General Solutions of Ordinary Differential Equations and Division by Zero Calculus - New Type Examples

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Abstract: We examined many examples of the relation between general solutions with singular points in ordinary differential equations and division by zero calculus, however, here we will introduce a new type example that was appeared from some general solution of an ordinary differential equation.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

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1 A new type example

We examined many examples of the relation between general solutions in ordinary differential equations with singular points and division by zero calculus (see, in particular [3]), however, here we will introduce a new type example that was appeared from some general solution of an ordinary differential equation.

We recall that for the ordinary differential equation with constants a and b

$$(ae^y + bx)\frac{dy}{dx} = 1, \tag{1.1}$$

we obtain the general solutions for any constant C

$$x = Ce^{by} + \frac{a}{1-b}e^y, \quad b \neq 1 \tag{1.2}$$

and for $b = 1$

$$x = Ce^y + aye^y \tag{1.3}$$

([2], page 165, 38).

The problem is:

How to derive the solution (1.3) from the general solution (1.2) by the division by zero calculus?

We wonder how to derive (1.3) from (1.2).

We recall that for the ordinary differential equation with a constant A

$$y'' = Ax^n, \tag{1.4}$$

we obtain the general solution with constants C_j, for $n \neq -1, -2$

$$y = \frac{Ax^{n+2}}{(n+1)(n+2)} + C_1x + C_2. \tag{1.5}$$

For $n = -2$, we obtain

$$y = A \log x + C_1x + C_2 \tag{1.6}$$

and for $n = -1$

$$y = A\int \log x dx + C_1x + C_2 = Ax(\log x - 1) + C_1x + C_2 \tag{1.7}$$

([2], page 307, 1). Now we can obtain (1.6) and (1.7) from (1.5) directly by the division by zero calculus. We obtained many and many examples. See the references.

However, for the function

$$\frac{a}{1-b}e^y, \tag{1.8}$$

by the division by zero calculus we obtain the formal result for $b = 1$

$$-a\frac{\partial e^y}{\partial b}|_{b=1}. \tag{1.9}$$

This will be a mysterious formula. Therefore, we wonder how to obtain (1.3).

2 Result

For the function

$$\frac{1}{1-b}e^y, \tag{2.1}$$

we will consider in this way

$$\frac{1}{1-b}e^y = \frac{e^y - e^{by}}{1-b} + \frac{1}{1-b}e^{by}. \tag{2.2}$$

Note that

$$\frac{e^y - e^{by}}{1-b}|_{b=1} = ye^y. \tag{2.3}$$

Therefore, if we can include the constant

$$\frac{a}{1-b} \tag{2.4}$$

to the general constant C, then we can obtain the desired result (1.3).

Indeed, from the identity

$$x = Ce^{by} + \frac{a}{1-b}e^y = \left(C + a\frac{1}{1-b}\right)e^{by} + a\frac{e^y - e^{by}}{1-b}, \tag{2.5}$$

by the division by zero $1/0 = 0$, we obtain the desired result (1.3).

Note that for the function

$$\frac{1}{1-b}e^{by} \tag{2.6}$$

we apply the division by zero $1/0 = 0$ separately in the terms

$$\frac{1}{1-b} \tag{2.7}$$

and

$$e^{by}. \tag{2.8}$$

We do not consider the division by zero calculus for (2.6). We gave a reasonable interpretation for the natural derivation of (1.3) from (1.2).

Could we consider the problem in the following way?

From the identity

$$\begin{aligned} x &= \left(C + a\frac{1}{1-b}\right)e^{by} + a\frac{e^y - e^{by}}{1-b} \\ &= Ce^{by} + a\frac{e^y - e^{by}}{1-b}, \end{aligned} \tag{2.9}$$

by putting $b = 1$ we have the desired result, by changing any constant C. However, in this logic we have a delicate problem, because its C is depending on b. However, we will feel some here.

3 Conclusion

This example (1.2) shows that the formula (1.9) is not almighty, we have some delicate case for the division by zero calculus.

4 Essence of division by zero calculus

We state the essence of division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{4.1}$$

we will **define**

$$f(a) = C_0. \tag{4.2}$$

For the correspondence (4.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (4.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

However, we can consider the more general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{4.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [1, 3, 4, 5].

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

Note that

$$(x^n)' = nx^{n-1}$$

and so

$$\left(\frac{x^n}{n}\right)' = x^{n-1}.$$

Here, we obtain the right result for $n = 0$

$$(\log x)' = \frac{1}{x}$$

by the division by zero calculus.

References

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