$\mathbb{R}$-Algebras of dimension 1, 2, 4 or 8

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Abstract : We will show that it is possible to find a few parameters that describes several "interesting" $\mathbb{R}$-Algebras of dimension 4 and 8 (we will recall the very easy and well known 1 and 2 cases).

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I Introduction

We will deal with finite dimensional hypercomplex numbers, that is to say, $\mathbb{R}$-algebras whose elements can be written (in dimension n) :

$$x = \sum_{i=0}^{n-1} a_i e_i$$

Where $a_i \in \mathbb{R}$, and e_i are elements of a basis of the underlying vector space in particular $e_0 = 1$; by bilinearity of the multiplication, such an algebra is fully described by the multiplication table of a basis.

II Dimension 1

This case is very simple : there exists only one $\mathbb{R}$-Algebra of dimension 1 : $\mathbb{R}$ itself, the natural basis is $\langle\ 1\ \rangle$.

III Dimension 2

The dimension 2 is very well known, a basis of such an algebra is $\langle\ 1, e\ \rangle$, and the multiplication table is given by :

$\cdot$	1	e
1	1	e
e	e	e^2

$\mathbb{R}^2$

Where $e^2 = a + b \cdot e$ for some a and b two real numbers. This algebra depends only on the value of e^2, and it is well known that there exists only 3 different such algebras (up to isomorphism) according to the value of $e^2 \in \{-1, 0, 1\}$, when $e = -1$, it is usually noted i, when $e = 0$, it is usually noted ε, when $e = 1$, it is usually noted j :

$\cdot$	1	i
1	1	i
i1	i	-1

Complex Numbers : $\mathbb{C}$

$\cdot$	1	ε
1	1	ε
ε	ε	0

Dual Numbers : $\mathbb{D}_1$

$\cdot$	1	j
1	1	j
j	j	1

Perplex[1] Numbers : $\mathbb{C}$̸

IV Dimension 4

Starting with dimension 4 there are too many possibilities to give a complete list of the $\mathbb{R}$-algebras, so we add some constraints[2] :

Elements of a $\mathbb{R}$-algebra of dimension 4 can be written : $z = a_0 + a_1 \cdot e_1 + a_2 \cdot e_2 + a_3 \cdot e_3$, where $a_i \in \mathbb{R}$. For sake of simplicity we will not use the $\cdot$ symbol anymore.

1. The Perplex Numbers are sometimes called Split Complex (hence the symbol : $\mathbb{C}$̸) or Hyperbolic Numbers.
2. I found here, in the hypercomplex group, the root of this idea but modified it a bit.

IV.1 Parametrization for Dimension 4

The following rules are very easy to encode in a spreadsheet, to experiment the different possibilities and to create new algebras (particularly the algebras not mentionned here (a few), but beware of the existence of an isomorphism with well known algebra.

① 1 is the unit
② $e_3 = e_1 e_2$
③ $e_1^2 \in \{-1, 0, 1\}$ and $e_2^2 \in \{-1, 0, 1\}$
④ There exists $\alpha \in \{-1, 1\}$ (hence $\alpha^2 = 1$) such that, for $1 \le n, m \le 3 : e_n(e_n e_m) = \alpha(e_n^2)e_m$ and $(e_n e_m)e_m = \alpha e_n(e_m^2)$
⑤ There exists $\gamma \in \{-1, 1\}$ (hence $\gamma^2 = 1$)such that $e_1 e_2 = \gamma e_2 e_1$.

We can prove that $e_1 e_3 = \gamma e_3 e_1$ (and $e_2 e_3 = \gamma e_3 e_2$) :

$$e_1e_3 \overset{②}{=} e_1(e_1e_2) \overset{④}{=} \alpha(e_1^2)e_2 \overset{③}{=} \alpha e_2(e_1^2) \overset{④}{=} \alpha^2(e_2e_1)e_1 \overset{⑤}{=} \gamma(e_1e_2)e_1 \overset{②}{=} \gamma\ e_3e_1$$

QED

This result will, also, be referred to as rule ⑤.

Such an algebra is fully described by the multiplication table of a basis, we will use $\langle\ 1, i, j, k\ \rangle$, rather than $\langle\ 1, e_1, e_2, e_3\ \rangle$.

The cells with a blank background are obtained by a straigtforward application of the rules, the proofs for the cells with a blue background are given there : Proofs, and the cells with an orange background are obtained from the blue or blank ones applying only rule ⑤.

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	i^2	k	$\alpha\, i^2 j$
j	j	γk	j^2	$\alpha\gamma j^2 i$
k	k	$\alpha\gamma\, i^2 j$	$\alpha j^2 i$	$\alpha\gamma j^2 i^2$

Generic $\mathbb{R}^4$-algebra

We will note such an algebra $\mathbb{R}(1^2, j^2, \alpha, \gamma)$.
At first glance there are $3^2 2^2 = 36$ different algebras of this type but as i and j play the same role we can impose $i \le j$ wich gives 24 possibilities, some of them being isomorphic as seen in the following table.

In the following table, $\mathcal{CD}(A, \lambda)$ is the result of the Cayley-Dickson construction from an algebra A with parameter λ defined by :
Let $B = A \oplus A$, so B is a vector space whose dimension is $2n$, addition and scalar multiplication are naturally defined, but the multiplication and the involution ($\overline{x}$ is the conjugate of x) must be specified :

$$(x,\ x') \times (y,\ y') = (xy - \lambda\overline{y'}x',\ y'x + x'\overline{y})$$

$$\overline{(x,\ x')} = (\overline{x},\ -x')$$

i^2	j^2	α	γ	Symbol	Name
-1	-1	-1	-1	$\not\mathbb{M}$	Split Hyperbolic Quaternions
-1	1	-1	-1		
-1	-1	1	-1	$\mathbb{H}$	Quaternions ($\simeq \mathcal{CD}(\mathbb{C},1) \simeq C\ell(0,\,2)$)
-1	-1	1	1	$\mathbb{C}_2$	BiComplex Numbers ($\simeq \mathbb{C} \otimes \mathbb{C} \simeq \not\mathbb{C} \otimes \mathbb{C}$)
-1	1	1	1		
-1	0	1	-1	$\mathcal{CD}(\mathbb{C},0)$	"Anticommutative" Dual Complex ($\simeq \mathcal{CD}(\mathbb{D}_1,1)$)
-1	0	1	1	$\mathbb{D}_2$	Dual Complex ($\simeq \mathbb{C} \otimes \mathbb{D}_1$)
-1	1	1	-1	$\not\mathbb{H}$	CoQuaternions
1	1	1	-1		($\simeq C\ell(1,\,1) \simeq C\ell(2,\,0) \simeq \mathcal{CD}(\mathbb{C},-1) \simeq \mathcal{CD}(\not\mathbb{C},-1) \simeq \mathcal{CD}(\not\mathbb{C},1)$)
0	0	-1	-1	Λ_2	Grassmann Numbers on 2 generators ($\simeq \mathcal{CD}(\mathbb{D}_1,0)$)
0	0	1	-1		
0	0	-1	1	$\mathfrak{D}_2$	Hyper Dual Numbers ($\simeq \mathbb{D}_1 \otimes \mathbb{D}_1$)
0	0	1	1		
0	1	1	-1	$\mathcal{CD}(\not\mathbb{C},0)$	"Anticommutative" Dual Perplex ($\simeq \mathcal{CD}(\mathbb{D}_1,-1)$)
0	1	1	1	$\not\mathbb{C}\otimes\mathbb{D}_1$	Dual Perplex Numbers
1	1	-1	-1	$\mathbb{M}$	Hyperbolic Quaternions
1	1	1	1	$\not\mathbb{C}_2$	BiPerplex Numbers ($\simeq \not\mathbb{C} \otimes \not\mathbb{C}$)

IV.2 Proofs for Dimension 4

The sign $=$ alone means that the proof uses a previous result, or $\alpha^2 = \gamma^2 = 1$, or the fact that real numbers commute with every element.
The sign $=$ with a circled number above means that the proof uses the rule bearing that same number.

ik	$\overset{\textcircled{2}}{=}$	$i(ij)$			$\overset{\textcircled{4}}{=}$	$\alpha(i^2)\ j$
ki	$\overset{\textcircled{5}}{=}$	γik			$=$	$\alpha\gamma(i^2)\ j$
ji	$\overset{\textcircled{5}}{=}$	γij			$\overset{\textcircled{2}}{=}$	$\gamma\ k$
jk	$\overset{\textcircled{2}}{=}$	$j(ij)$	$\overset{\textcircled{5}}{=}$	$\gamma j(ji)$	$\overset{\textcircled{4}}{=}$	$\alpha\gamma(j^2)\ i$
kj	$\overset{\textcircled{5}}{=}$	γjk	$=$	$\gamma\alpha\gamma j^2 i$	$=$	$\alpha j^2\ i$
$(k^2)i$	$\overset{\textcircled{4}}{=}$	$\alpha k(ki)$	$=$	$k\gamma(i^2)j$	$=$	$\gamma\alpha j^2 i^2\ i$
			$\Rightarrow$	k^2	$=$	$\gamma\alpha j^2 i^2$

IV.3 Multiplication tables for Dimension 4

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	j
j	j	-k	-1	-i
k	k	-j	i	1

Split Hyperbolic Quaternions
$\mathbb{R}^4$(-1, ±1, -1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Quaternions
$\mathbb{R}^4$(-1, -1, 1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	k	-1	-i
k	k	-j	-i	1

BiComplexes
$\mathbb{R}^4$(-1, ±1, 1, 1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	0	0
k	k	j	0	0

Anticommutative Dual Complex
$\mathbb{R}^4$(-1, 0, 1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	k	0	0
k	k	-j	0	0

Dual Complex
$\mathbb{R}^4$(-1, 0, 1, 1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	1	-i
k	k	j	i	1

CoQuaternions
$\mathbb{R}^4$(±1, 1, 1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	0	k	0
j	j	-k	0	0
k	k	0	0	0

2-Grassmann
$\mathbb{R}^4$(0, 0, ±1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	0	k	0
j	j	k	0	0
k	k	0	0	0

2-Hyper-Dual Numbers
$\mathbb{R}^4$(0, 0, ±1, 1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	0	k	0
j	j	-k	1	-i
k	k	0	i	0

Anticommutative Dual Perplex
$\mathbb{R}^4$(0, 1, 1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	0	k	0
j	j	k	1	i
k	k	0	i	0

Dual Perplex Numbers
$\mathbb{R}^4$(0, 1, 1, 1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	1	k	-j
j	j	-k	1	i
k	k	j	-i	1

Hyperbolic Quaternions
$\mathbb{R}^4$(1, 1, -1, -1)

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	1	k	j
j	j	k	1	i
k	k	j	i	1

BiPerplex Numbers
$\mathbb{R}^4$(1, 1, 1, 1)

V Dimension 8

With dimension 8 we will need more parameters ...

Elements of a $\mathbb{R}$-algebra of dimension 8 can be written : $z = a_0 + a_1 \cdot e_1 + a_2 \cdot e_2 + a_3 \cdot e_3 + a_4 \cdot e_4 + a_5 \cdot e_5 + a_6 \cdot e_6 + a_7 \cdot e_7$, where $a_i \in \mathbb{R}$. For sake of simplicity we will not use the $\cdot$ symbol anymore.

V.1 Parametrization for Dimension 8

The following rules are also very easy to encode in a spreadsheet, to experiment the different possibilities and to create new algebras (particularly the algebras not mentionned here (there are many of them)), but beware of the existence of an isomorphism with well known algebra.

① 1 is the unit
② $e_3 = e_1 e_2$
③ $e_1^2 \in \{-1, 0, 1\}$ and $e_2^2 \in \{-1, 0, 1\}$
④ There exists $\alpha \in \{-1, 1\}$ (hence $\alpha^2 = 1$) such that, for $1 \leq n, m \leq 7$: $e_n(e_n e_m) = \alpha(e_n^2)e_m$ and $(e_n e_m)e_m = \alpha e_n(e_m^2)$
⑤ There exists $\gamma \in \{-1, 1\}$ (hence $\gamma^2 = 1$) such that $e_1 e_2 = \gamma e_2 e_1$.
⑥ $e_5 = e_1 e_4$, $e_6 = e_2 e_4$, $e_7 = e_3 e_4$.
⑦ $e_4^2 \in \{-1, 0, 1\}$
⑧ There exists $\beta \in \{-1, 1\}$ (hence $\beta^2 = 1$) such that, for $1 \leq n \leq 2$: $e_n e_4 = \beta e_4 e_n$.
⑨ There exists $\delta \in \{-1, 1\}$ (hence $\delta^2 = 1$) such that, for $1 \leq n, m \leq 3, n \neq m$: $e_n(e_m e_4) = \delta(e_n e_m)e_4$, $(e_4 e_n)e_m = \delta e_4(e_n e_m)$, $(e_n e_4)e_m = \delta e_n(e_4 e_m)$ and, $(e_n e_4)(e_m e_4) = \delta e_n(e_4(e_m e_4))$.

From rule ⑨, we can prove for $1 \leq n < 3$, $(e_n e_4)(e_m e4) = \alpha\beta\delta e_4^2(e_n e_m)$:

$$(e_n e_4)(e_m e4) \overset{⑨}{=} \delta e_n(e_4(e_m e_4)) \overset{⑧}{=} \beta\delta e_n(e_4(e_4 e_m)) \overset{④}{=} \beta\delta\alpha e_n(e_4^2 e_m) \overset{③}{=} \alpha\beta\delta e_4^2\ (e_n e_m)$$

QED

This result will, also, be referred to as rule ⑨.

Such an algebra is fully described by the multiplication table of a basis, we will use $\langle\ 1, i, j, k, e_4, e_5, e_6, e_7\ \rangle$, rather than $\langle\ 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\ \rangle$.

The cells with a blank background are obtained by a straigtforward application of the rules, or the previous results in dimension 4 ; the proofs for the cells with a blue background are given there : Proofs, and the cells with an orange background are obtained from the blue or blank ones applying only rule ⑧.

	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	i^2	k	$\alpha i^2\ j$	e_5	$\alpha i^2\ e_4$	$\delta\ e_7$	$\alpha\delta i^2\ e_6$
j	j	$\gamma\ k$	j^2	$\alpha\gamma j^2\ i$	e_6	$\gamma\delta\ e_7$	$\alpha j^2\ e_4$	$\alpha\gamma\delta j^2\ e_5$
k	k	$\alpha\gamma i^2\ j$	$\alpha j^2\ i$	$\alpha\gamma i^2 j^2$	e_7	$\alpha\gamma\delta i^2\ e_6$	$\alpha\delta j^2\ e_5$	$\gamma i^2 j^2\ e_4$
e_4	e_4	$\beta\ e_5$	$\beta\ e_6$	$\delta\ e_7$	e_4^2	$\alpha\beta e_4^2\ i$	$\alpha\beta e_4^2\ j$	$\alpha\delta e_4^2\ k$
e_5	e_5	$\alpha\beta i^2\ e_4$	$\beta\ e_7$	$\alpha\delta i^2\ e_6$	$\alpha e_4^2\ i$	$\alpha\beta i^2 e_4^2$	$\alpha\beta\delta e_4^2\ k$	$i^2 e_4^2\ j$
e_6	e_6	$\beta\gamma\ e_7$	$\alpha\beta j^2\ e_4$	$\alpha\gamma\delta j^2\ e_5$	$\alpha e_4^2\ j$	$\alpha\beta\delta\gamma e_4^2\ k$	$\alpha\beta j^2 e_4^2$	$\gamma j^2 e_4^2\ i$
e_7	e_7	$\alpha\beta\gamma i^2\ e_6$	$\alpha\beta j^2\ e_5$	$\gamma\delta i^2 j^2\ e_4$	$\alpha e_4^2\ k$	$\gamma\beta\delta i^2 e_4^2\ j$	$\beta\delta j^2 e_4^2\ i$	$\gamma\delta i^2 j^2 e_4^2$

Generic $\mathbb{R}^8$-algebra

We will note such an algebra $\mathbb{R}^8(i^2, j^2, e_4^2, \alpha, \beta, \gamma, \delta)$.

At first glance there are $3^3 2^4 = 432$ different algebras of this type but as i, j and e_4 play the same role we can impose $i^2 \leq j^2 \leq e_4^2$. which gives 160 possibilities, some (many) of them being isomorphic.

i^2	j^2	e_4^2	α	β	γ	δ	Symbol	Name
-1	-1	-1	1	-1	-1	-1	$\mathbb{O}$	Octonions
-1	-1	-1	1	1	-1	1	$\mathbb{B}$	BiQuaternions
-1	-1	-1	1	1	1	1	$\mathbb{C}_3$	TriComplex Numbers
-1	-1	0	1	1	-1	1	$\mathbb{D}_4$	Dual Quaternions
-1	-1	1	1	-1	-1	-1	$\not{\mathbb{O}}$	Split Octonions
-1	-1	1	1	1	-1	1	$\not{\mathbb{B}}$	Split BiQuaternions
-1	1	0	1	1	-1	1	$\not{\mathbb{D}}_4$	Dual Split Quaternions
0	0	0	1	-1	-1	1	Λ_3	Grassmann Numbers
0	0	0	1	1	1	1	$\mathfrak{D}_3$	Hyper Dual Numbers
1	1	-1	1	-1	-1	1	$C\ell(2, 1)$	Clifford Algebra
1	1	1	1	-1	-1	1	$C\ell(3, 0)$	Clifford Algebra
1	1	1	1	1	1	1	$\not{\mathbb{C}}_3$	TriPerplex Numbers [3]

All the $\mathbb{R}$-algebra of dimension 8 that I know of can be described with these 7 parameters, but the Moreno Octonions (which would need an 8^{th} parameter to split the incidence of α).

3. The TriPerplex Numbers are also called "Proto-octonions".

V.2 Proofs for Dimension 8

As in the case of dimension 4 :
The sign $=$ alone means that the proof uses a previous result, or $\alpha^2 = \gamma^2 = 1$, or the fact that real numbers commute with every element.
The sign $=$ with a circled number above it means that the proof uses the rule bearing that same number.

ie_5	$\overset{\textcircled{6}}{=}$	$i(ie_4)$									$\overset{\textcircled{4}}{=}$	$\alpha(i^2)\ e_4$
ie_6	$\overset{\textcircled{6}}{=}$	$i(je_4)$	$\overset{\textcircled{9}}{=}$	$\delta(ij)e_4$	$\overset{\textcircled{2}}{=}$	δke_4					$\overset{\textcircled{6}}{=}$	$\delta\ e_7$
ie_7	$\overset{\textcircled{6}}{=}$	$i(ke_4)$	$\overset{\textcircled{9}}{=}$	$\delta(ik)e_4$	$=$	$\delta\alpha i^2 je_4$					$\overset{\textcircled{6}}{=}$	$\alpha\delta i^2\ e_6$
je_5	$\overset{\textcircled{6}}{=}$	$j(ie_4)$	$\overset{\textcircled{9}}{=}$	$\delta(ji)e_4$	$=$	$\delta\gamma ke_4$					$\overset{\textcircled{6}}{=}$	$\gamma\delta\ e_7$
je_6	$\overset{\textcircled{6}}{=}$	$j(je_4)$									$\overset{\textcircled{4}}{=}$	$\alpha(j^2)\ e_4$
je_7	$\overset{\textcircled{6}}{=}$	$j(ke_4)$	$\overset{\textcircled{9}}{=}$	$\delta(jk)e_4$	$=$	$\delta\gamma\alpha j^2 ie_4$					$\overset{\textcircled{6}}{=}$	$\alpha\gamma\delta j^2\ e_5$
ke_5	$\overset{\textcircled{6}}{=}$	$k(ie_4)$	$\overset{\textcircled{9}}{=}$	$\delta(ki)e_4$	$=$	$\delta\alpha\gamma i^2 je_4$					$\overset{\textcircled{6}}{=}$	$\alpha\gamma\delta i^2\ e_6$
ke_6	$\overset{\textcircled{6}}{=}$	$k(je_4)$	$\overset{\textcircled{9}}{=}$	$\delta(kj)e_4$	$=$	$\delta\alpha j^2 ie_4$					$\overset{\textcircled{6}}{=}$	$\alpha\delta j^2\ e_5$
ke_7	$\overset{\textcircled{6}}{=}$	$k(ke_4)$	$\overset{\textcircled{4}}{=}$	$\alpha(k^2)e_4$	$=$	$\alpha\gamma\alpha i^2 j^2 e_4$					$=$	$\gamma i^2 j^2\ e_4$
e_4k	$\overset{\textcircled{2}}{=}$	$e_4(ij)$	$\overset{\textcircled{9}}{=}$	$\delta(e_4i)j$	$\overset{\textcircled{8}}{=}$	$\delta\beta(ie_4)j$	$\overset{\textcircled{9}}{=}$	$\beta i(e_4j)$	$\overset{\textcircled{8}}{=}$	$i(je_4)$		
	$\overset{\textcircled{9}}{=}$	$\delta(ij)e_4$	$\overset{\textcircled{2}}{=}$	δke_4							$\overset{\textcircled{9}}{=}$	$\delta\ e_7$
e_4e_5	$\overset{\textcircled{6}}{=}$	$e_4(ie_4)$	$\overset{\textcircled{8}}{=}$	$\beta e_4(e_4i)$							$\overset{\textcircled{4}}{=}$	$\alpha\beta e_4^2\ i$
e_4e_6	$\overset{\textcircled{6}}{=}$	$e_4(je_4)$	$\overset{\textcircled{8}}{=}$	$\beta e_4(e_4j)$							$\overset{\textcircled{4}}{=}$	$\alpha\beta e_4^2\ j$
e_4e_7	$\overset{\textcircled{6}}{=}$	$e_4(ke_4)$	$=$	$\delta e_4(e_4k)$							$\overset{\textcircled{4}}{=}$	$\alpha\delta e_4^2\ k$
e_5i	$\overset{\textcircled{6}}{=}$	$(ie_4)i$	$\overset{\textcircled{8}}{=}$	$\beta(e_4i)i$							$\overset{\textcircled{4}}{=}$	$\alpha\beta i^2\ e_4$
e_5j	$\overset{\textcircled{6}}{=}$	$(ie_4)j$	$\overset{\textcircled{8}}{=}$	$\beta(e_4i)j$	$\overset{\textcircled{9}}{=}$	$\beta\delta e_4(ij)$	$\overset{\textcircled{2}}{=}$	$\beta\delta e_4k$	$=$	βke_4	$\overset{\textcircled{6}}{=}$	$\beta\ e_7$
e_5k	$\overset{\textcircled{6}}{=}$	$(ie_4)k$	$\overset{\textcircled{8}}{=}$	$\beta(e_4i)k$	$\overset{\textcircled{9}}{=}$	$\beta\delta e_4(ik)$	$=$	$\beta\delta\alpha i^2 e_4j$	$\overset{\textcircled{8}}{=}$	$\delta\alpha i^2 je_4$	$\overset{\textcircled{6}}{=}$	$\alpha\delta i^2\ e_6$
e_5e_4	$\overset{\textcircled{6}}{=}$	$(ie_4)e_4$									$\overset{\textcircled{4}}{=}$	$\alpha e_4^2\ i$
$(e_5^2)i$	$\overset{\textcircled{4}}{=}$	$\alpha e_5(e_5i)$	$=$	$\beta i^2 e_5e_4$	$=$	$\alpha\beta i^2 e_4^2 i$			$\Rightarrow$	e_5^2	$=$	$\alpha\beta i^2 e_4^2$
e_5e_6	$\overset{\textcircled{6}}{=}$	$(ie_4)(je_4)$	$\overset{\textcircled{9}}{=}$	$\alpha\beta\delta e_4^2(ij)$							$\overset{\textcircled{2}}{=}$	$\alpha\beta\delta e_4^2\ k$
e_5e_7	$\overset{\textcircled{6}}{=}$	$(ie_4)(ke_4)$	$\overset{\textcircled{9}}{=}$	$\delta i(e_4(ke_4))$	$\overset{\textcircled{6}}{=}$	$\delta i(e_4e_7)$	$=$	$\delta i(\alpha\delta e_4^2 k)$	$=$	$\alpha e_4^2(\alpha(i^2)j)$	$=$	$e_4^2 i^2\ j$
e_6i	$\overset{\textcircled{6}}{=}$	$(je_4)i$	$\overset{\textcircled{8}}{=}$	$\beta(e_4j)i$	$\overset{\textcircled{9}}{=}$	$\beta\delta e_4(ji)$	$=$	$\beta\delta\gamma e_4k$			$=$	$\beta\gamma\ e_7$
e_6j	$\overset{\textcircled{6}}{=}$	$(je_4)j$	$\overset{\textcircled{8}}{=}$	$\beta(e_4j)j$							$\overset{\textcircled{4}}{=}$	$\alpha\beta j^2\ e_4$
e_6k	$\overset{\textcircled{6}}{=}$	$(je_4)k$	$\overset{\textcircled{8}}{=}$	$\beta(e_4j)k$	$\overset{\textcircled{9}}{=}$	$\beta\delta e_4(jk)$	$=$	$\alpha\gamma\beta\delta j^2 e_4i$	$\overset{\textcircled{8}}{=}$	$\alpha\gamma\delta j^2 ie_4$	$\overset{\textcircled{6}}{=}$	$\alpha\gamma\delta j^2\ e_5$
e_6e_4	$\overset{\textcircled{6}}{=}$	$(je_4)e_4$									$\overset{\textcircled{4}}{=}$	$\alpha e_4^2\ j$
e_6e_5	$\overset{\textcircled{6}}{=}$	$(je_4)(ie_4)$	$\overset{\textcircled{9}}{=}$	$\alpha\beta\delta e_4^2(ji)$							$=$	$\alpha\beta\delta e_4^2\gamma\ k$
$(e_6^2)j$	$\overset{\textcircled{4}}{=}$	$\alpha e_6(e_6j)$	$=$	$\beta j^2 e_6e_4$	$=$	$\beta j^2\alpha e_4^2 j$			$\Rightarrow$	e_6^2	$=$	$\alpha\beta j^2 e_4^2$
e_6e_7	$\overset{\textcircled{6}}{=}$	$(je_4)(ke_4)$	$\overset{\textcircled{9}}{=}$	$\delta j(e_4(ke_4))$	$\overset{\textcircled{6}}{=}$	$\delta j(e_4e_7)$	$=$	$\delta j(\alpha\delta e_4^2 k)$			$=$	$\gamma j^2 e_4^2\ i$
e_7i	$\overset{\textcircled{6}}{=}$	$(ke_4)i$	$=$	$\delta(e_4k)i$	$\overset{\textcircled{9}}{=}$	$e_4(ki)$	$=$	$\alpha\gamma i^2 e_4j$	$=$	$\alpha\gamma\beta i^2 je_4$	$\overset{\textcircled{6}}{=}$	$\alpha\beta\gamma i^2\ e_6$
e_7j	$\overset{\textcircled{6}}{=}$	$(ke_4)j$	$=$	$\delta(e_4k)j$	$\overset{\textcircled{9}}{=}$	$e_4(kj)$	$=$	$\alpha j^2 e_4i$	$=$	$\alpha\beta j^2 ie_4$	$\overset{\textcircled{6}}{=}$	$\alpha\beta j^2\ e_5$
e_7k	$\overset{\textcircled{6}}{=}$	$(ke_4)j$	$=$	$\delta(e_4k)k$	$\overset{\textcircled{4}}{=}$	$\alpha\delta e_4k^2$					$=$	$\gamma\delta j^2 i^2\ e_4$
e_7e_4	$\overset{\textcircled{6}}{=}$	$(ke_4)e_4$									$\overset{\textcircled{4}}{=}$	$\alpha e_4^2\ k$
e_7e_5	$\overset{\textcircled{6}}{=}$	$(ke_4)(ie_4)$	$\overset{\textcircled{9}}{=}$	$\delta k(e_4(ie_4))$	$\overset{\textcircled{6}}{=}$	$\delta k(e_4e_5)$	$=$	$\delta k(\alpha\beta e_4^2 i)$	$=$	$\alpha\beta\delta e_4^2 ki$	$=$	$\beta\gamma\delta i^2 e_4^2\ j$
e_7e_6	$\overset{\textcircled{6}}{=}$	$(ke_4)(je_4)$	$\overset{\textcircled{9}}{=}$	$\delta k(e_4(je_4))$	$\overset{\textcircled{6}}{=}$	$\delta k(e_4e_6)$	$=$	$\delta k(\alpha\beta e_4^2 j)$	$=$	$\alpha\beta\delta e_4^2 kj$	$=$	$\beta\delta e_4^2 j^2\ i$
$(e_7^2)k$	$\overset{\textcircled{4}}{=}$	$\alpha e_7(e_7k)$	$=$	$\alpha\delta\gamma j^2 i^2 e_7e_4$	$=$	$\delta\gamma j^2 i^2 e_4^2 k$			$\Rightarrow$	e_7^2	$=$	$\gamma\delta j^2 i^2 e_4^2$

V.3 Multiplication tables for Dimension 8

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	-j	e_5	-e_4	-e_7	e_6
j	j	-k	-1	i	e_6	e_7	-e_4	-e_5
k	k	j	-i	-1	e_7	-e_6	e_5	-e_4
e_4	e_4	-e_5	-e_6	-e_7	-1	i	j	k
e_5	e_5	e_4	-e_7	e_6	-i	-1	-k	j
e_6	e_6	e_7	e_4	-e_5	-j	k	-1	-i
e_7	e_7	-e_6	e_5	e_4	-k	-j	i	-1

Octonions
$\mathbb{R}^8$(-1, -1, -1, 1, -1, -1, -1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	-j	e_5	-e_4	e_7	-e_6
j	j	-k	-1	i	e_6	-e_7	-e_4	e_5
k	k	j	-i	-1	e_7	e_6	-e_5	-e_4
e_4	e_4	e_5	e_6	e_7	-1	-i	-j	-k
e_5	e_5	-e_4	e_7	-e_6	-i	1	-k	j
e_6	e_6	-e_7	-e_4	e_5	-j	k	1	-i
e_7	e_7	e_6	-e_5	-e_4	-k	-j	i	1

BiQuaternions
$\mathbb{R}^8$(-1, -1, -1, 1, 1, -1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	-j	e_5	-e_4	e_7	-e_6
j	j	k	-1	-i	e_6	e_7	-e_4	-e_5
k	k	-j	-i	1	e_7	-e_6	-e_5	e_4
e_4	e_4	e_5	e_6	e_7	-1	-i	-j	-k
e_5	e_5	-e_4	e_7	-e_6	-i	1	-k	j
e_6	e_6	e_7	-e_4	-e_5	-j	-k	1	i
e_7	e_7	-e_6	-e_5	e_4	-k	j	i	-1

Tricomplex Numbers
$\mathbb{R}^8$(-1, -1, -1, 1, 1, 1, 1)

·	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	$-j$	e_5	$-e_4$	e_7	$-e_6$
j	j	$-k$	-1	i	e_6	$-e_7$	$-e_4$	e_5
k	k	j	$-i$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	e_4	e_5	e_6	e_7	0	0	0	0
e_5	e_5	$-e_4$	e_7	$-e_6$	0	0	0	0
e_6	e_6	$-e_7$	$-e_4$	e_5	0	0	0	0
e_7	e_7	e_6	$-e_5$	$-e_4$	0	0	0	0

Dual Quaternions

$\mathbb{R}^8$(-1, -1, 0, 1, 1, -1, 1)

·	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	$-j$	e_5	$-e_4$	$-e_7$	e_6
j	j	$-k$	-1	i	e_6	e_7	$-e_4$	$-e_5$
k	k	j	$-i$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	1	$-i$	$-j$	$-k$
e_5	e_5	e_4	$-e_7$	e_6	i	1	k	$-j$
e_6	e_6	e_7	e_4	$-e_5$	j	$-k$	1	i
e_7	e_7	$-e_6$	e_5	e_4	k	j	$-i$	1

Split Octonions

$\mathbb{R}^8$(-1, -1, 1, 1, -1, -1, -1)

·	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	$-j$	e_5	$-e_4$	e_7	$-e_6$
j	j	$-k$	-1	i	e_6	$-e_7$	$-e_4$	e_5
k	k	j	$-i$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	e_4	e_5	e_6	e_7	1	i	j	k
e_5	e_5	$-e_4$	e_7	$-e_6$	i	-1	k	$-j$
e_6	e_6	$-e_7$	$-e_4$	e_5	j	$-k$	-1	i
e_7	e_7	e_6	$-e_5$	$-e_4$	k	j	$-i$	-1

Split BiQuaternions

$\mathbb{R}^8$(-1, -1, 1, 1, 1, -1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	-1	k	-j	e_5	-e_4	e_7	-e_6
j	j	-k	1	-i	e_6	-e_7	e_4	-e_5
k	k	j	i	1	e_7	e_6	e_5	e_4
e_4	e_4	e_5	e_6	e_7	0	0	0	0
e_5	e_5	-e_4	e_7	-e_6	0	0	0	0
e_6	e_6	-e_7	e_4	-e_5	0	0	0	0
e_7	e_7	e_6	e_5	e_4	0	0	0	0

Dual Split Quaternions
$\mathbb{R}^8$(-1, 1, 0, 1, 1, -1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	0	k	0	e_5	0	e_7	0
j	j	-k	0	0	e_6	-e_7	0	0
k	k	0	0	0	e_7	0	0	0
e_4	e_4	-e_5	-e_6	e_7	0	0	0	0
e_5	e_5	0	-e_7	0	0	0	0	0
e_6	e_6	e_7	0	0	0	0	0	0
e_7	e_7	0	0	0	0	0	0	0

3-Grassmann Numbers
$\mathbb{R}^8$(0, 0, 0, 1, -1, -1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	0	k	0	e_5	0	e_7	0
j	j	k	0	0	e_6	e_7	0	0
k	k	0	0	0	e_7	0	0	0
e_4	e_4	e_5	e_6	e_7	0	0	0	0
e_5	e_5	0	e_7	0	0	0	0	0
e_6	e_6	e_7	0	0	0	0	0	0
e_7	e_7	0	0	0	0	0	0	0

3-Hyper Dual Numbers
$\mathbb{R}^8$(0, 0, 0, 1, 1, 1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	1	k	j	e_5	e_4	e_7	e_6
j	j	-k	1	-i	e_6	-e_7	e_4	-e_5
k	k	-j	i	-1	e_7	-e_6	e_5	-e_4
e_4	e_4	-e_5	-e_6	e_7	-1	i	j	-k
e_5	e_5	-e_4	-e_7	e_6	-i	1	k	-j
e_6	e_6	e_7	-e_4	-e_5	-j	-k	1	i
e_7	e_7	e_6	-e_5	-e_4	-k	-j	i	1

Clifford Algebra $C\ell(2,1)$

$\mathbb{R}^8$(1, 1, -1, 1, -1, -1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	1	k	j	e_5	e_4	e_7	e_6
j	j	-k	1	-i	e_6	-e_7	e_4	-e_5
k	k	-j	i	-1	e_7	-e_6	e_5	-e_4
e_4	e_4	-e_5	-e_6	e_7	1	-i	-j	k
e_5	e_5	-e_4	-e_7	e_6	i	-1	-k	j
e_6	e_6	e_7	-e_4	-e_5	j	k	-1	-i
e_7	e_7	e_6	-e_5	-e_4	k	j	-i	-1

Clifford Algebra $C\ell(3,0)$

$\mathbb{R}^8$(1, 1, 1, 1, -1, -1, 1)

$\cdot$	1	i	j	k	e_4	e_5	e_6	e_7
1	1	i	j	k	e_4	e_5	e_6	e_7
i	i	1	k	j	e_5	e_4	e_7	e_6
j	j	k	1	i	e_6	e_7	e_4	e_5
k	k	j	i	1	e_7	e_6	e_5	e_4
e_4	e_4	e_5	e_6	e_7	1	i	j	k
e_5	e_5	e_4	e_7	e_6	i	1	k	j
e_6	e_6	e_7	e_4	e_5	j	k	1	i
e_7	e_7	e_6	e_5	e_4	k	j	i	1

Proto-Octonions

$\mathbb{R}^8$(1, 1, 1, 1, 1, 1, 1)