Briefly about the Notion of a Quaternionic Holomorphic Function

Michael Parfenov
mwparfenov@gmail.com

Abstract The so-called essentially adequate notion of quaternionic holomorphy is briefly considered. This calls into question the known statement of R. Penrose that there is no satisfactory quaternionic analogue of the notion of a holomorphic function.

Keywords: *quaternionic holomorphic functions, quaternionic analysis, quaternionic generalization of Cauchy-Riemann's equations*

The situation with regard to the derivative notion in quaternionic analysis seems somewhat confused (see, for example, references in [6]. It has even led R. Penrose to say [1, p.201], "…quaternions are not really so mathematically "nice" as they seem at first sigh. They are relatively poor "magicians"; and, certainly, they are no match for complex numbers in this regard. The reason appears to be that there is no satisfactory quaternionic analogue of the notion of a holomorphic function." The most popular approach to the quaternionic holomorphic functions problem is shortly represented in [2]. Since then little has changed. Now we'd like to consider an essence of simpler approach, allowing to call into question the above statement of R. Penrose.

Theory of quaternionic differentiability and quaternionic holomorphic functions can be built on principles fully similar (essentially adequate) to ones of complex holomorphic functions.

The general definition of a derivative can be based on the following main idea, viz.: each point of any real line is at the same time a point of some plane and 3D space as a whole, and therefore any characterization of differentiability (and its relations) at a point must be the same regardless of whether we think of that point as a point on the real axis or a point in the complex plane, or a point in three-dimensional space. It follows that a quaternionic derivative of a quaternionic function $\psi(p)$ must be defined similar to complex derivative in the plain as a limit of a difference quotient of $\Delta\psi(p)$ by Δp when Δp converges to zero along any direction in the quaternionic space, where $\psi(p) = \psi_1(x,y,z,u) + \psi_2(x,y,z,u)i + \psi_3(x,y,z,u)j + \psi_4(x,y,z,u)k$ is a quaternionic function of real components x, y, z, u of an independent quaternionic variable $p = x + yi + zj + uk$. The functions $\psi_1(x,y,z,u), \psi_2(x,y,z,u), \psi_3(x,y,z,u), \psi_4(x,y,z,u)$ are real-valued quantities and $i, j. k$ are the base quaternions of the quaternion space $\mathbb{H}$.

Using the Cayley–Dickson construction (doubling procedure) [5, p.42] we have

$$p = a + b \cdot j \subset \mathbb{H}, \quad \psi(p) = \phi_1(a,b,\bar{a},\bar{b}) + \phi_2(a,b,\bar{a},\bar{b}) \cdot j \subset \mathbb{H},$$

where

$$a = x + yi, \; b = z + ui, \tag{1}$$

$$\phi_1(a,b,\bar{a},\bar{b}) = \psi_1 + \psi_2 i, \quad \phi_2(a,b,\bar{a},\bar{b}) = \psi_3 + \psi_4 i$$

are compex quantities, the "$\cdot$" and overbar signs denote, respectively, quaternionic multiplication and complex (or quaternionic if needed) conjugation.

Since the quaternion algebra is a noncommutative algebra with division, there can exist the left and the right definitions of a quaternionic derivative [2, p.19]:

$$\psi'_{left} = \lim_{\Delta p \to 0} [(\Delta p)^{-1} \cdot \{\psi(p+h) - \psi(p)\}],$$
$$\psi'_{right} = \lim_{\Delta p \to 0} [\{\psi(p+h) - \psi(p)\} \cdot (\Delta p)^{-1}].$$

Each of them by itself is incomplete (may be called "non-essentially adequate"), since each of underlying algebras, viz.: with only the "left" or only the "right" multiplication does not represent *all arbitrary rotations* of vectors in 3D space. In other words, we have to refuse to consider only the left or only the right approach (regarding another as equivalent [2, p.19]) when defining a quaternionic derivative. The left and the right derivatives should be considered only together.

In complex analysis each holomorphic function is considered as the complex potential [3, p. 328; 8, p.1]. The derivative of it represents some unambiguous planar steady state vector field (electrical field, fluid flow et al) and vice versa, that is, the derivative is unambiguous. Given this, the quaternionic derivative must also correspond some unambiguous steady state vector field in dimension more than 2 and hence must be also unambiguous [8]. Thus we are forced to require the equality of the left and right derivatives, i.e.

$$\psi'_{left} = \psi'_{right}$$

in an initial domain of definition $G_4 \subset \mathbb{H}$ to associate the quaternionic derivative with physical reality. It is reasonable forced requirement.

At that the limit of the difference quotient is required to be independent not only of directions to approach a limiting point $\Delta p = 0$ (as in complex analysis), but also of the manner of quaternionic division: on the left or on the right. Such an independence for a derivative can be called the "independence of the way of its computation". Based on that it is possible to introduce the following

Definition 1. A single-valued quaternionic function $\psi(p) : G_4 \to \mathbb{H}$ is quaternionic-differentiable at a point $p \in G_4 \subset \mathbb{H}$ if there exists a limiting value of the difference quotient $\frac{\Delta\psi}{\Delta p}$ as $\Delta p \to 0$, and this value is independent of the way of its computation [4, p.13].

Definition 2. A quaternionic function is said to be a quaternionic holomorphic (briefly, $\mathbb{H}$-holomorphic) function at a point p, if it has a quaternionic derivative independent of a way of its computation in some open connected neighborhood $G_4 \subset \mathbb{H}$ of a point $p \in \mathbb{H}$ [4, p.14].

In the Cayley–Dickson doubling form the Definition 1 leads to the following formulation of the necessary and sufficient conditions for $\psi(p)$ to be $\mathbb{H}$-holomorphic [4, p.21; 7, p.15]:

Definition 3. It is assumed that the constituents $\phi_1(a, b, \bar{a}, \bar{b})$ and $\phi_2(a, b, \bar{a}, \bar{b})$ of a quaternionic function $\psi(p) = \psi(a, b) = \phi_1 + \phi_2 j$ possess continuous first-order partial derivatives with respect to $a, \bar{a}, b,$ and $\bar{b}$ in some open connected neighborhood $G_4 \subset \mathbb{H}$ of a point $p \in G_4$. Then a function $\psi(p)$ is said to be $\mathbb{H}$-holomorphic and denoted by $\psi_H(p)$ at a point p, if and only if the functions $\phi_1(a, b, \bar{a}, \bar{b})$ and $\phi_2(a, b, \bar{a}, \bar{b})$ satisfy in $G_3 \subset G_4$ the following quaternionic generalization of Cauchy-Riemann's equations:

$$\begin{cases} 1)\ (\partial_a \phi_1| = (\partial_{\bar{b}} \overline{\phi_2}|, \quad 2)\ (\partial_a \phi_2| = -(\partial_{\bar{b}} \overline{\phi_1}|, \\ 3)\ (\partial_a \phi_1| = (\partial_b \phi_2|, \quad 4)\ (\partial_{\bar{a}} \phi_2| = -(\partial_{\bar{b}} \phi_1|. \end{cases} \qquad (2)$$

Here ∂_i, $i = a, \bar{a}, b, \bar{b}$, denotes the partial derivative with respect to i. The brackets $(..|$ with the closing vertical bar indicate that the transition $a = \bar{a} = x$ (to 3D space) has been already performed in expressions enclosed in brackets. Thus, $\mathbb{H}$ -holomorphy conditions (2) are defined so that during the check of the quaternionic holomorphy of any quaternionic function we have to do the transition $a = \bar{a} = x$ in already calculated expressions for the partial derivatives of the functions ϕ_1 and ϕ_2. However, this doesn't mean that we deal with triplets in general, since the transition $a = \bar{a} = x$ (or $y = 0$) cannot be initially done for quaternionic variables and functions. Any quaternionic function remains the same 4-dimensional quaternionic function regardless of whether we check its holomorphy or not. Simply put, the $\mathbb{H}$-holomorphic functions are 4-dimensional quaternionic functions whose derivatives met equations (2) after the transition to 3D space. In other words, they are those quaternionic functions (in $\mathbb{H}$ space), whose the left and right derivatives become equal after the transition to 3D space. That's not surprising, since unambiguous conservative physical fields represented by derivatives exist just in 3D space.

Example 1. The quaternionic function $\psi(p) = p^2 = p \cdot p = (a + b \cdot j) \cdot (a + b \cdot j) = \phi_1(a, b, \bar{a}, \bar{b}) + \phi_2(a, b, \bar{a}, \bar{b}) \cdot j$. By the direct quaternionic multiplication we obtain the following expressions for the components of the Cayley–Dickson doubling form:

$$\phi_1(a, b, \bar{a}, \bar{b}) = a^2 - b\bar{b}\,, \quad \phi_2(a, b, \bar{a}, \bar{b}) = (a + \bar{a})b.$$

Correspondingly, the complex conjugate functions are $\overline{\phi_1}(a, b, \bar{a}, \bar{b}) = \bar{a}^2 - \bar{b}b$ and $\overline{\phi_2}(a, b, \bar{a}, \bar{b}) = (\bar{a} + a)\bar{b}$.

When obtaining the components $\phi_1(a, b, \bar{a}, \bar{b})$ and $\phi_2(a, b, \bar{a}, \bar{b})$, we took into account that $j\alpha = \bar{\alpha}j$ for any $\alpha \in \mathbb{C}$ [5, p. 42] and $j^2 = -1$.

Now we calculate the partial derivatives:
$\partial_a\phi_1 = 2a$, $\partial_{\bar{b}}\overline{\phi}_2 = \bar{a} + a$, $\partial_a\phi_2 = b$, $\partial_{\bar{b}}\overline{\phi}_1 = -b$ for equations (2-1,2) and $\partial_b\phi_2 = a + \bar{a}$, $\partial_{\bar{a}}\phi_2 = b$, $\partial_{\bar{b}}\phi_1 = -b$ for equations (2-3,4). By subsituting $a = x$ and $\bar{a} = x$ into expressions for calculated partial derivatives we have

$$\begin{cases} 1)\ (\partial_a\phi_1| = (\partial_{\bar{b}}\overline{\phi}_2| = 2x, \quad 2)\ (\partial_a\phi_2| = -(\partial_{\bar{b}}\overline{\phi}_1| = b, \\ 3)\ (\partial_a\phi_1| = (\partial_b\phi_2| = 2x, \quad 4)\ (\partial_{\bar{a}}\phi_2| = -(\partial_{\bar{b}}\phi_1| = b \end{cases}$$

in coordinates x, z, u of 3D space. We see that equations (2) are fulfilled and the (4-dimensional!) quaternionic function $\psi_H(p) = p^2$ is $\mathbb{H}$-holomorphic.

Example 2. The quaternionic function $\psi(p) = e^p = \phi_1 + \phi_2 \cdot j$, where e is the base of the natural logarithm. We represent the quaternion variable $p = x + yi + zj + uk$ as a sum of real and imaginary parts: $p = x + vr$, where $v = \sqrt{y^2 + z^2 + u^2}$ is a real-valued function, $r = \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}$ can be considered as a purely imaginary unit quaternion. Since $r^2 = -1$ as well as x and v are real-valued, the quaternionic formula $p = x + vr$ is algebraically equivalent to the complex formula $z = x + yi$. Then, using the quaternionic analogue of Euler's formula: $e^{vr} = \cos v + r\sin v$, where $r = \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}} = \frac{yi}{v} + \frac{zj+uk}{v} = \frac{yi}{v} + \frac{(z+ui)}{v} \cdot j$, we have
:

$$\psi(p) = \phi_1 + \phi_2 \cdot j = e^p = e^{(x+vr)} = e^x e^{vr} = e^x(\cos v + r\sin v)$$
$$= e^x\left(\cos v + \frac{yi\sin v}{v}\right) + e^x\frac{(z+ui)\sin v}{v} \cdot j,$$

whence $\phi_1 = e^x\left(\cos v + \frac{yi\sin v}{v}\right)$, $\phi_2 = e^x\frac{(z+ui)\sin v}{v}$.

Using relations $b = z + ui$ and $x = \frac{a+\bar{a}}{2}$, $y = \frac{a-\bar{a}}{2i}$, following from (1), we obtain the expressions for ϕ_1 and ϕ_2 as functions of $a, \bar{a}, b, \bar{b}$:

$$\phi_1 = 2\beta\cos v + \frac{\beta(a-\bar{a})\sin v}{v}, \ \phi_2 = \frac{2\beta b\sin v}{v},$$

where

$$\beta = \frac{e^{\frac{a+\bar{a}}{2}}}{2}, |p| = \sqrt{x^2 + y^2 + z^2 + u^2} = \sqrt{a\bar{a} + b\bar{b}}, v = \frac{\sqrt{4|p|^2 - (a+\bar{a})^2}}{2} \tag{3}$$

are real-valued.

The partial derivatives of the functions ϕ_1 and ϕ_2 in the case of $\psi(p) = e^p$ are the following:

$$\partial_a\phi_1 = \beta\left[\cos v + \frac{(a-\bar{a}+1)\sin v}{v} - \frac{(a-\bar{a})^2(v\cos v - \sin v)}{4v^3}\right];$$
$$\partial_b\phi_2 = \partial_{\bar{b}}\overline{\phi}_2 = \beta\left[\frac{2\sin v}{v} + \frac{b\bar{b}(v\cos v - \sin v)}{v^3}\right];$$
$$\partial_a\phi_2 = -\partial_{\bar{b}}\phi_1 = \beta b\left[\frac{\sin v}{v} - \frac{(a-\bar{a})(v\cos v - \sin v)}{2v^3}\right];$$
$$\partial_{\bar{a}}\phi_2 = -\partial_{\bar{b}}\overline{\phi}_1 = \beta b\left[\frac{\sin v}{v} + \frac{(a-\bar{a})(v\cos v - \sin v)}{2v^3}\right].$$

Performing the transition $a = \bar{a} = x$ and taking into consideration that $b\bar{b} = |b|^2$, and the relations $v = |b|$ and $\beta = \frac{e^x}{2}$ are true after the transition, we obtain as follows:

$$1)\ (\partial_a\phi_1| = (\partial_{\bar{b}}\overline{\phi}_2| = \frac{e^x(\cos|b| + |b|^{-1}\sin|b|)}{2};\quad 2)\ (\partial_a\phi_2| = -(\partial_{\bar{b}}\overline{\phi}_1| = \frac{e^x b|b|^{-1}\sin|b|}{2};$$
$$3)\ (\partial_a\phi_1| = (\partial_b\phi_2| = \frac{e^x(\cos|b| + |b|^{-1}\sin|b|)}{2};\quad 4)\ (\partial_{\bar{a}}\phi_2| = -(\partial_{\bar{b}}\phi_1| = \frac{e^x b|b|^{-1}\sin|b|}{2}.$$

We see that equations (2) hold and the function $\psi_H(p) = e^p$ is $\mathbb{H}$-holomorphic.

Quaternionic derivative. It was proved [4, p.30; 6, p.18] that the quaternionic generalization of the complex derivatives has the following expression for the full quaternionic derivatives (uniting the left and right derivatives) of the k'th order:

$$\psi_H^{(k)}(p) = \phi_1^{(k)} + \phi_2^{(k)} \cdot j,$$

where the constituents $\phi_1^{(k)}$ and $\phi_2^{(k)}$ are expressed as follows:

$$\phi_1^{(k)} = \partial_a\phi_1^{(k-1)} + \partial_{\bar{a}}\phi_1^{(k-1)}, \ \phi_2^{(k)} = \partial_a\phi_2^{(k-1)} + \partial_{\bar{a}}\phi_2^{(k-1)};$$

$\phi_1^{(k-1)}$ and $\phi_2^{(k-1)}$ are the constituents of the $(k-1)'$th derivative of $\psi_H(p)$, represented in the Cayley–Dickson doubling form as $\psi(p)^{(k-1)} = \phi_1^{(k-1)} + \phi_2^{(k-1)} \cdot j, k \geq 1$; and $\phi_1^{(0)} = \phi_1(a, b)$, $\phi_2^{(0)} = \phi_2(a, b)$, $k = 1$.

. *If a quaternion function $\psi_H(p)$ is once $\mathbb{H}$-differentiable in G_4, then it possesses full derivatives of all orders in G_4, each one $\mathbb{H}$- differentiable* [4, p.31].

The first derivative of the considered in Example 1 function $\psi_H(p) = p^2$ is the following:

$$\psi_H^{(1)}(p) = (p^2)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)} \cdot j = (\partial_a \phi_1 + \partial_{\bar{a}} \phi_1) + (\partial_a \phi_2 + \partial_{\bar{a}} \phi_2) \cdot j$$
$$= (2a + 0) + (b + b) \cdot j = 2a + 2b \cdot j = 2p.$$

For the first derivative of the considered in Example 2 function $\psi_H(p) = e^p$ we have as follows:

$$\partial_{\bar{a}} \phi_1 = \beta \left[\frac{(v \cos v - \sin v)}{v} + \frac{(a-\bar{a})^2 (v \cos v - \sin v)}{4v^3}\right],$$

$$\psi_H^{(1)}(p) = (e^p)^{(1)} = \phi_1^{(1)} + \phi_2^{(1)} \cdot j = (\partial_a \phi_1 + \partial_{\bar{a}} \phi_1) + (\partial_a \phi_2 + \partial_{\bar{a}} \phi_2) \cdot j$$
$$= \left\{\beta \left[\cos v + \frac{(a-\bar{a}+1) \sin v}{v} - \frac{(a-\bar{a})^2 (v \cos v - \sin v)}{4v^3}\right] + \beta \left[\frac{(v \cos v - \sin v)}{v} + \frac{(a-\bar{a})^2 (v \cos v - \sin v)}{4v^3}\right]\right\}$$
$$+ \left\{\beta b \left[\frac{\sin v}{v} - \frac{(a-\bar{a})(v \cos v - \sin v)}{2v^3}\right] + \beta b \left[\frac{\sin v}{v} + \frac{(a-\bar{a})(v \cos v - \sin v)}{2v^3}\right]\right\} \cdot j$$
$$= 2\beta \cos v + \frac{\beta (a-\bar{a}) \sin v}{v} + \frac{2\beta b \sin v}{v} \cdot j = e^p.$$

We see that the expressions for the first full quaternionic derivatives of the functions $\psi_H(p) = p^2$ and $\psi_H(p) = e^p$ are similar to corresponding ones in real and complex analysis.. The same there is for the derivatives of higher orders [4, p. 33].

Some properties of $\mathbb{H}$-holomorphic functions. *It was established that the class* [7, p.22] *of $\mathbb{H}$-holomorphic functions possesses the remarkable properties, some of them are the following:*

1) Each $\mathbb{H}$-holomorphic function $\psi_H(p)$ can be constructed (without change of a functional dependence form) from its complex holomorphic analog $\psi_C(\xi)$ by replacing a complex variable $\xi \in G_2 \subset C$ as a whole in an expression for $\psi_C(\xi)$ by a quaternionic variable $p \in G_4 \subset \mathbb{H}$, where G_4 is defined (except, possibly, at certain singularities) by the relation $G_4 \supset G_2$ in the sense that G_2 exactly follows from G_4 upon the transition from p to ξ [4, p.28 ; 7, p.15].

2) All expressions for the full quaternionic derivatives of the H-holomorphic functions and rules for their differentiation are the same ones as the expressions and rules for corresponding derivatives of complex holomorphic analogs [4, p.33; 7, p.15, p.17]. For example, if the first derivative of the complex holomorphic function $\psi_C(\xi) = \xi^3$ is $3\xi^2$, then the first derivative of the $\mathbb{H}$-holomorphic analog $\psi_H(p) = p^3$ is $3p^2$.

3) Algebraic properties of the H-holomorphic functions are fully similar (essentially adequate) to ones of the complex holomorphic functions: the quaternionic multiplication of these quaternionic functions behaves as commutative [7, p.18], the left quotient equals the right one, the rules for differentiating sums, products, ratios, inverses, and compositions are the same as in complex analysis. One can just verify these properties, constructing $\mathbb{H}$-holomorphic functions from their complex holomorphic counterparts.

4) The considered concept of quaternionic holomorphy allows us to investigate steady state vector fields in 3D space, each of them corresponds to a $\mathbb{H}$-holomorphic function considered as a quaternionic potential [7, p. 23; 8, p. 7].

Example 3. Here we show that the quaternionic multiplication of the quaternionic holomorphic functions $\psi_H(p) = p^2$ and $\psi_H(p) = e^p$ behaves as commutative. Let the functions $f = f_1 + f_2 \cdot j$ and $g = g_1 + g_2 \cdot j$ be arbitrary quaternionic functions in the Cayley–Dickson construction. The quaternionic multiplication [5, p. 43; 7, p.18] of these functions can be writen as follows:

$$f \cdot g = (f_1 + f_2 \cdot j) \cdot (g_1 + g_2 \cdot j) = R(f \cdot g) + I(f \cdot g) \cdot j,$$

where

$$R(f \cdot g) = f_1 g_1 - f_2 \overline{g_2}\,, \quad I(f \cdot g) = f_2 \overline{g_1} + f_1 g_2 \qquad (4)$$

are the designations of the parts of the quaternionic product.

Consider the quaternionic product $p^2 \cdot e^p$. In this case we have (see Examples 1,2) $f_1 = a^2 - b\bar{b}$; $f_2 = (a + \bar{a})b$; $g_1 = 2\beta \cos v + \frac{\beta(a-\bar{a}) \sin v}{v}$ and $g_2 = \frac{2\beta b \sin v}{v}$. Subsituting these into (4) we obtain the following expressions:

$$R(p^2 \cdot e^p\ ) = f_1 g_1 - f_2 \overline{g_2} = (a^2 - b\bar{b})[2\beta \cos v + \frac{\beta(a-\bar{a}) \sin v}{v}] - (a + \bar{a}) b \frac{2\beta \bar{b} \sin v}{v},$$

$$I(p^2 \cdot e^p\ ) = f_2 \overline{g_1} + f_1 g_2 = (a + \bar{a}) b \left[2\beta \cos v + \frac{\beta(\bar{a}-a) \sin v}{v}\right] + (a^2 - b\bar{b}) \frac{2\beta b \sin v}{v},$$

where (as well as in the sequel) it is taken into account that in accordance with (3) we have $\beta = \bar{\beta},\ v = \bar{v}$.

Consider the quaternionic product $e^p \cdot p^2$. In this case we have $f_1 = 2\beta\cos v + \frac{\beta(a-\bar{a})\sin v}{v}$, $f_2 = \frac{2\beta b \sin v}{v}$; $g_1 = a^2 - b\bar{b}$, $g_2 = (a+\bar{a})b$. Subsituting these into (4) we obtain the following expressions:

$$R(e^p \cdot p^2) = f_1 g_1 - f_2\overline{g_2} = \left[2\beta\cos v + \frac{\beta(a-\bar{a})\sin v}{v}\right](a^2 - b\bar{b}) - \frac{2\beta b \sin v}{v}(\bar{a}+a)\bar{b},$$

$$I(e^p \cdot p^2) = f_2\overline{g_1} + f_1 g_2 = \frac{2\beta b \sin v}{v}(\bar{a}^2 - \bar{b}b) + [2\beta\cos v + \frac{\beta(a-\bar{a})\sin v}{v}](a+\bar{a})b.$$

We see that

$$R(p^2 \cdot e^p) = R(e^p \cdot p^2).$$

Now we verify that $I(p^2 \cdot e^p) = I(e^p \cdot p^2)$, i.e. $(a+\bar{a})b\left[2\beta\cos v + \frac{\beta(\bar{a}-a)\sin v}{v}\right] + (a^2 - b\bar{b})\frac{2\beta b \sin v}{v} = \frac{2\beta b \sin v}{v}(\bar{a}^2 - b\bar{b}) + [2\beta\cos v + \frac{\beta(a-\bar{a})\sin v}{v}](a+\bar{a})b$.

Opening the brackets and simplifying expressions, we obtain

$$2a^2\beta b\sin v + \beta\bar{a}\,(a+\bar{a})b\sin v - \beta a\,(a+\bar{a})b\sin v = 2\bar{a}^2\beta b\sin v + \beta a\,(a+\bar{a})b\sin v - \beta\bar{a}\,(a+\bar{a})b\sin v,$$

whence it follows that we have the identity $(a^2+\bar{a}^2)\beta b\sin v = (\bar{a}^2 + a^2)\beta b\sin v$, i.e

$$I(p^2 \cdot e^p) = I(e^p \cdot p^2)$$

is fulfilled.

Thus, it is shown that in principle non-commutative quaternionic multiplication behaves as commutative in the case of quaternionic multiplication of the H-holomorphic functions p^2 *and* e^p.

As shown above, quaternionic generalization (2) of complex Cauchy-Riemann's equations is based on the requirement of unambiguousness of a quaternionic derivative, which is needed to investigate steady state vector fields in 3D space. It is not superfluous to note that the physical formulation of a problem played initially an important role in the theory of complex-differentiable functions, and the so-called complex Cauchy-Riemann equations were found [9] as early as in 1752 in d'Alembert's doctrine about planar fluid flow.

References

[1] Penrose, R., *The Road to Reality: A Complete Guide to the Laws of the Universe*, Random House, 2016.

[2] Gorgone, S., *An Overview and Analysis of Quaternions*, Department of Mathematics, Ithaca College, 2013.

[3] Mathews, J. H., Howell, R. W., *Complex Analysis for Mathematics and Engineering*, 3rd ed., Jones and Bartlett Publishers, 1997.

[4] Parfenov, M., Adequate Quaternionic Generalization of Complex Differentiability, [*viXra: Functions and Analysis*], 2016. Available: vixra.org/abs/1609.0006

[5] Kantor, I. L., Solodovnikov, A. S., *Hypercomplex Numbers. An Elementary Introduction to Algebras,* Springer-Verlag, New York-Berlin-Heidelberg-London-Paris-Tokyo, 1989

[6] Parfenov, M., "On Properties of Holomorphic Functions in Quaternionic Analysis", *American Journal of Mathematical Analysis*, Vol. 5, No. 1, pp. 17-24, 2017. Available: http://pubs.sciepub.com/ajma/5/1/4/index.html#Cor

[7] Parfenov, M., "Essentially Adequate Concept of Holomorphic Functions in Quaternionic Analysis." *American Journal of Mathematical Analysis,* vol. **8**, No. 1, 14-30, 2020. Available: pubs.sciepub.com/ajma/8/1/3/

[8] Parfenov, M., "A Quaternionic Potential Conception with Applying to 3D Potential Fields", *American Journal of Mathematical Analysis*, Vol. 7, No. 1, p.p. 1-10, 2019. Available: pubs.sciepub.com/ajma/7/1/1/index.html

[9] Remmert, R., *Funktionentheorie 1*, Springer Lehrbuch, Ser. Grundwissen Math., 4. Aufl., Springer, 1991.