ASYMPTOTICS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A SPECTRAL PARAMETER

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Abstract. The main goal of this paper is to construct the so-called Birkhoff-type solutions for linear ordinary differential equations with a spectral parameter. Such solutions play an important role in direct and inverse problems of spectral theory. In Section 1, we construct the Birkhoff-type solutions for n-th order differential equations. Section 2 is devoted to first-order systems of differential equations.

Keywords: Birkhoff-type solutions; linear ODU; higher-order differential operators; first-order differential systems; asymptotics of solutions.

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The main goal of this paper is to construct the so-called Birkhoff-type solutions for linear ordinary differential equations with a spectral parameter. Such solutions play an important role in many problems of the spectral theory (see, for example, [1] and references therein). Moreover, they also appear in the inverse problem theory ([2]-[3]).

This paper contains two sections. In Section 1 we construct the Birkhoff-type solutions for n-th order differential equations, and Section 2 is devoted to first-order systems of differential equations. Other results related to this area one can find in [4]-[10].

I. Birkhoff-type solutions for arbitrary order differential equations

1.1. In this section we study the differential equation of order $n \geq 2$:

$$y^{(n)} + \sum_{m=0}^{n-2} p_m(x) y^{(m)} = \rho^n y, \quad 0 \leq x \leq T \leq \infty \tag{1.1}$$

on the finite interval $(T < \infty)$ or on the half-line $(T = \infty)$. Here ρ is the spectral parameter, and $p_m(x) \in L(0,T)$ are complex-valued integrable functions.

Our goal here is to construct a fundamental system of solutions (FSS) $\{y_k(x,\rho)\}_{k=\overline{1,n}}$ of equation (1.1) such that

$$y_k(x,\rho) \sim \exp(\rho R_k x), \quad |\rho| \to \infty,$$

where $R_1, R_2, \ldots, R_n$ are the roots of the equation $R^n - 1 = 0$. We note that the functions $\{\exp(\rho R_k x)\}_{k=\overline{1,n}}$ form the FSS for the "simplest" equation $y^{(n)} = \rho^n y$, when in (1.1) $p_m(x) = 0$, $m = \overline{0, n-2}$.

It is easy to see that the ρ-plane can be partitioned into sectors S of angle $\dfrac{\pi}{n}$ $\Big(\arg\rho \in \Big(\dfrac{\mu\pi}{n}, \dfrac{(\mu+1)\pi}{n}\Big),\ \mu = \overline{0, 2n-1}\Big)$ in which the roots $R_1, R_2, \ldots, R_n$ of the equation $R^n - 1 = 0$ can be numbered in such a way that

$$Re(\rho R_1) < Re(\rho R_2) < \ldots < Re(\rho R_n), \quad \rho \in S. \tag{1.2}$$

Fix $\alpha \in [0,T)$, $k=\overline{1,n}$, and a sector S with the property (1.2). Consider the following integro-differential equation with respect to $y_k(x,\rho)$, $x\in[\alpha,T]$, $\rho\in\overline{S}$:

$$y_k(x,\rho)=\exp(\rho R_k x)-\frac{1}{n\rho^{n-1}}\int_\alpha^x \Big(\sum_{j=1}^k R_j\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt$$

$$+\frac{1}{n\rho^{n-1}}\int_x^T \Big(\sum_{j=k+1}^n R_j\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt. \qquad (1.3)$$

Remark 1.1. We note that the general solution of the differential equation

$$y^{(n)}=\rho^n y+f(x), \quad 0\le x\le T,$$

has the form

$$y(x,\rho)=\sum_{j=1}^n C_j\exp(\rho R_j x)+\frac{1}{n\rho^{n-1}}\sum_{j=1}^n\int_{\gamma_j}^x R_j\exp(\rho R_j(x-t))f(t)\,dt,$$

where $\gamma_j\in[0,T]$ are arbitrary fixed numbers. Clearly, (1.3) corresponds to the case

$$C_j=\begin{cases}0, & j\ne k,\\ 1, & j=k,\end{cases}\qquad \gamma_j=\begin{cases}\alpha, & j\le k,\\ T, & j>k.\end{cases}$$

Let us now transform (1.3) to a system of linear integral equations. Differentiating (1.3) with respect to x , we get for $\nu=\overline{0,n-1}$,

$$y_k^{(\nu)}(x,\rho)=(\rho R_k)^\nu\exp(\rho R_k x)$$

$$-\frac{1}{n\rho^{n-1}}\int_\alpha^x \Big(\sum_{j=1}^k R_j(\rho R_j)^\nu\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt+$$

$$\frac{1}{n\rho^{n-1}}\int_x^T \Big(\sum_{j=k+1}^n R_j(\rho R_j)^\nu\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt. \qquad (1.4)$$

Denote

$$z_{\nu k}(x,\rho)=(\rho R_k)^{-\nu}\exp(-\rho R_k x)y_k^{(\nu)}(x,\rho).$$

Then

$$y_k^{(\nu)}(x,\rho)=(\rho R_k)^\nu\exp(\rho R_k x)z_{\nu k}(x,\rho),$$

and (1.4) implies

$$z_{\nu k}(x,\rho)=$$

$$1-\frac{1}{n\rho^{n-1}}\int_\alpha^x \Big(\sum_{j=1}^k R_j^{\nu+1}R_k^{-\nu}\exp(\rho(R_j-R_k)(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)(\rho R_k)^m z_{mk}(t,\rho)\Big)\,dt$$

$$+\frac{1}{n\rho^{n-1}}\int_x^T \Big(\sum_{j=k+1}^n R_j^{\nu+1}R_k^{-\nu}\exp(\rho(R_j-R_k)(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)(\rho R_k)^m z_{mk}(t,\rho)\Big)\,dt, \quad (1.5)$$

or

$$z_{\nu k}(x,\rho)=1+\sum_{m=0}^{n-2}\int_\alpha^T A_{\nu mk}(x,t,\rho)z_{mk}(t,\rho)\,dt,\quad \nu=\overline{0,n-1}, \tag{1.6}$$

where

$$A_{\nu mk}(x,t,\rho)=\begin{cases} -\dfrac{p_m(t)}{n\rho^{n-1-m}}\displaystyle\sum_{j=1}^{k} R_j^{\nu+1}R_k^{m-\nu}\exp(\rho(R_j-R_k)(x-t)), & x\ge t,\\ \dfrac{p_m(t)}{n\rho^{n-1-m}}\displaystyle\sum_{j=k+1}^{n} R_j^{\nu+1}R_k^{m-\nu}\exp(\rho(R_j-R_k)(x-t)), & x<t. \end{cases} \tag{1.7}$$

For fixed $\rho\in\overline{S}$ and $k=\overline{1,n}$, we consider (1.6) as a system of linear integral equations with respect to $z_{\nu k}(x,\rho)$, $x\in[\alpha,T]$. By virtue of (1.7) and (1.2) we have

$$\int_\alpha^T |A_{\nu mk}(x,t,\rho)|\,dt\ \le \frac{1}{n|\rho|^{n-1-m}}\int_\alpha^T |p_m(t)|\,dt. \tag{1.8}$$

Denote

$$\rho_\alpha:=\max_{m=\overline{0,n-2}}\Big(2\int_\alpha^T |p_m(t)|\,dt\Big)^{\frac{1}{n-1-m}}.$$

It follows from (1.8) that for $|\rho|\ge\rho_\alpha,\ \rho\in\overline{S},\ x\in[\alpha,T],\ k=\overline{1,n},\ \nu=\overline{0,n-1}:$

$$\sum_{m=0}^{n-2}\max_{\alpha\le x\le T}\int_\alpha^T |A_{\nu mk}(x,t,\rho)|\,dt\ \le\frac{1}{2}. \tag{1.9}$$

Solving (1.6) by the method of successive approximations and using (1.9), we obtain that for $|\rho|\ge\rho_\alpha,\ \rho\in\overline{S},\ x\in[\alpha,T],\ k=\overline{1,n},$ system (1.6) has a unique solution $z_{\nu k}(x,\rho),\ \nu=\overline{0,n-1},$ and $|z_{\nu k}(x,\rho)|\le 2$. Substituting this estimate into the right-hand side of (1.6) and using (1.8), we get

$$|z_{\nu k}(x,\rho)-1|\le\frac{C}{|\rho|},\quad |\rho|\ge\rho_\alpha,\ \rho\in\overline{S},\ x\in[\alpha,T],\ k=\overline{1,n},\ \nu=\overline{0,n-1}.$$

In other words, for $|\rho|\to\infty,\ \rho\in\overline{S}$,

$$z_{\nu k}(x,\rho)=1+O\Big(\frac{1}{\rho}\Big),\quad k=\overline{1,n},\ \nu=\overline{0,n-1}, \tag{1.10}$$

uniformly in $x\in[\alpha,T]$. Moreover, one can obain more precise asymptotic formulae for the functions $z_{\nu k}(x,\rho)$ than (1.10). Indeed, substituting (1.10) into the right-hand side of (1.5), we calculate

$$z_{\nu k}(x,\rho)=1-\frac{1}{n\rho}\int_\alpha^x\Big(\sum_{j=1}^{k}R_j^{\nu+1}R_k^{n-\nu-2}\exp(\rho(R_j-R_k)(x-t))\Big)p_{n-2}(t)\,dt$$

$$+\frac{1}{n\rho}\int_x^T\Big(\sum_{j=k+1}^{n}R_j^{\nu+1}R_k^{n-\nu-2}\exp(\rho(R_j-R_k)(x-t))\Big)p_{n-2}(t)\,dt+O\Big(\frac{1}{\rho^2}\Big).$$

The terms with $j \neq k$ give us $o\Big(\dfrac{1}{\rho}\Big)$ as $|\rho| \to \infty,\ \rho \in \overline{S},$ uniformly in $x \in [\alpha, T]$. Hence

$$z_{\nu k}(x,\rho) = 1 + \frac{\beta_1(x)}{\rho R_k} + o\Big(\frac{1}{\rho}\Big), \quad |\rho| \to \infty,\ \rho \in \overline{S},\ x \in [\alpha, T],\ \nu = \overline{0, n-1}, \tag{1.11}$$

where

$$\beta_1(x) = -\frac{1}{n}\int_\alpha^x p_{n-2}(t)\,dt. \tag{1.12}$$

Thus, for $y_k^{(\nu)}(x,\rho),\ k = \overline{1,n},\ \nu = \overline{0,n-1},$ we obtain the asymptotics

$$y_k^{(\nu)}(x,\rho) = (\rho R_k)^\nu \exp(\rho R_k x)\Big(1 + \frac{\beta_1(x)}{\rho R_k} + o\Big(\frac{1}{\rho}\Big)\Big), \quad |\rho| \to \infty,\ \rho \in \overline{S}, \tag{1.13}$$

uniformly in $x \in [\alpha, T]$.

Furthermore, since $z_{\nu k}(x,\rho)$ are solutions of (1.5), the functions $y_k^{(\nu)}(x,\rho)$ satisfy (1.4). Consequently, the functions $y_k(x,\rho)$ are solutions of the differential equation (1.1). Thus, we arrive at the following theorem.

Theorem 1.1. *Fix $\alpha \in [0,T]$ and a sector S with the property (1.2). For $\rho \in \overline{S},\ |\rho| \geq \rho_\alpha,\ x \in [0,T],$ there exists a FSS $\{y_k(x,\rho)\}_{k=\overline{1,n}}$ of the equation (1.1) such that:*
1) the functions $y_k^{(\nu)}(x,\rho),\ \nu = \overline{0,n-1}$ are continuous for $x \in [0,T],\ \rho \in \overline{S},\ |\rho| \geq \rho_\alpha$;
2) for each $x \in [0,T]$, the functions $y_k^{(\nu)}(x,\rho),\ \nu = \overline{0,n-1}$ are analytic with respect to $\rho \in S,\ |\rho| \geq \rho_\alpha$;
3) uniformly for $x \in [\alpha, T]$, the asymptotic formula (1.13) holds, where $\beta_1(x)$ is defined by (1.12);
4) as $|\rho| \to \infty,\ \rho \in \overline{S}$,

$$\det[y_k^{(\nu-1)}(x,\rho)]_{\nu,k=\overline{1,n}} = \rho^{\frac{n(n-1)}{2}} \det[R_k^{\nu-1}]_{\nu,k=\overline{1,n}}\Big(1 + o\Big(\frac{1}{\rho}\Big)\Big).$$

1.2. In this section we discuss the possibility to obtain more precise asymptotic formulae than (1.13). For this purpose we need some smoothness for the coefficients of equation (1.1). Denote by $W_N[a,b]$ the set of functions $f(x),\ x \in [a,b]$ such that the functions $f^{(\nu)}(x),\ \nu = \overline{0, N-1}$ are absolutely continuous, and $f^{(\nu)}(x) \in L(a,b),\ \nu = \overline{0,N}$. For $N \leq 0$ we put $W_N[a,b] = L(a,b)$.

Suppose that $p_{n-2}(x) \in W_1[\alpha, T]$. Substituting (1.11) into the right-hand side of (1.5), we get

$$z_{\nu k}(x,\rho) = 1 - \frac{1}{n(\rho R_k)}\int_\alpha^x p_{n-2}(t)\Big(1 + \frac{\beta_1(t)}{\rho R_k}\Big)\,dt - \frac{1}{n(\rho R_k)^2}\int_\alpha^x p_{n-3}(t)\,dt$$

$$-\frac{1}{n\rho}\int_\alpha^x \Big(\sum_{j=1}^{k-1} R_j^{\nu+1} R_k^{n-\nu-2} \exp(\rho(R_j - R_k)(x-t))\Big) p_{n-2}(t)\,dt$$

$$+\frac{1}{n\rho}\int_x^T \Big(\sum_{j=k+1}^{n} R_j^{\nu+1} R_k^{n-\nu-2} \exp(\rho(R_j - R_k)(x-t))\Big) p_{n-2}(t)\,dt + o\Big(\frac{1}{\rho^2}\Big).$$

Integration by parts yields

$$z_{\nu k}(x,\rho)=1-\frac{1}{n(\rho R_k)}\int_\alpha^x p_{n-2}(t)\,dt-\frac{1}{n(\rho R_k)^2}\int_\alpha^x\Big(p_{n-3}(t)+p_{n-2}(t)\beta_1(t)\Big)\,dt$$

$$-\frac{1}{n(\rho R_k)^2}\sum_{\substack{j=1,\\ j\neq k}}^{n}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}p_{n-2}(x)+\frac{1}{n(\rho R_k)^2}\sum_{j=1}^{k-1}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}\exp(\rho(R_j-R_k)(x-\alpha))p_{n-2}(\alpha)$$

$$+\frac{1}{n(\rho R_k)^2}\sum_{j=k+1}^{n}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}\exp(\rho(R_j-R_k)(x-T))p_{n-2}(T)+o\Big(\frac{1}{\rho^2}\Big).$$

This asymptotic formula contains the terms with exponentials, and it is not convenient for applications. In order to obtain the asymptotics without terms having exponentials, we should modify slightly the original equation (1.3). More precisely, instead of (1.3) we consider the following equation:

$$y_k(x,\rho)=\sum_{j=1}^{n}C_j\exp(\rho R_jx)-\frac{1}{n\rho^{n-1}}\int_\alpha^x\Big(\sum_{j=1}^{k}R_j\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt$$

$$+\frac{1}{n\rho^{n-1}}\int_x^T\Big(\sum_{j=k+1}^{n}R_j\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt, \tag{1.14}$$

where

$$C_j=\begin{cases}-\dfrac{p_{n-2}(\alpha)}{n(\rho R_k)^2}\dfrac{R_j}{R_k-R_j}\exp(\rho(R_k-R_j)\alpha), & \text{for } j=\overline{1,k-1},\\ 1, & \text{for } j=k,\\ -\dfrac{p_{n-2}(T)}{n(\rho R_k)^2}\dfrac{R_j}{R_k-R_j}\exp(\rho(R_k-R_j)T), & \text{for } j=\overline{k+1,n}.\end{cases}$$

This implies

$$y_k^{(\nu)}(x,\rho)=\sum_{j=1}^{n}C_j(\rho R_j)^\nu\exp(\rho R_jx)$$

$$-\frac{1}{n\rho^{n-1}}\int_\alpha^x\Big(\sum_{j=1}^{k}R_j(\rho R_j)^\nu\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt$$

$$+\frac{1}{n\rho^{n-1}}\int_x^T\Big(\sum_{j=k+1}^{n}R_j(\rho R_j)^\nu\exp(\rho R_j(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)y_k^{(m)}(t,\rho)\Big)\,dt. \tag{1.15}$$

Denote

$$z_{\nu k}(x,\rho)=(\rho R_k)^{-\nu}\exp(-\rho R_kx)y_k^{(\nu)}(x,\rho).$$

Then (1.15) becomes

$$z_{\nu k}(x,\rho)=1-\frac{p_{n-2}(\alpha)}{n(\rho R_k)^2}\sum_{j=1}^{k-1}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}\exp(\rho(R_j-R_k)(x-\alpha))$$

$$-\frac{p_{n-2}(T)}{n(\rho R_k)^2}\sum_{j=k+1}^{n}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}\exp(\rho(R_j-R_k)(x-T))$$

$$-\frac{1}{n\rho^{n-1}}\int_\alpha^x\Big(\sum_{j=1}^{k}R_j^{\nu+1}R_k^{-\nu}\exp(\rho(R_j-R_k)(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)(\rho R_k)^m z_{mk}(t,\rho)\Big)\,dt$$

$$+\frac{1}{n\rho^{n-1}}\int_x^T\Big(\sum_{j=k+1}^{n}R_j^{\nu+1}R_k^{-\nu}\exp(\rho(R_j-R_k)(x-t))\Big)\Big(\sum_{m=0}^{n-2}p_m(t)(\rho R_k)^m z_{mk}(t,\rho)\Big)\,dt,$$

$$\nu=\overline{0,n-1}. \tag{1.16}$$

Solving the system (1.16) by the method of successive approximations and repeating the preceding arguments, we obtain that system (1.16) has a unique solution $z_{\nu k}(x,\rho)$ such that (1.10) holds. Substituting (1.10) into the right-hand side of (1.16), we arrive at (1.11). Substituting now (1.11) into the right-hand side of (1.16) we get by the same way as above that

$$z_{\nu k}(x,\rho)=1+\frac{\beta_1(x)}{\rho R_k}+\frac{\beta_{2\nu}(x)}{(\rho R_k)^2}+o\Big(\frac{1}{\rho^2}\Big),\quad |\rho|\to\infty,\ \rho\in\overline{S},\ x\in[\alpha,T], \tag{1.17}$$

where

$$\beta_{2\nu}(x)=-\frac{1}{n}\int_\alpha^x\Big(p_{n-3}(t)+p_{n-2}(t)\beta_1(t)\Big)\,dt-\frac{1}{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}p_{n-2}(x).$$

Since

$$\frac{1}{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{R_j^{\nu+1}R_k^{-\nu}}{R_k-R_j}=-\frac{n-1}{2}+\nu,$$

the asymptotic formula (1.17) takes the form

$$z_{\nu k}(x,\rho)=1+\frac{\beta_1(x)}{\rho R_k}+\frac{\beta_2(x)+\nu\beta_1'(x)}{(\rho R_k)^2}+o\Big(\frac{1}{\rho^2}\Big),\quad |\rho|\to\infty,\ \rho\in\overline{S},\ x\in[\alpha,T], \tag{1.18}$$

where

$$\beta_2(x)=-\frac{1}{n}\int_\alpha^x\Big(p_{n-3}(t)+p_{n-2}(t)\beta_1(t)\Big)\,dt-\frac{n-1}{2}\beta_1'(x).$$

Hence

$$y_k^{(\nu)}(x,\rho)=(\rho R_k)^\nu\exp(\rho R_k x)\Big(1+\frac{\beta_1(x)}{\rho R_k}+\frac{\beta_2(x)+\nu\beta_1'(x)}{(\rho R_k)^2}+o\Big(\frac{1}{\rho^2}\Big)\Big).$$

By the same way one can obtain the following more general assertion.

Theorem 1.2. *Fix $\alpha\in[0,T]$, $N\geq 0$ and a sector S with the property (1.2). Assume that $p_m(x)\in W_{N+m-n+2}[\alpha,T]$, $m=\overline{0,n-2}$. Then for $\rho\in\overline{S}$, $|\rho|\geq\rho_\alpha$, $x\in[0,T]$, there exists a FSS $\{y_k(x,\rho)\}_{k=\overline{1,n}}$ of the equation (1.1) such that:*
1) the functions $y_k^{(\nu)}(x,\rho)$, $\nu=\overline{0,n-1}$, are continuous for $x\in[0,T]$, $\rho\in\overline{S}$, $|\rho|\geq\rho_\alpha$;
2) for each $x\in[0,T]$, the functions $y_k^{(\nu)}(x,\rho)$, $\nu=\overline{0,n-1}$, are analytic with respect to

$\rho \in S,\ |\rho| \ge \rho_\alpha$;
3) as $|\rho| \to \infty,\ \rho \in \overline{S}$, uniformly for $x \in [\alpha, T]$,

$$y_k(x,\rho) = \exp(\rho R_k x)\Big(1 + \sum_{s=1}^{N+1} \frac{\beta_s(x)}{(\rho R_k)^s} + o\Big(\frac{1}{\rho^{N+1}}\Big)\Big), \tag{1.19}$$

$$y_k^{(\nu)}(x,\rho) = (\rho R_k)^\nu \exp(\rho R_k x)\Big(1 + \sum_{s=1}^{N+1} \frac{\beta_{s\nu}(x)}{(\rho R_k)^s} + o\Big(\frac{1}{\rho^{N+1}}\Big)\Big),$$

where

$$\beta_{s\nu}(x) = \sum_{r=0}^{\nu} C_\nu^r \beta_{s-r}^{(r)}(x), \quad C_\nu^r := \frac{\nu!}{r!(\nu-r)!},$$

$$\beta_0(x) = 1, \quad \beta_s(x) = 0 \quad \textit{for } s < 0,$$

$$\beta_s'(x) = -\frac{1}{n}\Big(\sum_{r=2}^{n} C_n^r \beta_{s+1-r}^{(r)}(x) + \sum_{m=0}^{n-2} p_m(x)\beta_{s-n+m+1,m}(x)\Big) \quad \text{for } s \ge 1. \tag{1.20}$$

In particular, (1.20) yields

$$\beta_1'(x) = -\frac{1}{n} p_{n-2}(x),$$

$$\beta_2'(x) = -\frac{1}{n}\Big(C_n^2 \beta_1''(x) + p_{n-2}(x)\beta_1(x) + p_{n-3}(x)\Big),$$

$$\beta_3'(x) = -\frac{1}{n}\Big(C_n^2 \beta_2''(x) + C_n^3 \beta_1'''(x) + p_{n-2}(x)\Big(\beta_2(x) + C_{n-2}^1 \beta_1'(x)\Big) + p_{n-3}(x)\beta_1(x) + p_{n-4}(x)\Big),$$

$$\ldots\ldots$$

The reccurent formula (1.20) for the coefficients $\beta_s(x), s \ge 1$, can be obtained by substitution (1.19) into (1.1). We note that $\beta_s(x) \in W_{N-s+2}[\alpha, T]$.

II. First-order systems of differential equations

2.1. Consider the following system:

$$LY(x) := \frac{1}{\rho} Y'(x) - A(x,\rho) Y(x), \quad 0 \le x \le T \le \infty, \tag{2.1}$$

where $Y = [y_\nu]_{\nu=\overline{1,n}}$ is a column-vector, and the matrix $A(x,\rho)$ has the form

$$A(x,\rho) = A_{(0)} + \sum_{\mu=1}^{\infty} \frac{A_{(\mu)}(x)}{\rho^\mu}, \quad A_{(\mu)} = [a_{(\mu)\nu j}]_{\nu,j=\overline{1,n}}, \tag{2.2}$$

We assume that
(i_1) A_0 is a constant matrix, and its eigenvalues $R_1, R_2, \ldots, R_n$ are such that $R_k \ne 0,\ R_j \ne R_k\ (j \ne k)$;
(i_2) for $\mu \ge 1,\ \nu, j = \overline{1,n},\quad a_{(\mu)\nu j}(x) \in L(0,T)$, and $\|a_{(\mu)\nu j}(x)\|_{L(0,T)} \le C a_*^\mu,\ a_* \ge 0$.

We shall say that $L \in \Lambda_0$, if (i_1)-(i_2) hold. We also consider the classes $\Lambda_N \subset \Lambda_0,\ (N \ge 1)$ with additional smoothness properties of $A_{(\mu)}(x)$. More precisely, we shall say that $L \in \Lambda_N$ if (i_1)-(i_2) hold, and $a_{(\mu)\nu j}(x) \in W_{N-\mu+1}[0,T]$ for $\mu = \overline{1,N}$, $\nu, j = \overline{1,n}$. Note that $N = 1$ is the most popular case in applications.

2.2. In Sections 2.2-2.3 we provide some formal calculations in order to show ideas. For the explicit results see Section 2.4.

Acting formally one can seek solutions $Y_k(x,\rho)$, $k = \overline{1,n}$, of system (2.1) in the form

$$Y_k(x,\rho) = \exp(\rho R_k x) \sum_{\mu=0}^{\infty} \frac{g_{(\mu)k}(x)}{\rho^\mu}, \tag{2.3}$$

where $g_{(\mu)k}(x) = [g_{(\mu)\nu k}(x)]_{\nu=\overline{1,n}}$ are column-vectors. Substituting (2.3) into (2.1) we get formally

$$LY_k(x,\rho) = \exp(\rho R_k x) \sum_{\mu=0}^{\infty} \frac{1}{\rho^\mu} \Big\{ \Big(R_k g_{(\mu)k}(x) + g'_{(\mu-1)k}(x) \Big)$$

$$-\Big(A_{(0)} g_{(\mu)k}(x) + A_{(1)}(x) g_{(\mu-1)k}(x) + \ldots + A_{(\mu)}(x) g_{(0)k}(x) \Big) \Big\} = 0$$

(here we put $g_{(\mu)k}(x) = 0$ for $\mu < 0$). This yields the following reccurent formulae for constructing the coefficients $g_{(\mu)k}(x)$:

$$\left.\begin{aligned} A_{(0)} g_{(0)k}(x) - R_k g_{(0)k}(x) = 0, \\ A_{(0)} g_{(\mu)k}(x) - R_k g_{(\mu)k}(x) = \\ g'_{(\mu-1)k}(x) - \Big(A_{(1)}(x) g_{(\mu-1)k}(x) + \ldots + A_{(\mu)}(x) g_{(0)k}(x) \Big),\ \mu \ge 1. \end{aligned}\right\} \tag{2.4}$$

For example, if $A_{(0)} = diag[R_k]_{k=\overline{1,n}}$, then one can take

$$g_{(0)\nu k}(x) = Q_k(x)\delta_{\nu k}, \quad Q_k(x) = \exp\Big(\int_0^x a_{(1)kk}(\xi)\, d\xi \Big), \tag{2.5}$$

where $\delta_{\nu k}$ is the Kronecker symbol. Then

$$g_{(1)\nu k}(x) = \frac{a_{(1)\nu k}(x) Q_k(x)}{R_k - R_\nu}, \quad \nu \ne k,$$

$$g'_{(1)kk}(x) = a_{(2)kk}(x) Q_k(x) + \sum_{\nu=1}^{n} a_{(1)k\nu}(x) g_{(1)\nu k}(x),$$

$$\ldots\ldots$$

2.3. Denote

$$L^* Z(x) := \frac{1}{\rho} Z'(x) + Z(x) A(x,\rho),$$

where $Z = [z_1, \ldots, z_n]$ is a row-vector. Then

$$ZLY + L^* ZY = \frac{1}{\rho} \frac{d}{dx}(ZY), \tag{2.6}$$

and consequently,

$$\int_a^b ZLY\, dx = \frac{1}{\rho} (ZY)\Big|_a^b - \int_a^b L^* ZY\, dx. \tag{2.7}$$

Let $U(x) = [u_{jk}(x)]_{j,k=\overline{1,n}}$ be an absolutely continuous non-degenerate matrix, and let $V(x) = [v_{jk}(x)]_{j,k=\overline{1,n}}$ be the inverse matrix: $V(x) = (U(x))^{-1}$. Denote

$$U_k(x) = \begin{bmatrix} u_{1k}(x) \\ \dots \\ u_{nk}(x) \end{bmatrix}, \qquad V_j(x) = [v_{j1}(x), \dots, v_{jn}(x)].$$

Let us show that system (2.1) is equivalent to the integral equation

$$Y(x) = \sum_{j=1}^{n} U_j(x)\Big(I_j + \rho \int_{\gamma_j}^{x} L^* V_j(t) Y(t)\, dt\Big), \quad \gamma_j \in [0, T], \tag{2.8}$$

where I_j are arbitrary constants.

Indeed, assume that $Y(x)$ satisfies (2.1), i.e. $LY(x) = 0$. Then, by virtue of (2.7),

$$(V_j Y)\Big|_{\gamma_j}^{x} = \rho \int_{\gamma_j}^{x} L^* V_j(t) Y(t)\, dt, \quad j = \overline{1, n}, \tag{2.9}$$

where $\gamma_j \in [0, T]$ are arbitrary fixed numbers. This implies

$$V_j(x) Y(x) = I_j + \rho \int_{\gamma_j}^{x} L^* V_j(t) Y(t)\, dt, \quad j = \overline{1, n}, \tag{2.10}$$

where $I_j = V_j(\gamma_j) Y(\gamma_j)$. Since $V = U^{-1}$, we arrive at (2.8).

Inversly, if Y(x) satisfies (2.8) with certain constants I_j and $\gamma_j \in [0, T]$, then (2.10) is valid. In particular, this yields $I_j = V_j(\gamma_j) Y(\gamma_j)$, and consequently, one gets (2.9). Taking now (2.7) into account, we obtain

$$\int_{\gamma_j}^{x} V_j(t) LY(t)\, dt = 0, \quad j = \overline{1, n},$$

i.e. $V_j(x) LY(x) = 0$, $j = \overline{1, n}$. Since $\det V(x) \neq 0$, we get $LY(x) = 0$, i.e. $Y(x)$ is a solution of the equation (2.1).

Using the integral equation (2.8) with concrete $U(x), I_j$ and γ_j, one can obtain various solutions of system (2.1) having desirable properties.

2.4. In this section we construct the Birkhoff-type solutions for system (2.1). The main result is formulated in Theorem 2.1.

Fix a sector S in the ρ-plane such that

$$Re(\rho R_1) < \dots < Re(\rho R_n), \quad \rho \in S. \tag{2.11}$$

Theorem 2.1. *Fix $N \geq 0$, and assume that $L \in \Lambda_N$. Then, there exists $\rho_* > 0$ such that for $\rho \in \overline{S}$, $|\rho| \geq \rho_*$, $x \in [0, T]$, system (2.1) has a FSS $\{Y_k(x, \rho)\}_{k=\overline{1,n}}$ with the following properties:*
1) $Y_k(x, \rho)$ are continuous for $x \in [0, T]$, $\rho \in \overline{S}$, $|\rho| \geq \rho_*$;

2) for each $x \in [0,T]$, $Y_k(x,\rho)$ *are analytic with respect to* $\rho \in S$, $|\rho| \ge \rho_*$;
3) for $\rho \in \overline{S}$, $|\rho| \ge \rho_*$, $x \in [0,T]$,

$$Y_k(x,\rho) = \exp(\rho R_k x)\Big(\sum_{\mu=0}^{N} \frac{g_{(\mu)k}(x)}{\rho^\mu} + \frac{\varepsilon_k(x,\rho)}{\rho^N}\Big), \tag{1.12}$$

where the column-vectors $g_{(\mu)k}(x) = [g_{(\mu)\nu k}(x)]_{\nu=\overline{1,n}}$ *are defined by (2.4), and for the vector* $\varepsilon_k(x,\rho) = [\varepsilon_{\nu k}(x,\rho)]_{\nu=\overline{1,n}}$,

$$\lim_{\substack{|\rho|\to\infty \\ \rho\in\overline{S}}} \max_{0\le x\le T} |\varepsilon_{\nu k}(x,\rho)| = 0, \quad \nu, k = \overline{1,n},$$

i.e. uniformly in $x \in [0,T]$, $\varepsilon_k(x,\rho) = o(1)$ *as* $|\rho| \to \infty$, $\rho \in \overline{S}$.

Proof. Without loss of generality we consider here the case when $A_{(0)}$ is a diagonal matrix:

$$A_{(0)} = diag[R_k]_{k=\overline{1,n}}.$$

The general case is studied in the end of the proof by reduction to the diagonal case.

Define the matrix $U(x,\rho) = [u_{jk}(x,\rho)]_{j,k=\overline{1,n}}$ with the columns $U_k(x,\rho) = [u_{jk}(x,\rho)]_{j=\overline{1,n}}$ by the formula

$$U_k(x,\rho) = \exp(\rho R_k x) \sum_{\mu=0}^{N} \frac{g_{(\mu)k}(x)}{\rho^\mu}, \quad k = \overline{1,n}, \tag{2.13}$$

where $g_{(\mu)k}(x)$ satisfy (2.4) and (2.5) holds. Let $V(x,\rho) = (U(x,\rho))^{-1}$ be inverse matrix with the rows $V_j(x,\rho) = [v_{j1}(x,\rho), \dots, v_{jn}(x,\rho)]$. For each fixed $k = \overline{1,n}$ and $\rho \in \overline{S}$, we consider the integral equation

$$Y_k(x,\rho) = U_k(x,\rho) + \rho \int_0^x \Big(\sum_{j=1}^{k} U_j(x,\rho) L^* V_j(t,\rho)\Big) Y_k(t,\rho)\, dt$$

$$-\rho \int_x^T \Big(\sum_{j=k+1}^{n} U_j(x,\rho) L^* V_j(t,\rho)\Big) Y_k(t,\rho)\, dt, \tag{2.14}$$

with respect to the column-vector Y_k. We note that (2.14) is a particular case of (2.8) when

$$I_j = \begin{cases} 0, & j \ne k, \\ 1, & j = k, \end{cases} \qquad \gamma_j = \begin{cases} 0, & j \le k, \\ T, & j > k. \end{cases}$$

In order to study the solvability of (2.14) we need some preliminary calculations. Denote by Γ_a the set of functions $\theta(x,\rho)$ of the form

$$\theta(x,\rho) = \sum_{\mu=0}^{\infty} \frac{\theta_{(\mu)}(x)}{\rho^\mu},$$

where $\theta_{(\mu)}(x) \in L(0,T)$, and $\|\theta_{(\mu)}(x)\|_{L(0,T)} \le C a^\mu$.

It follows from (2.13), (2.4) and (2.5) that

$$LU_k(x,\rho) = \frac{1}{\rho^{N+1}} \exp(\rho R_k x)\Big(H_{(0)k}(x) + \frac{H_{(1)k}(x,\rho)}{\rho}\Big), \tag{2.15}$$

where $H_{(\mu)k} = [H_{(\mu)\nu k}]_{\nu=\overline{1,n}}$ are column-vectors such that

$$H_{(0)k}(x) = g'_{(N)k}(x) - \Big(A_{(1)}(x)g_{(N)k}(x) + \ldots + A_{(N+1)}(x)g_{(0)k}(x)\Big), \tag{2.16}$$

$$H_{(1)\nu k}(x,\rho) \in \Gamma_{a_*}. \tag{2.17}$$

In particular, (2.16) yields

$$H_{(0)kk}(x) = 0, \quad H_{(0)\nu k}(x) \in L[0,T],\ \nu \neq k. \tag{2.18}$$

The next step is to calculate the inverse matrix $V(x,\rho) = (U(x,\rho))^{-1}$. Since

$$g_{(0)k}(x) = [\delta_{\nu k} Q_k(x)]_{\nu=\overline{1,n}},$$

it follows from (2.13) that

$$\Big(\det U(x,\rho)\Big)^{-1} = \exp\Big(-\rho x \sum_{k=1}^{n} R_k\Big)\Big(\prod_{k=1}^{n} Q_k(x)\Big)^{-1}\Big(1 + \frac{\theta(x,\rho)}{\rho}\Big),$$

and $\theta(x,\rho) \in \Gamma_{a_1}$ for a certain $a_1 > 0$. Therefore,

$$V_j(x,\rho) = \exp(-\rho R_j x)\Big(g^*_{(0)j}(x) + \frac{g^*_{(1)j}(x,\rho)}{\rho}\Big), \quad j = \overline{1,n}, \tag{2.19}$$

where $g^*_{(\mu)j} = [g^*_{(\mu)j1}, \ldots, g^*_{(\mu)jn}]$, $\mu = 0,1$, are row-vectors such that

$$g^*_{(0)j\nu}(x) = \Big(Q_j(x)\Big)^{-1}\delta_{j\nu}, \quad g^*_{(1)j\nu}(x,\rho) \in \Gamma_{a_1}. \tag{2.20}$$

By virtue of (2.15), (2.17)-(2.20),

$$VLU = \Big[\frac{1}{\rho^{N+1}}\Big(h_{(0)jk}(x) + \frac{h_{(1)jk}(x,\rho)}{\rho}\Big)\exp(\rho(R_k - R_j)x)\Big]_{j,k=\overline{1,n}}, \tag{2.21}$$

where

$$h_{(0)jk}(x) = \Big(Q_j(x)\Big)^{-1} H_{(0)jk}(x),\ h_{(1)jk}(x,\rho) \in \Gamma_{a_2}, \quad j,k = \overline{1,n},\ a_2 = \max(a_*, a_1). \tag{2.22}$$

In particular, $h_{(0)kk}(x) = 0$.

Furthermore, since $VU = E$ is the indentity matrix, it follows from (2.6) that

$$VLU + L^* VU = 0,$$

i.e.

$$L^* V = -VLUV. \tag{2.23}$$

Using (2.23) and (2.19)-(2.22) we calculate

$$L^* V_j(x,\rho) = \frac{1}{\rho^{N+1}}\exp(-\rho R_j x)\Big(\omega_{(0)j}(x) + \frac{\omega_{(1)j}(x,\rho)}{\rho}\Big), \quad j = \overline{1,n}, \tag{2.24}$$

where $\omega_{(\mu)j} = [\omega_{(\mu)j1}, \ldots, \omega_{(\mu)jn}]$, $\mu = 0, 1$, are row-vectors such that

$$\omega_{(0)jk}(x) = -\Big(Q_j(x)Q_k(x)\Big)^{-1} H_{(0)jk}(x),\ \omega_{(1)jk}(x,\rho) \in \Gamma_{a_2}, \quad j,k = \overline{1,n}. \tag{2.25}$$

In particular,

$$\omega_{(0)kk}(x) = 0, \quad k = \overline{1,n}. \tag{2.26}$$

Denote

$$W_k^0(x,\rho) = \sum_{\mu=0}^{N} \frac{g_{(\mu)k}(x)}{\rho^\mu}. \tag{2.27}$$

By the replacement

$$Y_k(x,\rho) = \exp(\rho R_k x) W_k(x,\rho), \tag{2.28}$$

we transform the integral equation (2.14) to the form

$$W_k(x,\rho) = W_k^0(x,\rho) +$$

$$\frac{1}{\rho^N}\int_0^x \Big(\sum_{j=1}^{k} \exp\Big(\rho(R_j - R_k)(x-t)\Big) W_j^0(x,\rho)\Big(\omega_{(0)j}(t) + \frac{\omega_{(1)j}(t,\rho)}{\rho}\Big)\Big) W_k(t,\rho)\,dt$$

$$-\frac{1}{\rho^N}\int_x^T \Big(\sum_{j=k+1}^{n} \exp\Big(\rho(R_j - R_k)(x-t)\Big) W_j^0(x,\rho)\Big(\omega_{(0)j}(t) + \frac{\omega_{(1)j}(t,\rho)}{\rho}\Big)\Big) W_k(t,\rho)\,dt, \tag{2.29}$$

or

$$W_k(x,\rho) = W_k^0(x,\rho) + \frac{1}{\rho^N}\int_0^T B_k(x,t,\rho) W_k(t,\rho)\,dt, \tag{2.30}$$

where the matrix $B_k = [B_{k\nu s}]_{\nu,s=\overline{1,n}}$ has the form

$$B_k(x,t,\rho) = \begin{cases} \displaystyle\sum_{j=1}^{k} \exp\Big(\rho(R_j - R_k)(x-t)\Big) W_j^0(x,\rho)\Big(\omega_{(0)j}(t) + \frac{\omega_{(1)j}(t,\rho)}{\rho}\Big), & x \ge t, \\ \displaystyle -\sum_{j=k+1}^{n} \exp\Big(\rho(R_j - R_k)(x-t)\Big) W_j^0(x,\rho)\Big(\omega_{(0)j}(t) + \frac{\omega_{(1)j}(t,\rho)}{\rho}\Big), & x < t. \end{cases} \tag{2.31}$$

Solving (2.30) by the method of successive approximations and using (2.24)-(2.27), we obtain that there exists $\rho_* > 0$ such that for $\rho \in \overline{S}$, $|\rho| \ge \rho_*$, $x \in [0,T]$, $k = \overline{1,n}$, the integral equation (2.30) has a unique solution $W_k(x,\rho)$ having the following asymptotics

$$W_k(x,\rho) = W_k^0(x,\rho) + o\Big(\frac{1}{\rho^N}\Big), \quad |\rho| \to \infty,\ \rho \in \overline{S}, \tag{2.32}$$

uniformly for $x \in [0,T]$.

Indeed, let $N \ge 1$. It follows from (2.31), (2.11), (2.25) and (2.27) that

$$\max_{0 \le x \le T} \int_0^T |B_{k\nu s}(x,t,\rho)|\,dt \le C, \quad |\rho| \ge a_3,\ \rho \in \overline{S},\ k,\nu,s = \overline{1,n},\ a_3 > a_2.$$

Then the method of successive approximations gives

$$W_k(x,\rho) = W_k^0(x,\rho) + O\Big(\frac{1}{\rho^N}\Big), \quad |\rho| \to \infty,\ \rho \in \overline{S},\ k = \overline{1,n}, \tag{2.33}$$

uniformly in $x \in [0,T]$. Substituting (2.33) into the right-hand side of (2.29) we get

$$W_k(x,\rho) = W_k^0(x,\rho) + \frac{1}{\rho^N} \int_0^x \Big(\sum_{j=1}^k \exp\Big(\rho(R_j - R_k)(x-t)\Big) g_{(0)j}(x)\omega_{(0)j}(t) g_{(0)k}(t)\,dt$$

$$-\frac{1}{\rho^N} \int_x^T \Big(\sum_{j=k+1}^n \exp\Big(\rho(R_j - R_k)(x-t)\Big) g_{(0)j}(x)\omega_{(0)j}(t) g_{(0)k}(t)\,dt + O\Big(\frac{1}{\rho^{N+1}}\Big).$$

The terms with $j \neq k$ give us $o\Big(\frac{1}{\rho^N}\Big)$ as $\rho \to \infty$, $\rho \in \overline{S}$ uniformly in $x \in [0,T]$. The term with $j = k$ is equal to zero, since according to (2.5) and (2.26),

$$\omega_{(0)k}(t) g_{(0)k}(t) = 0.$$

Thus, for $N \geq 1$, (2.32) is proved. The case $N = 0$ requires different calculations (see [7], [9] for details).

Using (2.28), (2.32) and (2.27), we arrive at (2.12), and consequently Theorem 2.1 is proved for the case when $A_{(0)}$ is a diagonal matrix.

Now we study the general case when $A_{(0)}$ is an arbitrary matrix with the property (i_1). Let

$$\Omega_k = [\Omega_{\nu k}]_{\nu=\overline{1,n}}, \quad k = \overline{1,n},$$

be eigenvectors of $A_{(0)}$ for the eigenvalues R_k, $k = \overline{1,n}$. Consider the matrix

$$\Omega = [\Omega_1, \dots, \Omega_n] = [\Omega_{\nu k}]_{\nu,k=\overline{1,n}}.$$

By the replacement

$$Y(x) = \Omega \tilde{Y}(x),$$

we transform system (2.1) to the form

$$\frac{1}{\rho}\tilde{Y}'(x) = \tilde{A}(x,\rho)\tilde{Y}(x), \tag{2.34}$$

where

$$\tilde{A}(x,\rho) = \Omega^{-1} A(x,\rho)\Omega.$$

Clearly,

$$\tilde{A}(x,\rho) = \tilde{A}_{(0)} + \sum_{\mu=0}^{\infty} \frac{\tilde{A}_{(\mu)}(x)}{\rho^\mu}, \quad \tilde{A}_{(\mu)} = [\tilde{a}_{(\mu)\nu j}]_{\nu,j=\overline{1,n}},$$

and

$$\tilde{A}_{(\mu)} = \Omega^{-1} A_{(\mu)}\Omega, \quad \tilde{A}_{(0)} = diag[R_k]_{k=\overline{1,n}}.$$

For system (2.34), Theorem 2.1 has been already proved. Thus, Theorem 2.1 holds also for an arbitrary $L \in \Lambda_N$. Moreover,

$$g_{(0)k}(x) = \tilde{Q}_k(x)\Omega_k, \quad \tilde{Q}_k(x) = \exp\Big(\int_0^x \tilde{a}_{(1)kk}(\xi)\,d\xi\Big).$$

2.5. In order to prove Theorem 2.1 one can also use the following arguments. Fix a sector S with the property (2.11) and consider system (2.1) for $\rho \in \overline{S}$:

$$\frac{1}{\rho} Y' = A(x,\rho)Y \tag{2.35}$$

with respect to the matrix $Y = [y_{jk}]_{j,k=\overline{1,n}}$. Let for definiteness, $A_{(0)} = diag[R_k]_{k=\overline{1,n}}$.

By the replacement

$$Y = U\xi, \quad \xi = [\xi_{jk}]_{j,k=\overline{1,n}},$$

where the matrix U is defined by (2.13), we reduce (2.35) to the system

$$\frac{1}{\rho} \xi' = -VLU\xi \tag{2.36}$$

with respect to ξ. For the matrix VLU we have the representation (2.21). Then (2.36) becomes

$$\xi'_{jk} = -\frac{1}{\rho^N} \sum_{\nu=1}^{n} h_{j\nu}(x,\rho) \exp\Big(\rho(R_\nu - R_j)x\Big)\xi_{\nu k}, \quad j,k = \overline{1,n}, \tag{2.37}$$

where

$$h_{j\nu}(x,\rho) = h_{(0)j\nu}(x) + \frac{h_{(1)j\nu}(x,\rho)}{\rho}.$$

Consider the integral equations

$$\xi_{jk}(x,\rho) = \delta_{jk} - \frac{1}{\rho^N} \int_{\gamma_{jk}}^{x} \sum_{\nu=1}^{n} h_{j\nu}(t,\rho) \exp\Big(\rho(R_\nu - R_j)t\Big)\xi_{\nu k}(t,\rho)\,dt, \ \gamma_{jk} = \begin{cases} 0, & j \le k, \\ T, & j > k. \end{cases} \tag{2.38}$$

Clearly, if $\{\xi_{jk}\}$ is a solution of (2.38), then $\{\xi_{jk}\}$ satisfy (2.37).

By the replacement

$$\xi_{jk}(x,\rho) = \exp\Big(\rho(R_k - R_j)x\Big)\eta_{jk}(x,\rho),$$

we reduce (2.38) to the system

$$\eta_{jk}(x,\rho) = \delta_{jk} - \frac{1}{\rho^N} \int_{\gamma_{jk}}^{x} \exp\Big(\rho(R_j - R_k)(x-t)\Big) \sum_{\nu=1}^{n} h_{j\nu}(t,\rho)\eta_{\nu k}(t,\rho)\,dt, \ j,k = \overline{1,n}. \tag{2.39}$$

Solving (2.39) by the method of successive approximations we obtain that there exists $\rho_* > 0$ such that for $\rho \in \overline{S}$, $|\rho| \ge \rho_*$, $x \in [0,T]$, system (2.39) has a unique solution $\eta_{jk}(x,\rho)$ having the asymtotics

$$\eta_{jk}(x,\rho) = \delta_{jk} + o\Big(\frac{1}{\rho^N}\Big), \rho \to \infty, \ \rho \in \overline{S}, \tag{2.40}$$

uniformly in $x \in [0,T]$. Sinse

$$y_{jk}(x,\rho) = \sum_{\nu=1}^{n} u_{j\nu}(x,\rho)\eta_{\nu k}(x,\rho) \exp\Big(\rho(R_k - R_\nu)x\Big),$$

we infer for the columns $Y_k = [y_{jk}]_{j,k=\overline{1,n}}$:

$$Y_k(x,\rho) = \exp(\rho R_k x) \sum_{\nu=1}^{n} W_\nu^0(x,\rho)\eta_{\nu k}(x,\rho).$$

Taking (2.40) into account we arrive at the assertions of Theorem 2.1.

2.8. In this section we consider the n-th order differential equation of the form

$$\ell y := y^{(n)} + \sum_{k=0}^{n-1} P_k(x,\rho) y^{(k)} = 0, \quad 0 \le x \le T \le \infty, \tag{2.41}$$

where

$$P_k(x,\rho) = \rho^{n-k} p_{kk} + \rho^{n-k+1} p_{k,k+1}(x) + \ldots + p_{kn}(x).$$

We assume that
(i_1) $p_{kk},\ k=\overline{0,n-1}$ are constants, $p_{00} \neq 0$, and the roots $R_1,\ldots,R_n$ of the characteristic polynomial

$$F(R) = \sum_{k=0}^{n} p_{kk} R^k, \quad p_{nn} := 1,$$

are simple;
(i_2) $p_{k,k+j}(x) \in L(0,T),\ k=\overline{0,n-1},\ j\ge 1$.

We shall say that $\ell \in \Lambda_0'$ if (i_1)-(i_2) hold. If additionally, for a certain $N \ge 1$, $p_{k,k+j}(x) \in W_{N-j+1}[0,T],\ j=\overline{1,N}$, we shall say that $\ell \in \Lambda_N'$.

Fix $N \ge 0$ and a sector S in the ρ-plane with the property (2.11).

Theorem 2.2. *Assume that $\ell \in \Lambda_N'$. Then there exists $\rho_* > 0$ such that for $\rho \in \overline{S},\ |\rho| \ge \rho_*,\ x \in [0,T]$, equation (2.41) has a FSS $\{y_k(x,\rho)\}_{k=\overline{1,n}}$ with the following properties:*
1) the functions $y_k^{(\nu-1)}(x,\rho),\ k,\nu=\overline{1,n}$ are continuous for $x\in[0,T],\ \rho\in\overline{S},\ |\rho|\ge\rho_*$;
2) for each $x\in[0,T]$, the functions $y_k^{(\nu-1)}(x,\rho),\ k,\nu=\overline{1,n}$, are analytic with respect to $\rho \in S,\ |\rho|\ge\rho_*$;
3) as $|\rho|\to\infty,\ \rho\in\overline{S}$, uniformly in $x\in[0,T]$,

$$y_k^{(\nu-1)}(x,\rho) = (\rho R_k)^{\nu-1} \exp(\rho R_k x)\Big(\sum_{\mu=0}^{N} \frac{G_{(\mu)\nu k}(x)}{\rho^\mu} + o\Big(\frac{1}{\rho^N}\Big)\Big), \quad k,\nu=\overline{1,n}, \tag{2.42}$$

where

$$G_{(\mu)\nu k}(x) = \sum_{j=0}^{\nu-1} C_{\nu-1}^j R_k^{\nu-1-j} G_{(\mu-j)0k}^{(j)},$$

and $G_{(\mu)0k}(x)$ can be calculated by substitution (2.42) into (2.41). In particular,

$$G_{(0)\nu k} = \exp\Big(\int_0^x \omega_k(\xi)\,d\xi\Big), \quad \omega_k(x) = -\frac{1}{F'(R_k)} \sum_{j=0}^{n-1} p_{j,j+1}(x) R_k^j. \tag{2.43}$$

Proof. We transform (2.41) to the form

$$\frac{y^{(n)}}{\rho^n} + \sum_{k=0}^{n-1} \mathcal{P}_k(x,\rho) \frac{y^{(k)}}{\rho^k} = 0, \tag{2.44}$$

where

$$\mathcal{P}_k(x,\rho) = \frac{1}{\rho^{n-k}} P_k(x,\rho) = \sum_{\mu=0}^{n-k} \frac{p_{k,k+\mu}(x)}{\rho^\mu}.$$

Denote

$$y_1 = y,\ y_2 = \frac{1}{\rho}\, y', \dots, y_n = \frac{1}{\rho}\, y^{(n-1)}.$$

Then equation (2.44) is equivalent to the system

$$\frac{1}{\rho} \begin{bmatrix} y_1' \\ y_2' \\ \dots \\ y_{n-1}' \\ y_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -\mathcal{P}_0(x,\rho) & -\mathcal{P}_1(x,\rho) & -\mathcal{P}_2(x,\rho) & \dots & -\mathcal{P}_{n-1}(x,\rho) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{n-1} \\ y_n \end{bmatrix}. \tag{2.45}$$

System (2.45) is a particular case of (2.1) with

$$A(x,\rho) = A_{(0)} + \sum_{\mu=1}^{n} \frac{A_{(\mu)}(x)}{\rho^\mu},$$

$$A_{(0)} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -p_{00} & -p_{11} & -p_{22} & \dots & -p_{n-1,n-1} \end{bmatrix},$$

$$A_{(1)}(x) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ -p_{01}(x) & -p_{12}(x) & \dots & -p_{n-1,n}(x) \end{bmatrix},$$

$$A_{(2)}(x) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ -p_{02}(x) & -p_{13}(x) & \dots & -p_{n-2,n}(x) & 0 \end{bmatrix},$$

$$A_{(n)}(x) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ -p_{0n}(x) & 0 & \dots & 0 \end{bmatrix}.$$

Clearly, $\Omega_k = [R_k^{\nu-1}]_{\nu=\overline{1,n}}$, $k=\overline{1,n}$ are eigenvectors of $A_{(0)}$ for the eigenvalues R_k. Thus, Theorem 2.2 follows from Theorem 2.1 with $\tilde a_{(1)kk}(x) = \omega_k(x)$, $\tilde Q_k(x) = G_k(x)$. □

References

[1] Naimark M.A., Linear Differential Operators, 2nd ed., Nauka, Moscow, 1969; English transl. of 1st ed., Parts I,II, Ungar, New York, 1967, 1968.

[2] Beals R., Deift P. and Tomei C., Direct and Inverse Scattering on the Line, Math. Surveys and Monographs, v.28. Amer. Math. Soc. Providence: RI, 1988.

[3] Yurko V.A., Inverse Spectral Problems for Differential Operators and their Applications, Gordon and Breach, New York, 1999.

[4] Tamarkin J.D., On Certain General Problems of Theory of Ordinary Linear Differential Equations, Petrograd, 1917.

[5] Rasulov M.L., Contour Integral Method, Nauka, Moscow, 1964. English transl. Amsterdam, 1967.

[6] Vagabov A.I., Asymptotic behavior of solutions of differential equations with respect to a parameter, and applications. Dokl. Akad. Nauk 326 (1992), no.2, 219-223; transl. in Russian Acad. Sci. Dokl. Math. 46 (1993), no.2, 240-244.

[7] Vagabov A.I., On sharpening an asymptotic theorem of Tamarkin. Diff. Uravneniya 29 (1993), no.1, 41-49; transl. in Diff. Equations 29 (1993), no.1, 33-41.

[8] Rykhlov V., Asymptotics of a system of solutions of a differential equation of general form with a parameter. Ukrain. Mat. Zh. 48 (1996), no. 1, 96-108.

[9] Rykhlov V., Asymptotical formulas for solutions of linear differential systems of the first order, Results Math. 36 (1999), 342-353.

[10] Savchuk A.M., Shkalikov A.A. Asymptotic analysis of solutions of ordinary differential equations with distribution coefficients, Sb. Math. 211 (2020), no. 11, 1623-1659.

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