THE LOCAL PRODUCT AND LOCAL PRODUCT SPACE

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Abstract. In this note we introduce the notion of the local product on a sheet and associated space. As an application we prove under some special conditions the following inequalities

$$2\pi\frac{|\log(\langle\vec{a},\vec{b}\rangle)|}{(||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4})|\langle\vec{a},\vec{b}\rangle|}\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[4s+3]{\sum_{i=1}^{n}x_i^{4s+3}}dx_1dx_2\cdots dx_n\bigg|$$

$$\leq\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(-i\frac{\sqrt[4s+3]{\sum_{j=1}^{n}x_j^{4s+3}}}{||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4}}\bigg)dx_1dx_2\cdots dx_n\bigg|$$

and

$$\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(i\frac{\sqrt[4s+3]{\sum_{j=1}^{n}x_j^{4s+3}}}{||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4}}\bigg)dx_1dx_2\cdots dx_n\bigg|$$

$$\leq 2\pi\frac{|\langle\vec{a},\vec{b}\rangle|\times|\log(\langle\vec{a},\vec{b}\rangle)|}{(||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4})}\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[4s+3]{\sum_{i=1}^{n}x_i^{4s+3}}dx_1dx_2\cdots dx_n\bigg|$$

for all $s\in\mathbb{N}$, where $\langle,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi iq}$.

1. Introduction

The notion of an inner product and associated space is so rife in the literature that there is hardly any formal introduction. The inner product space tends to offer a useful terrain for achieving a large class of mathematical results, ranging from identities to inequalities. The result in this setting is often always the best possible. A typical instance is the Cauchy-Schwartz achieved in the setting of the Hilbert space [1]. In this paper we introduce the notion of the local product and the induced local product space. This space turns out to be a special type of a complex inner product space. We exploit this space to obtain the following inequalities

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Theorem 1.1. *Let $\vec{a},\vec{b}\in \mathbb{R}^n$ such that $\langle \vec{a},\vec{b}\rangle >1$ and $a_i,b_i\geq 1$ for all $1\leq i\leq n$, then the lower bound holds*

$$2\pi\frac{|\log(\langle \vec{a},\vec{b}\rangle)|}{(||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4})|\langle \vec{a},\vec{b}\rangle|}\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[4s+3]{\sum_{i=1}^{n}x_i^{4s+3}}dx_1dx_2\cdots dx_n\bigg|$$
$$\leq \bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(-i\frac{\sqrt[4s+3]{\sum\limits_{j=1}^{n}x_j^{4s+3}}}{||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4}}\bigg)dx_1dx_2\cdots dx_n\bigg|$$

for all $s\in \mathbb{N}$, where $\langle ,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi iq}$.

Theorem 1.2. *Let $\vec{a},\vec{b}\in \mathbb{R}^n$ such that $\langle \vec{a},\vec{b}\rangle >1$ and $a_i,b_i\geq 1$ for all $1\leq i\leq n$, then the upper bound holds*

$$\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(i\frac{\sqrt[4s+3]{\sum\limits_{j=1}^{n}x_j^{4s+3}}}{||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4}}\bigg)dx_1dx_2\cdots dx_n\bigg|$$
$$\leq 2\pi\frac{|\langle \vec{a},\vec{b}\rangle|\times|\log(\langle \vec{a},\vec{b}\rangle)|}{(||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4})}\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[4s+3]{\sum_{i=1}^{n}x_i^{4s+3}}dx_1dx_2\cdots dx_n\bigg|$$

for all $s\in \mathbb{N}$, where $\langle ,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi iq}$.

2. The local product and associated space

In this note we introduce and study the notion of the **local product** and associated space.

Definition 2.1. Let $\vec{a},\vec{b}\in \mathbb{C}^n$ and $f:\mathbb{C}\longrightarrow \mathbb{C}$ be continuous on $\cup_{j=1}^{n}[|a_j|,|b_j|]$. Let $(\mathbb{C}^n,\langle ,\rangle)$ be a complex inner product space. Then by the k^{th} local product of $\vec{a}$ with $\vec{b}$ on the sheet f, we mean the bi-variate map $\mathcal{G}_f^k:(\mathbb{C}^n,\langle ,\rangle)\times(\mathbb{C}^n,\langle ,\rangle)\longrightarrow \mathbb{C}$ such that

$$\mathcal{G}_f^k(\vec{a};\vec{b})=f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}f\circ\mathbf{e}\bigg((i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^{k}}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n$$

where $\langle ,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi iq}$. We denote an inner product space with a k^{th} **local product** defined over a sheet f as the k^{th} local product space over a sheet f. We denote this space with the triple $(\mathbb{C}^n,\langle ,\rangle,\mathcal{G}_f^k(;))$.

The k^{th} local product is in some sense a universal map induced by a sheet. In other words a local product can be constructed by carefully choosing the sheet. By taking our sheet to be the constant function $f := 1$ we obtain the local product

$$\begin{aligned}\mathcal{G}_1^k(\vec{a};\vec{b}) &= \int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} dx_1dx_2\cdots dx_n\\ &= \prod_{i=1}^{n}|b_i|-|a_i|.\end{aligned}$$

Similarly, if we take our sheet to be $f = \log$, then under the condition that $\langle\vec{a},\vec{b}\rangle \neq 0$, we obtain the induced local product

$$\mathcal{G}_{\log}^k(\vec{a};\vec{b}) = 2\pi \times (i)^{k+1}\frac{\log(\langle\vec{a},\vec{b}\rangle)}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[k]{\sum_{j=1}^{n}x_j^k}dx_1dx_2\cdots dx_n.$$

By taking the sheet $f = \mathrm{Id}$ to be the identity function, then we obtain in this setting the associated local product

$$\mathcal{G}_{\mathrm{Id}}^k(\vec{a};\vec{b}) = \langle\vec{a},\vec{b}\rangle\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\left(\frac{(-i)^k\sqrt[k]{\sum_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\right)dx_1dx_2\cdots dx_n.$$

Again, by taking the sheet $f = \mathrm{Id}^{-1}$ with $\langle a,b\rangle \neq 0$, then we obtain the corresponding induced k^{th} local product

$$\mathcal{G}_{\mathrm{Id}^{-1}}^k(\vec{a};\vec{b}) = \frac{1}{\langle\vec{a},\vec{b}\rangle}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\left(-\frac{(-i)^k\sqrt[k]{\sum_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\right)dx_1dx_2\cdots dx_n.$$

3. Properties of the local product product

In this section we study some properties of the local product on a fixed sheet.

Proposition 3.1. *The following holds*

(i) *If f is linear such that $\langle a,b\rangle = -\langle b,a\rangle$ then*

$$\mathcal{G}_f^k(\vec{a};\vec{b}) = (-1)^{n+1}\mathcal{G}_f^k(\vec{b};\vec{a}).$$

(ii) *Let $f,g : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $f(t) \leq g(t)$ for any $t \in [1,\infty)$. Then $|\mathcal{G}_f(\vec{a};\vec{b})| \leq |\mathcal{G}_g(\vec{a};\vec{b})|$.*

Proof. (i) By the linearity of f, we can write

$$\mathcal{G}_f^k(\vec{a};\vec{b}) = f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ \mathbf{e}\bigg((-i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n$$

$$= f(\langle \vec{a},\vec{b}\rangle)\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|} f\circ \mathbf{e}\bigg((-i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n$$

$$= f(-\langle b,a\rangle)(-1)^n\int\limits_{|b_n|}^{|a_n|}\int\limits_{|b_{n-1}|}^{|a_{n-1}|}\cdots\int\limits_{|b_1|}^{|a_1|} f\circ \mathbf{e}\bigg((-i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n$$

$$= (-1)^{n+1}f(\langle b,a\rangle)\int\limits_{|b_n|}^{|a_n|}\int\limits_{|b_{n-1}|}^{|a_{n-1}|}\cdots\int\limits_{|b_1|}^{|a_1|} f\circ \mathbf{e}\bigg((-i)^k\frac{\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n$$

$$= (-1)^{n+1}\mathcal{G}_f^k(\vec{b};\vec{a}).$$

(ii) Property (ii) follows very easily from the inequality $f(t)\leq g(t)$.

□

4. **Applications of the local product**

In this section we explore some applications of the local product.

Theorem 4.1. *Let $\vec{a}=(a_1,a_2,\ldots,a_n), \vec{b}=(b_1,b_2,\ldots,b_n)\in \mathbb{R}^n$ such that $\langle \vec{a},\vec{b}\rangle > 1$ and $a_i,b_i\geq 1$ for all $1\leq i\leq n$, then the lower bound holds*

$$2\pi\frac{|\log(\langle \vec{a},\vec{b}\rangle)|}{(||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4})|\langle \vec{a},\vec{b}\rangle|}\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[4s+3]{\sum_{i=1}^{n}x_i^{4s+3}}dx_1dx_2\cdots dx_n\bigg|$$

$$\leq \bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(-i\frac{\sqrt[4s+3]{\sum\limits_{j=1}^{n}x_j^{4s+3}}}{||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4}}\bigg)dx_1dx_2\cdots dx_n\bigg|$$

for all $s\in\mathbb{N}$, where $\langle,\rangle$ denotes the inner product and where $\mathbf{e}(q)=e^{2\pi i q}$.

Proof. Let $f:\mathbb{R}\longrightarrow\mathbb{R}^+$ and $\vec{a},\vec{b}\in\mathbb{R}^n$ such that $\langle\vec{a},\vec{b}\rangle>1$. We note that

$$\mathcal{G}_{\log}^k(\vec{a};\vec{b}) = 2\pi\times(i)^{k+1}\frac{\log(\langle\vec{a},\vec{b}\rangle)}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[k]{\sum_{j=1}^{n}x_j^k}dx_1dx_2\cdots dx_n.$$

By taking the sheet $f = \mathrm{Id}$ to be the identity function, then we obtain in this setting the associated local product

$$\mathcal{G}_{\mathrm{Id}}^{k}(\vec{a};\vec{b}) = \langle \vec{a},\vec{b}\rangle \int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(\frac{(i)^k\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n.$$

Since $\log < \mathrm{Id}$ for all $t\in[1,\infty)$, it follows that $\mathcal{G}_{\log}^{4s+3}(\vec{a};\vec{b}) \leq \mathcal{G}_{\mathrm{Id}}^{4s+3}(\vec{a};\vec{b})$ by taking $k = 4s+3$ for all $s\in\mathbb{N}$ and the inequality follows from this inequality. □

Remark 4.2. Next we obtain another inequality which controls the multiple integral of an exponential function by the multiple integral of their powers.

Theorem 4.3. *Let $\vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ such that $\langle\vec{a},\vec{b}\rangle > 1$ and $a_i, b_i \geq 1$ for all $1 \leq i \leq n$, then the upper bound holds*

$$\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(i\frac{\sqrt[4s+3]{\sum\limits_{j=1}^{n}x_j^{4s+3}}}{||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4}}\bigg)dx_1dx_2\cdots dx_n\bigg|$$
$$\leq 2\pi\frac{|\langle\vec{a},\vec{b}\rangle|\times|\log(\langle\vec{a},\vec{b}\rangle)|}{(||\vec{a}||^{4s+4}+||\vec{b}||^{4s+4})}\bigg|\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[4s+3]{\sum_{i=1}^{n}x_i^{4s+3}}dx_1dx_2\cdots dx_n\bigg|$$

for all $s\in\mathbb{N}$, where $\langle,\rangle$ denotes the inner product and where $\mathbf{e}(q) = e^{2\pi iq}$.

Proof. Let $f:\mathbb{R}\longrightarrow\mathbb{R}^+$ and $\vec{a},\vec{b}\in\mathbb{R}^n$ such that $\langle\vec{a},\vec{b}\rangle > 1$. We note that

$$\mathcal{G}_{\log}^{k}(\vec{a};\vec{b}) = 2\pi\times(i)^{k+1}\frac{\log(\langle\vec{a},\vec{b}\rangle)}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\sqrt[k]{\sum_{j=1}^{n}x_j^k}dx_1dx_2\cdots dx_n.$$

By taking the sheet $f = \mathrm{Id}^{-1}$ to be the reciprocal of the identity function, then we obtain in this setting the associated local product

$$\mathcal{G}_{\mathrm{Id}^{-1}}^{k}(\vec{a};\vec{b}) = \frac{1}{\langle\vec{a},\vec{b}\rangle}\int\limits_{|a_n|}^{|b_n|}\int\limits_{|a_{n-1}|}^{|b_{n-1}|}\cdots\int\limits_{|a_1|}^{|b_1|}\mathbf{e}\bigg(-\frac{(i)^k\sqrt[k]{\sum\limits_{j=1}^{n}x_j^k}}{||\vec{a}||^{k+1}+||\vec{b}||^{k+1}}\bigg)dx_1dx_2\cdots dx_n.$$

Since $\mathrm{Id}^{-1} < \log$ for all $t\in[1,\infty)$, it follows that $\mathcal{G}_{\mathrm{Id}^{-1}}^{4s+3}(\vec{a};\vec{b}) \leq \mathcal{G}_{\log}^{4s+3}(\vec{a};\vec{b})$ by taking $k = 4s+3$ for all $s\in\mathbb{N}$ and the inequality follows from this inequality. □

References

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