g(x) = \sup\{\psi(x) : \psi \in S\}$. This is also trivial.

Now what we want to prove the equation (3.9). With lemma (1.5) we can find a $\widetilde{\psi} \in C_c^+(X)$ with $\widetilde{\psi}(x) < g(x)\,,\ \forall\, x \in \operatorname{supp}\left(\widetilde{\psi}\right)$ and:

$$I_R\left(g - \widetilde{\psi}\right) = I_R(g) - I_R\left(\widetilde{\psi}\right) < I_R^*(g) - I_R\left(\widetilde{\psi}\right) < \varepsilon. \tag{3.10}$$

For all $x \in \operatorname{supp}\left(\widetilde{\psi}\right)$ exists because of (iii) a $\psi_x \in F$ with $\widetilde{\psi}(x) < \psi_x(x) \le g(x)$. We define $V_x := \left(\psi_x - \widetilde{\psi}\right)^{-1}(0, +\infty)$, thus V_x ist a neighbourhood of x with $\widetilde{\psi}(z) < \psi_x(z) \le g(z)\,,\ \forall\, z \in V_x$. Because $\operatorname{supp}\left(\widetilde{\psi}\right)$ is compact, thus exists $x_1, \dots, x_n$ with $\operatorname{supp}\left(\widetilde{\psi}\right) \subseteq \bigcup_{j=1}^{n} V_{x_j}$. Hence define $\psi = \max_{1\le j\le n} \psi_{x_j}$,thus it follows with (ii) that $\psi \in F$ with $\widetilde{\psi} \le \psi \le g$ and hence $I_R^*(g) - I_R(\psi) < \varepsilon$.
Hence we get:

$$I_R^*(g_U) = \sup\left\{I_R(\psi) : \psi \in C_c^+(U)\,,\ \psi \le g\right\} = I_R^*\big|_U(g) \tag{3.11}$$

which is correspond equation (3.9).
Let now $f : U \to \mathbb{C}$ any function. Thus it follows for all $h \in C_u^+(X)$:

$$(|f_U| \le h) \Leftrightarrow \left(|f| \le h\big|_U \in C_u^+(U)\right). \tag{3.12}$$

Actually, if $|f_U| \le h$:

$$|f_U| := \begin{cases} |f|\,, & x \in U, \\ 0, & x \in (X \setminus U) \end{cases} \tag{3.13}$$

and $h = h\big|_U,\ \forall\, x \in U$ holds $|f| \le h\big|_U$. Conversely, if $|f| \le h\big|_U$,we write $h_U := 1_U h \le h$ and obviously $h_U = \left(h\big|_U\right)_U$, thus $|f_U| \le \left(h\big|_U\right)_U = h_U = 1_U h \le h$. Obviously for all $x \in U$ holds $h_U = h = h\big|_U$. Thus it follows because of $|f_U| \le h_U \le h$:

$$\begin{aligned} \|f_U\|_{L^1,\mathrm{semi}} &:= \inf\left\{I_R^*(h) : h \in C_c^+(X)\,,\ |f_U| \le h\right\} \\ &= \inf\left\{I_R^*\left(h\big|_U\right) : h\big|_U \in C_c^+(U)\,,\ |f| \le h\big|_U\right\} \\ &= \inf\left\{I_R^*(h_U) : h \in C_c^+(U)\,,\ |f| \le h\right\} \\ &\overset{\text{eq.(3.9)}}{=} \inf\left\{I_R^*\big|_U(h) : h \in C_c^+(U)\,,\ |f| \le h\right\} \\ &:= \|f\|_{L^1,\mathrm{semi}}^U . \end{aligned} \tag{3.14}$$

We now show that for every function $\phi \in C_c(X)$ and for every $\varepsilon > 0$ there exists a $\psi \in C_c(U)$ with $\left\|\phi\big|_U - \psi\right\|_{L^1,\mathrm{semi}} < \varepsilon$. By breaking it down into real and imaginary parts, we can assume that ψ is real, and by breaking down the real function $\phi = \phi^+ - \phi^-$ into the positive and negative parts ϕ^+ and ϕ^+ we can then assume without loss of generality that $\phi \ge 0$, thus $\phi \in C_c^+(X)$. According to the above discussion we know that the trivial continuation of $\phi\big|_U \in C_u^+(U)$ is $\left(\phi\big|_U\right)_U = 1_U\phi := \phi_U \le \phi$ on X and $\phi_U = \phi = \phi\big|_U$ for all $x \in U$. Thus from lemma (1.4) we get:

$$I_R^*\big|_U\left(\phi\big|_U\right) = I_R^*\left(\left(\phi\big|_U\right)_U\right) = I_R^*(\phi_U) \le I_R^*(\phi) = I_R(\phi) < +\infty. \tag{3.15}$$

Claim 3:

Let $f \in C_u^+(X)$. Then f is integrable iff $I_R^*(f) < \infty$ and hence holds:

$$\int_X f\,\mathrm{d}_{I_R}x = I_R^*(f). \tag{3.16}$$

Proof:

We choose a sequence $(\phi_n)_{n\in\mathbb{N}} \subseteq C_c^+(X)$ with $\phi_n \le f,\ \forall\, n \in \mathbb{N}$ and $I_R(\phi_n) \to I_R^*(f)$. This is well-defined, because $I_R^*(f) = \sup\{I_R(\psi) : \psi \in C_c^+(X)\,,\ \psi \le f\}$. Then $(f - \phi_n) \in C_u^+(X)$ thus:

$$\|f - \phi_n\|_{L^1,\mathrm{semi}} = I_R^*(f - \phi_n) = I_R^*(f) - I_R^*(\phi_n) = I_R^*(f) - I_R(\phi_n) \to 0. \tag{3.17}$$

□ (Claim 3)

Is then $\psi \in C_c^+(U)$ with $\psi \leq \phi\big|_U$ and $I_R\big|_U(\psi) > I_R^*\big|_U\left(\phi\big|_U\right) - \varepsilon$, thus it follows from lemma (1.34) because of $\phi\big|_U - \psi \geq 0$:

$$\begin{aligned} \left\|\phi\big|_U - \psi\right\|_{L^1,\text{semi}} &= \int_U \left(\phi\big|_U(x) - \psi(x)\right) \mathrm{d}_{I_R|_U} x \\ &= \int_U \phi\big|_U(x) \,\mathrm{d}_{I_R|_U} x - \int_U \psi(x) \,\mathrm{d}_{I_R|_U} x \\ &= I_R^*\big|_U\left(\phi\big|_U\right) - I_R(\psi) < \varepsilon. \end{aligned} \tag{3.18}$$

Is f_U integrable over X and is $(\phi_n)_{n\in\mathbb{N}}$ a sequence in $C_c(X)$ with $\|f_U - \phi_n\|_{L^1,\text{semi}} \to 0$,thus we choose for every $n \in \mathbb{N}$ a $\psi_n \in C_c(U)$ with $\left\|\phi_n\big|_U - \psi_n\right\|_{L^1,\text{semi}}^U \leq 1/n$. Then:

$$\begin{aligned} \|f - \psi_n\|_{L^1,\text{semi}}^U &\leq \left\|f - \phi_n\big|_U\right\|_{L^1,\text{semi}}^U + \left\|\phi_n\big|_U - \psi_n\right\|_{L^1,\text{semi}}^U \\ \leq \quad & \|f_U - \phi_n\|_{L^1,\text{semi}} + \frac{1}{n} \to 0, \quad (n \to \infty). \end{aligned} \tag{3.19}$$

Hence f is also integrable over U in terms of $I_R\big|_U$. Conversely, if f is integrable over U, and if $(\psi_n)_{n\in\mathbb{N}}$ in $C_c^+(U)$ with $\|f - \psi_n\|_{L^1,\text{semi}}^U \to 0$, then because of equation (3.14):

$$\|f_U - (\psi_n)_U\|_{L^1,\text{semi}} = \|(f - \psi_n)_U\|_{L^1,\text{semi}} = \|f - \psi_n\|_{L^1,\text{semi}}^U \to 0. \tag{3.20}$$

Where we have taken $C_c(U)$ as a subset of $C_c(X)$. It then follows:

$$\int_X f_U(x) \,\mathrm{d}_{I_R} x = \lim_{n\to\infty} I_R\left((\psi_n)_U\right) = \lim_{n\to\infty} I_R\big|_U(\psi_n) = \int_U f(x) \,\mathrm{d}_{I_R|_U} x. \tag{3.21}$$

□ (Lemma 3.1)

Finally, we give the rule of change of variable in multiple integral and prove it.

Theorem 3.1 (Change of variable in multiple integral):
Let $U, V \subseteq \mathbb{R}^n$ open and let $\boldsymbol{\phi} : U \to V$ a C^1-diffeomorphism. Then for any function $f : V \to \mathbb{C}$ is integrable over V iff the function $\mathbf{x} \mapsto f(\boldsymbol{\phi}(\mathbf{x})) |\det D\boldsymbol{\phi}(x)|$ is integrable over U and hence:

$$\int_V f(\mathbf{y}) \,\mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{y} = \int_U f(\boldsymbol{\phi}(\mathbf{x})) |\det D\boldsymbol{\phi}(x)| \,\mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x}. \tag{3.22}$$

Proof:

We define first $\widehat{f}(\mathbf{x}) = f(\boldsymbol{\phi}(\mathbf{x})) |\det D\boldsymbol{\phi}(x)|$. Then the map $f \mapsto \widehat{f}$ is bijective. Thus for every $\widehat{f} : U \to \mathbb{C}$ a unique $f : V \to \mathbb{C}$ can be found such that $f \mapsto \widehat{f}$. Thus the map $f \mapsto \widehat{f}$ maps the space $C_c(V)\,(C_c(U))$ to the space $C_c^+(V)\,(C_c^+(U))$ such that for all $g, h : V \to \mathbb{R}_0^+$ holds $(g \leq h) \Leftrightarrow \left(\widehat{g} \leq \widehat{h}\right)$. Actually for all $\boldsymbol{\phi}(\mathbf{x}) \in V$ we can find if $g(\boldsymbol{\phi}(\mathbf{x})) \leq h(\boldsymbol{\phi}(\mathbf{x}))$:

$$\widehat{g}(\mathbf{x}) = g(\boldsymbol{\phi}(\mathbf{x})) |\det D\boldsymbol{\phi}(x)| \leq h(\boldsymbol{\phi}(\mathbf{x})) |\det D\boldsymbol{\phi}(x)| = \widehat{h}(\mathbf{x}). \tag{3.23}$$

With equation (1.28) and lemma (2.6) we know that:

$$\begin{aligned} \boldsymbol{\lambda}_n(\psi) &= \int_V \psi(\mathbf{y}) \,\mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{y} = \int_U \psi(\boldsymbol{\phi}(\mathbf{x})) |\det D\boldsymbol{\phi}(x)| \,\mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x} \\ &= \int_U \widehat{\psi}(\mathbf{x}) \,\mathrm{d}_{\boldsymbol{\lambda}_n} \mathbf{x} = \boldsymbol{\lambda}_n\left(\widehat{\psi}\right) \end{aligned} \tag{3.24}$$

for all $\psi \in C_c(V)$. Then for $g \in C_u^+(V)$ and with equation (1.8) we have:

$$\begin{aligned} \boldsymbol{\lambda}_n^*(g) &= \sup\left\{\boldsymbol{\lambda}_n(\phi) : \phi \in C_c^+(V),\ \phi \leq g\right\} \\ &= \sup\left\{\boldsymbol{\lambda}_n\left(\widehat{\phi}\right) : \widehat{\phi} \in C_c^+(U),\ \widehat{\phi} \leq \widehat{g}\right\} = \boldsymbol{\lambda}_n^*(\widehat{g}). \end{aligned} \tag{3.25}$$

Is then $f : V \to \mathbb{C}$ (or $f : V \to \mathbb{R}_0^+$) any function, with equation (3.14) we have:

$$\begin{aligned} \|f\|_{L^1,\text{semi}}^V &= \inf\left\{\boldsymbol{\lambda}_n^*(g) : g \in C_c^+(V),\ |f| \leq g\right\} \\ &= \inf\left\{\boldsymbol{\lambda}_n^*(\widehat{g}) : \widehat{g} \in C_c^+(U),\ \left|\widehat{f}\right| \leq \widehat{g}\right\} = \left\|\widehat{f}\right\|_{L^1,\text{semi}}^U. \end{aligned} \tag{3.26}$$

If then $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $C_c(V)$, then it holds:

$$\left(\|f-\varphi_n\|^V_{L^1,\text{semi}}\to 0\right)\Leftrightarrow\left(\left\|\widehat{f}-\widehat{\varphi}_n\right\|^U_{L^1,\text{semi}}\to 0\right). \tag{3.27}$$

So f is integrable over V iff $\widehat{f}$ is integrable over U , and it follows with definition (1.5), equation (1.28), lemma (2.6) and lemma (3.1):

$$\begin{aligned}&\int_U f(\boldsymbol{\phi}(\mathbf{x}))\left|\det D\boldsymbol{\phi}(x)\right|\,\mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x}=\int_U\widehat{f}(\mathbf{x})\,\mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x}=\lim_{n\to\infty}\boldsymbol{\lambda}_n(\widehat{\varphi}_n)\\ =\;&\lim_{n\to\infty}\int_U\widehat{\varphi}_n(\mathbf{x})\,\mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{x}=\lim_{n\to\infty}\int_V\varphi_n(\mathbf{y})\,\mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y}=\lim_{n\to\infty}\boldsymbol{\lambda}_n(\varphi_n)=\int_V f(\mathbf{y})\,\mathrm{d}_{\boldsymbol{\lambda}_n}\mathbf{y}.\end{aligned} \tag{3.28}$$

□

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