Image deblurring: a class of matrices approximating Toeplitz matrices

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Abstract

We deal with the image deblurring problem. We assume that the blur mask has large dimensions. To restore the images, we propose a GNC-type technique, in which a convex approximation of the energy function is first minimized. The computational cost of the GNC algorithm depends strongly on the cost of such a first minimization. So, we propose of approximating the Toeplitz symmetric matrices in the blur operator by means of suitable matrices. Such matrices are chosen in a class of matrices which can be expressed as a direct sum between a circulant and a reverse circulant matrix.

1 Introduction

The problem of restoring images consists of estimating the original image, starting from the observed image and the supposed blur. In our model, we suppose to know the blur mask. In general, this problem is ill–conditioned and/or ill–posed in the Hadamard sense (see also [36]). Thanks to known regularization techniques (see, e.g., [3, 21, 27]), it is possible to reduce this problem to a well–posed problem, whose solution is the minimum of the so-called *primal energy function*, which consists of the sum of two terms. The former indicates the faithfulness of the solution to the data, and the latter is in connection with the regularity properties of the solution (see also [21, 31]). In order to obtain more realistic restored images, the discontinuities in the intensity field is considered (see also [31]). Indeed, in images of real scenes, there are some discontinuities in correspondence with edges of several objects. To deal with such discontinuities, we consider some line variables (see also [31]). It is possible to minimize a priori the primal energy function in these variables, to determine a *dual energy function* (see, e.g., [10, 18, 27]), which treats implicitly discontinuities. Indeed, minimizing the dual energy function is more computationally efficient than minimizing directly the primal energy function. In general, the dual energy function has a quadratic term, related to the faithfulness with the data, and a not necessarily convex addend, the regularization term. In order to link these two kinds of energy functions, some suitable duality theorems are used (see, e.g., [3, 4, 6, 7, 9, 10, 27]).

In order to improve the quality of the reconstructed images, it is possible to consider a dual energy function which implicitly treats Boolean line variables. The proposed duality theorems can be used even with such a function. However, the related dual energy function is not necessarily convex. So, to minimize it, we use a GNC (Graduated Non-Convexity)-type technique, which considers as first convex approximation the proposed convex dual energy function (see also [4, 10, 39, 40, 41, 42, 45]).

It is possible to verify experimentally that the more expensive minimization is the first one, because the other ones just start with a good approximation of the solution. Hence, when we minimize the first convex approximation, we will approximate every block of the blur operator by matrices whose product can be computed by a suitable fast discrete transform. As every block is a symmetric Toeplitz matrix, we deal with determining a class of matrices easy to handle from the computational point of view, which yield a good approximation of the Toeplitz matrices.

Toeplitz-type linear systems arise from numerical approximation of differential equations. Moreover, in restoration of blurred images, it is often dealt with Toeplitz matrices (see, e.g., [25]).

So we investigate a particular class, which is a sum of two families of simultaneously diagonalizable real matrices, whose elements we call *β-matrices*. Such a class includes both circulant and reverse circulant matrices. Symmetric circulant matrices have several applications to ordinary and partial differential equations (see, e.g., [26, 28, 34, 35]), images and signal restoration (see, e.g., [14, 37]), graph theory (see, e.g., [19, 24, 31, 29, 32, 33]). Reverse circulant matrices have different applications, for instance in exponential data fitting and signal processing (see, e.g., [1, 2, 23, 43, 44]).

In Section 2 we present the problem of image deblurring and the related regularization technique; in Section 3 we present a GNC-type technique for the minimization of the energy function; in in Section 4 we investigate spectral properties of β-matrices; in Section 5 we deal with structural properties; in Section 6 we study the properties of the multiplications of our family of matrices; in Section 7 we determine some conditions in order that a β-matrix is invertible; in Section 8 we deal with the problem of approximating a real symmetric Toeplitz matrix by a β-matrix.

2 Regularization of the problem

The problem of image restoration consists of reconstructing the original image from an image blurred and/or corrupted by noise. In the sequel we will assume that all intensities of our involved pixels are put into one column, with the rule that $(i,j) < (i',j')$ if and only if $i < i'$ or $i = i'$ and $j < j'$. The direct problem is formulated as follows:

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{n},$$

where the n^2-dimensional vectors $\boldsymbol{x}$, $\boldsymbol{y}$ are respectively the original and the observed image. In particular, the elements of these vectors indicate the light intensity of pixels in the corresponding image. The n^2-dimensional vector $\boldsymbol{n}$ expresses the additive noise on the image, which we assume to be independent and identically distributed (i.i.d.) Gaussian, with zero mean and known variance. The $n^2 \times n^2$ matrix A is a linear operator, which represents the translation invariant blur acting on the image. To obtain a blurred image, each pixel of original image turns to be equal to a weighted average of its neighbors. Given a positive matrix $M \in \mathbb{R}^{(2h+1)\times(2h+1)}$, called *blur mask,* the entries of matrix A are defined by

$$a_{(i,j),(i+w,j+v)} = \begin{cases} m_{h+1+w,h+1+v}, & \text{if } |w|,|v| \le h, \\ 0, & \text{otherwise.} \end{cases}$$

Here, in lexicographic notation, the generic index $((i,j),(h,l))$ of matrix A is supposed to be equal to $((j-1)n+i,(l-1)n+h)$. The matrix A turns to be a block Toeplitz matrix with Toeplitz blocks. If we assume that the blur operator is uniform on each direction and is very wide (that is, $h \sim n$), then the matrix A is symmetric.

The image restoration problem consists of finding an estimation $\boldsymbol{x}$ of the unknown original image given the blurred image $\boldsymbol{y}$, the matrix A and the variance of the noise σ^2. This is an ill–posed inverse problem in the Hadamard sense.

A *clique* c of order k is the subset of points of a square grid on which the k–th order finite difference is defined. We denote by C_k the set of all cliques of order k. More precisely, we consider, for $k = 1$,

$$\begin{aligned} C_1 = \{c = \{(i,j),(h,l)\} : \; & i = h, j = l+1 \text{ or} \\ & i = h+1, j = l\}; \end{aligned}$$

for $k = 2$,

$$\begin{aligned} C_2 = \; & \{c = \{(i,j),(h,l),(r,q)\} : \\ & i = h = r, j = l+1 = q+2, \text{ or} \\ & i = h+1 = r+2, j = l = q\}; \end{aligned}$$

and for $k = 3$,

$$\begin{aligned} C_3 \; = & \{c = \{(i,j),(h,l),(r,q),(w,z)\} : \\ & i = h = r = w, j = l+1 = q+2 = z+3, \text{ or} \\ & i = h+1 = r+2 = w+3, j = l = q = z\}. \end{aligned}$$

We denote by $D_c^k\boldsymbol{x}$ the k–th order finite difference operator of the vector $\boldsymbol{x}$ associated with the clique c, that is, if $c = \{(i,j),(h,l)\} \in C_1$, then

$$D_c^1\boldsymbol{x} = x_{i,j} - x_{h,l};$$

if $c = \{(i,j),(h,l),(r,q)\} \in C_2$, then

$$D_c^2\boldsymbol{x} = x_{i,j} - 2x_{h,l} + x_{r,q};$$

and if $c = \{(i,j),(h,l),(r,q),(w,z\} \in C_3$, then

$$D_c^3 \boldsymbol{x} = x_{i,j} - 3x_{h,l} + 3x_{r,q} - x_{w,q}.$$

In [10] it has been shown that the use of second order difference operators allows to obtain significantly better results than those obtained by first order difference operators. On the other hand, in [10] it is noted that third order difference operators give slightly better results than those obtained with second order difference operators to the detriment of an excessive increase in computational costs. Therefore we will only use second order difference operators, and hence we refer to C and D_c as C_2 and D_c^2. We associate with each clique c a non–negative weight b_c, called *line variable*, which has the role of dropping the regularity constraints, where discontinuities could appear. In particular, the zero value is associated with a discontinuity of the considered image in correspondence with the clique c. In our model, the original image is considered idealistically as a pair $(\boldsymbol{x},\boldsymbol{b})$, where $\boldsymbol{x}$, $\boldsymbol{b}$ are the vectors of the grey intensity of pixels and of the set of all line components b_c, $c \in C$, respectively.

A regularized solution of the investigated problem is the minimizer of the following function, called *primal energy function*, defined by

$$E(\boldsymbol{x},\boldsymbol{b}) = \|\boldsymbol{y} - A\boldsymbol{x}\|^2 + \sum_{c\in C} \left[\lambda^2 (D_c\boldsymbol{x})^2 b_c + \beta(b_c)\right], \tag{1}$$

where β is a suitable non-increasing function, called *balancing function*, and $\|\cdot\|$ is the Euclidean norm. The first term in the right hand indicates the faithfulness of the solution to the data and the last one is a regularization term, which is related to a smoothness condition on $\boldsymbol{x}$. The scalar parameter λ^2 is in connection with the confidence to the data and the degree of regularization of the solutions. In particular, when λ^2 is close to zero, we represent a strong faithfulness to the data, while when λ^2 is very large we have a confidence to the a priori information.

To find the minimum of the primal energy function (1), we first minimize with respect to $\boldsymbol{b}$. So, the dual energy function $E_d(\boldsymbol{x})$ (see, e.g., [6, 7, 10, 27]) is given by

$$E_d(\boldsymbol{x}) = \inf_{\mathbf{b}\in B^{|C|}} E(\boldsymbol{x},\boldsymbol{b}), \tag{2}$$

where $|C|$ is the cardinality of the set C. Observe that, by [12, Theorem 1], E_d is well-defined. Observe that

$$E_d(\boldsymbol{x}) = \|\boldsymbol{y} - A\boldsymbol{x}\|^2 + \sum_{c\in C} g(D_c\boldsymbol{x}), \tag{3}$$

where

$$g(t) = \inf_{b\in B} (\lambda^2 b t^2 + \beta(b)), \tag{4}$$

is the *potential function*, which associates a cost with each value of the finite difference operator and does not depend on the involved clique (see also [27]).

In general, to reduce computational costs, for reconstructing images, it is more advisable to use the dual energy rather the primal energy, because a lower number of variables have to be determined. Thus, some versions of the duality theorem were given in [9, 10] for energy functions which do not include the constraint of avoiding parallel lines. For other versions existing in the literature see, e.g., [13, 18, 27].

3 GNC algorithm

In general, a function g satisfying duality theorems is not convex. So, neither is the dual energy function in 3. Thus, to minimize such a function, we use a GNC (*Graduated Non-Convexity*) algorithm (see also [4, 10, 39, 40, 41, 42, 45]). The solution of the algorithms for minimizing a non–convex function depends on the choice of the initial point. To give an advisable choice of such a point, the GNC technique finds a finite family of approximating functions $\{E_d^{(p)}\}_p$, whose the first one is convex and the last one is the original dual energy function. So, the following algorithm is applied:

`initialize` $\boldsymbol{x}$;
`while` $E_d^{(p)} \neq E_d$ `do`

- `find the minimum of the function` $E_d^{(p)}$ `starting from the initial point` $\boldsymbol{x}$;
- $\boldsymbol{x} = \arg\min E_d^{(p)}$;
- `update the parameter` p.

It is possible to verify experimentally that the more expensive minimization is the first one, because the other ones just start with a good approximation of the solution. Hence, in this thesis, when we minimize the first convex approximation, we propose to approximate every block of the operator A by means of matrices whose product can be computed by means a suitable fast discrete transform. Since every block of A is a symmetric Toeplitz matrix, we now deal with determining a class of matrices easy to handle from the computational point of view, that give a good approximation of the Toeplitz matrices.

4 Spectral characterization of β-matrices

We begin with presenting a new class of simultaneously diagonalizable matrices, so we define the following matrix. Let n be a fixed positive integer, and $Q_n = (q_{k,j}^{(n)})_{k,j}$, k, $j = 0, 1, \ldots, n-1$, where

$$q_{k,j}^{(n)} = \begin{cases} \alpha_j \cos\Big(\dfrac{2\pi k j}{n}\Big) & \text{if } 0 \le j \le \lfloor n/2 \rfloor, \\[2ex] \alpha_j \sin\Big(\dfrac{2\pi k (n-j)}{n}\Big) & \text{if } \lfloor n/2 \rfloor \le j \le n-1, \end{cases} \tag{5}$$

$$\alpha_j = \begin{cases} \dfrac{1}{\sqrt{n}} = \overline{\alpha} & \text{if } j=0, \text{ or } j = n/2 \text{ if } n \text{ is even}, \\[2ex] \sqrt{\dfrac{2}{n}} = \widetilde{\alpha} & \text{otherwise}, \end{cases} \tag{6}$$

and put

$$Q_n = \left(\mathbf{q}^{(0)} \,\middle|\, \mathbf{q}^{(1)} \,\middle|\, \cdots \,\middle|\, \mathbf{q}^{(\lfloor \frac{n}{2} \rfloor)} \,\middle|\, \mathbf{q}^{(\lfloor \frac{n+1}{2} \rfloor)} \,\middle|\, \cdots \,\middle|\, \mathbf{q}^{(n-2)} \,\middle|\, \mathbf{q}^{(n-1)}\right), \tag{7}$$

where

$$\mathbf{q}^{(0)} = \frac{1}{\sqrt{n}}\begin{pmatrix}1 \; 1 \; \cdots \; 1\end{pmatrix}^T = \frac{1}{\sqrt{n}}\mathbf{u}^{(0)}, \tag{8}$$

$$\begin{aligned}
\mathbf{q}^{(j)} &= \sqrt{\frac{2}{n}}\left(1 \;\; \cos\left(\frac{2\pi j}{n}\right) \cdots \cos\left(\frac{2\pi j(n-1)}{n}\right)\right)^T = \sqrt{\frac{2}{n}}\mathbf{u}^{(j)}, \\
\mathbf{q}^{(n-j)} &= \sqrt{\frac{2}{n}}\left(0 \;\sin\left(\frac{2\pi j}{n}\right) \cdots \sin\left(\frac{2\pi j(n-1)}{n}\right)\right)^T = \sqrt{\frac{2}{n}}\mathbf{v}^{(j)},
\end{aligned} \tag{9}$$

$j = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$. Moreover, when n is even, set

$$\mathbf{q}^{(n/2)} = \frac{1}{\sqrt{n}}\begin{pmatrix}1 \; -1 \;\; 1 \; -1 \cdots -1\end{pmatrix}^T = \frac{1}{\sqrt{n}}\mathbf{u}^{(n/2)}. \tag{10}$$

In [38] it is proved that all columns of Q_n are orthonormal, and thus Q_n is an orthonormal matrix.

Now we define the following function. Given $\boldsymbol{\lambda} \in \mathbb{C}^n$, $\boldsymbol{\lambda} = (\lambda_0\, \lambda_1 \cdots \lambda_{n-1})^T$, set

$$\operatorname{diag}(\boldsymbol{\lambda}) = \Lambda = \begin{pmatrix}
\lambda_0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \lambda_1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \lambda_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_{n-2} & 0 \\
0 & 0 & 0 & \ldots & 0 & \lambda_{n-1}
\end{pmatrix},$$

where $\Lambda \in \mathbb{C}^{n\times n}$ is a diagonal matrix.

A vector $\boldsymbol{\lambda} \in \mathbb{R}^n$, $\boldsymbol{\lambda} = (\lambda_0\, \lambda_1 \cdots \lambda_{n-1})^T$ is said to be *symmetric* (resp., *asymmetric*) iff $\lambda_j = \lambda_{n-j}$ (resp., $\lambda_j = -\lambda_{n-j}$) $\in \mathbb{R}$ for every $j = 0, 1, \ldots, \lfloor n/2 \rfloor$.

Let Q_n be as in (7), and $\mathscr{G}_n$ be the space of the matrices *simultaneously diagonalizable* by Q_n, that is

$$\mathscr{G}_n = \operatorname{sd}(Q_n) = \{Q_n \Lambda Q_n^T : \Lambda = \operatorname{diag}(\boldsymbol{\lambda}),\, \boldsymbol{\lambda} \in \mathbb{R}^n\}.$$

A matrix belonging to $\mathscr{G}_n$, $n \in \mathbb{N}$, is called *γ-matrix* (see also [8]). Moreover, we define the following classes by

$$\mathscr{C}_n = \{Q_n \Lambda Q_n^T : \Lambda = \operatorname{diag}(\boldsymbol{\lambda}),\, \boldsymbol{\lambda} \in \mathbb{R}^n,\, \boldsymbol{\lambda} \text{ is symmetric}\}, \tag{11}$$

$$\mathscr{B}_n = \{Q_n \Lambda Q_n^T : \Lambda = \operatorname{diag}(\boldsymbol{\lambda}),\, \boldsymbol{\lambda} \in \mathbb{R}^n,\, \boldsymbol{\lambda} \text{ is asymmetric}\},$$

$$\begin{aligned}
\mathscr{D}_n \;=\; &\{Q_n \Lambda Q_n^T : \Lambda = \operatorname{diag}(\boldsymbol{\lambda}),\, \boldsymbol{\lambda} \in \mathbb{R}^n,\, \boldsymbol{\lambda} \text{ is symmetric}, \\
&\lambda_0 = 0, \lambda_{n/2} = 0 \text{ if } n \text{ is even}\},
\end{aligned}$$

$$\begin{aligned}
\mathscr{E}_n \;=\; &\{Q_n \Lambda Q_n^T : \Lambda = \operatorname{diag}(\boldsymbol{\lambda}),\; \boldsymbol{\lambda} \in \mathbb{R}^n, \lambda_j = 0,\; j = 1, \ldots, n-1, \\
&j \neq n/2 \text{ when } n \text{ is even}\}.
\end{aligned}$$

Proposition 4.1. (see also [8])*The class $\mathscr{G}_n$ is a matrix algebra of dimension n.*

Proposition 4.2. (see also [8]) *The class $\mathscr{C}_n$ is a subalgebra of $\mathscr{G}_n$ of dimension* $\lfloor \frac{n}{2} \rfloor + 1$.

Proposition 4.3. (see also [8]) *The class $\mathscr{B}_n$ is a linear subspace of $\mathscr{G}_n$, and has dimension* $\lfloor \frac{n-1}{2} \rfloor$.

Similarly as in Propositions 4.1 and 4.2, it is possible to prove that $\mathscr{D}_n$ is a subalgebra of $\mathscr{G}_n$ of dimension $\lfloor \frac{n-1}{2} \rfloor$ and $\mathscr{E}_n$ is a subalgebra of $\mathscr{G}_n$ of dimension 1 when n is odd and 2 when n is even. Moreover, the following results hold.

Theorem 4.4. (see also [8]) *One has*

$$\mathscr{G}_n = \mathscr{C}_n \oplus \mathscr{B}_n, \tag{12}$$

where $\oplus$ is the orthogonal sum, and $\langle \cdot, \cdot \rangle$ denotes the Frobenius product, defined by

$$\langle G_1, G_2 \rangle = tr(G_1^T G_2), \qquad G_1, G_2 \in \mathscr{G}_n,$$

where $tr(G)$ is the trace of the matrix G.

Theorem 4.5. (see also [8]) *It is*

$$\mathscr{C}_n = \mathscr{D}_n \oplus \mathscr{E}_n, \tag{13}$$

where $\oplus$ is the orthogonal sum with respect to the Frobenius product.

Now we give a consequence of 4.4 and 4.5.

Corollary 4.5.1. *The following result holds:*

$$\mathscr{G}_n = \mathscr{B}_n \oplus \mathscr{D}_n \oplus \mathscr{E}_n.$$

We recall the definition of the classical Hartley matrix (see also [5] and the references therein). If n is odd, we have

$$H_n = \frac{1}{\sqrt{n}} \left(\mathbf{u}^{(0)} \;\; \mathbf{u}^{(1)} + \mathbf{v}^{(1)} \;\; \dots \;\; \mathbf{u}^{\left(\frac{n-1}{2}\right)} + \mathbf{v}^{\left(\frac{n-1}{2}\right)} \;\; \mathbf{u}^{\left(\frac{n-1}{2}\right)} - \mathbf{v}^{\left(\frac{n-1}{2}\right)} \;\; \dots \;\; \mathbf{u}^{(1)} - \mathbf{v}^{(1)} \right). \tag{14}$$

When n is even we get

$$H_n = \frac{1}{\sqrt{n}} \left(\mathbf{u}^{(0)} \;\; \mathbf{u}^{(1)} + \mathbf{v}^{(1)} \dots \mathbf{u}^{\left(\frac{n}{2}-1\right)} + \mathbf{v}^{\left(\frac{n}{2}-1\right)} \; \mathbf{u}^{\left(\frac{n}{2}\right)} \; \mathbf{u}^{\left(\frac{n}{2}-1\right)} - \mathbf{v}^{\left(\frac{n}{2}-1\right)} \dots \mathbf{u}^{(1)} - \mathbf{v}^{(1)} \right). \tag{15}$$

It is not difficult to see that

$$H_n = Q_n Y_n, \tag{16}$$

where

$$y_{k,j}^{(n)} = \begin{cases} 1 & \text{if } k=j=0, \\ \dfrac{1}{\sqrt{2}} & \text{if } k=j \text{ and } 1 \le k \le \dfrac{n-1}{2}, \\ \dfrac{1}{\sqrt{2}} & \text{if } k+j=n \text{ and } 1 \le k \le n-1, \\ -\dfrac{1}{\sqrt{2}} & \text{if } k=j \text{ and } \dfrac{n+1}{2} \le k \le n-1, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

if n is odd, and

$$y_{k,j}^{(n)} = \begin{cases} 1 & \text{if } k=j=0 \text{ or } k=j=\dfrac{n}{2}, \\ \dfrac{1}{\sqrt{2}} & \text{if } k=j \text{ and } 1 \le k \le \dfrac{n}{2}-1, \\ \dfrac{1}{\sqrt{2}} & \text{if } k+j=n \text{ and } 1 \le k \le n-1, \\ -\dfrac{1}{\sqrt{2}} & \text{if } k=j \text{ and } \dfrac{n}{2}+1 \le k \le n-1, \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

if n is even. Now, set

$$\mathscr{H}_n = \mathrm{sd}(H_n) = \{H_n \Lambda H_n^T : \Lambda = \mathrm{diag}(\boldsymbol{\lambda}), \, \boldsymbol{\lambda} \in \mathbb{R}^n\}. \tag{19}$$

It is not difficult to see that

$$\begin{aligned} \mathscr{C}_n &= \{Q_n \Lambda Q_n^T : \Lambda = \mathrm{diag}(\boldsymbol{\lambda}), \, \boldsymbol{\lambda} \in \mathbb{R}^n, \, \boldsymbol{\lambda} \text{ is symmetric}\} = \\ &= \{H_n \Lambda H_n^T : \Lambda = \mathrm{diag}(\boldsymbol{\lambda}), \, \boldsymbol{\lambda} \in \mathbb{R}^n, \, \boldsymbol{\lambda} \text{ is symmetric}\}. \end{aligned} \tag{20}$$

From (19) and (20) it follows that

$$\mathscr{H}_n = \mathscr{C}_n \oplus \mathscr{F}_n, \tag{21}$$

where

$$\mathscr{F}_n = \{H_n \Lambda H_n^T : \Lambda = \mathrm{diag}(\boldsymbol{\lambda}), \, \boldsymbol{\lambda} \in \mathbb{R}^n, \, \boldsymbol{\lambda} \text{ is asymmetric}\}.$$

Now we define the following class:

$$\mathscr{A}_n = \{F_n \Lambda F_n^* : \Lambda = \mathrm{diag}(\boldsymbol{\lambda}), \, \boldsymbol{\lambda} \in (\mathrm{i}\mathbb{R})^n, \, \boldsymbol{\lambda} \text{ is asymmetric}\}.$$

So we define the β*-matrices* as the matrices belonging to the following set:

$$\mathscr{V}_n = \mathscr{C}_n \oplus \mathscr{B}_n \oplus \mathscr{F}_n \oplus \mathscr{A}_n. \tag{22}$$

5 Structural characterizations of γ-matrices

In this section we show that $\mathscr{V}_n$ coincides with the direct sum of the sets of all real symmetric circulant matrices and of all reverse circulant matrices.

We consider the set of families

$$
\begin{aligned}
\mathscr{L}_{n,k} &= \{A \in \mathbb{R}^{n\times n} : \text{ there is } \mathbf{a} = (a_0\, a_1\, \dots\, a_{n-1})^T \in \mathbb{R}^n \text{ with } a_{l,j} = a_{(j+kl) \bmod n}\},\\
\mathscr{K}_{n,k} &= \{A \in \mathbb{R}^{n\times n} : \text{ there is a symmetric } \mathbf{a} = (a_0\, a_1\, \dots\, a_{n-1})^T \in \mathbb{R}^n\\
&\qquad \text{with } a_{l,j} = a_{(j+kl) \bmod n}\},\\
\mathscr{J}_{n,k} &= \Big\{A \in \mathbb{R}^{n\times n} : \text{ there is a symmetric } \mathbf{a} = (a_0\, a_1\, \dots\, a_{n-1})^T \in \mathbb{R}^n \text{ with}\\
&\qquad \sum_{t=0}^{n-1} a_t = 0, \sum_{t=0}^{n-1} (-1)^t a_t = 0 \text{ when } n \text{ is even, and } a_{l,j} = a_{(j+kl) \bmod n}\Big\},
\end{aligned}
$$

where $k \in \{1,2,\dots,n-1\}$.

When $k = n-1$, $\mathscr{L}_{n,n-1}$ is the class of all *real circulant matrices*, that is the family of those matrices $C \in \mathbb{R}^{n\times n}$ such that every row, after the first, has the elements of the previous one shifted cyclically one place right (see, e.g., [20]).

Given a vector $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{c} = (c_0\, c_1 \cdots c_{n-1})^T$, let us define

$$
\mathrm{circ}(\mathbf{c}) = C = \begin{pmatrix}
c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1}\\
c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2}\\
c_{n-2} & c_{n-1} & c_0 & \ddots & c_{n-4} & c_{n-3}\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
c_2 & c_3 & c_4 & \ddots & c_0 & c_1\\
c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0
\end{pmatrix},
$$

where $C \in \mathscr{L}_{n,n-1}$.

If i is the imaginary unit and $\omega_n = e^{\frac{2\pi \mathrm{i}}{n}}$, then the n-th roots of 1 are

$$
\omega_n^j = e^{\frac{2\pi j \mathrm{i}}{n}} = \cos\Big(\frac{2\pi j}{n}\Big) + \mathrm{i}\sin\Big(\frac{2\pi j}{n}\Big), \qquad j = 0,1,\dots,n-1.
$$

The *Fourier matrix* of dimension $n \times n$ is defined by $F_n = (f_{k,l}^{(n)})_{k,l}$, where

$$
f_{k,l}^{(n)} = \frac{1}{\sqrt{n}}\,\omega_n^{kl}, \qquad k,l = 0,1,\dots,n-1.
$$

Note that F_n is symmetric, and $F_n^{-1} = F_n^*$ (see, e.g., [20]).

Let $\mathscr{W}_n$ be the space of all real matrices *simultaneously diagonalizable* by F_n, that is

$$
\mathscr{W}_n = \mathrm{sd}(F_n) = \{F_n \Lambda F_n^* \in \mathbb{R}^{n\times n} : \Lambda = \mathrm{diag}(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \in \mathbb{C}^n\}.
$$

It is not difficult to see that $\mathscr{W}_n$ is a commutative matrix algebra.

Theorem 5.1. ([20, Theorems 3.2.2 and 3.2.3]) *The following result holds:*

$$
\mathscr{W}_n = \mathscr{L}_{n,n-1}.
$$

As a consequence of this theorem, we get that the n eigenvectors of every circulant matrix $C \in \mathbb{R}^{n\times n}$ are given by

$$\mathbf{w}^{(j)} = (1\ \omega_n^j\ \omega_n^{2j}\ \cdots\ \omega_n^{(n-1)j})^T,$$

and the eigenvalues of a matrix $C = \mathrm{circ}(\mathbf{c}) \in \mathscr{F}_n$ are expressed by

$$\lambda_j = \mathbf{c}^T \mathbf{w}^{(j)} = \sum_{k=0}^{n-1} c_k \omega_n^{jk}, \qquad j = 0, 1, \ldots, n-1.$$

Now we present some results about symmetric circulant real matrices. Observe that, if $C = \mathrm{circ}(\mathbf{c})$, with $\mathbf{c} \in \mathbb{R}^n$, then C is symmetric if and only if $\mathbf{c}$ is symmetric. Thus, the class of all real symmetric circulant matrices coincides with $\mathscr{K}_{n,n-1}$ and has dimension $\lfloor \frac{n}{2} \rfloor + 1$ over $\mathbb{R}$.

Theorem 5.2. (see, e.g., [19, §4], [38, Lemma 3]) *Let $C \in \mathscr{K}_{n,n-1}$. Then, the set of all eigenvectors of C can be expressed as $\{\mathbf{q}^{(0)}, \mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(n-1)}\}$, where $\mathbf{q}^{(j)}$, $j = 0, 1, \ldots, n-1$, is as in* (8), (9) *and* (10).

Note that from Theorem 5.2 it follows that the set of all real symmetric circulant matrices is contained in $\mathscr{G}_n$. The nest result holds.

Theorem 5.3. (see, e.g., [11, §1.2], [19, §4], [47, Theorem 1]) *Let $C = circ(\mathbf{c}) \in \mathscr{K}_{n,n-1}$. Then, the eigenvalues λ_j of C, $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, are given by*

$$\lambda_j = \mathbf{c}^T \mathbf{u}^{(j)}. \tag{23}$$

Moreover, for $j = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$ it is

$$\lambda_j = \lambda_{n-j}.$$

From Theorem 5.3 it follows that, if C is a real symmetric circulant matrix and $\boldsymbol{\lambda}^{(C)}$ is the set of its eigenvalues, then $\boldsymbol{\lambda}^{(C)}$ is symmetric, thanks to (23). Hence,

$$\mathscr{K}_{n,n-1} \subset \mathscr{C}_n. \tag{24}$$

Now we prove that $\mathscr{C}_n$ is contained in the class of all real symmetric circulant matrices $\mathscr{K}_{n,n-1}$. First, we give the following

Theorem 5.4. (see [8]) *Every matrix $C \in \mathscr{C}_n$ is circulant, that is*

$$\mathscr{C}_n \subset \mathscr{L}_{n,n-1}. \tag{25}$$

A consequence of Theorem 5.4 is the following

Corollary 5.4.1. (see [8]) *The class $\mathscr{C}_n$ is the set of all real symmetric circulant matrices, that is*

$$\mathscr{C}_n = \mathscr{K}_{n,n-1}. \tag{26}$$

If $k = 1$, then $\mathscr{L}_{n,1}$ is the set of all *real reverse circulant* (or *real anti-circulant*) *matrices*, that is the class of all matrices $B \in \mathbb{R}^{n\times n}$ such that every row, after the first, has the elements of the previous one shifted cyclically one place left (see, e.g., [20]). Given a vector $\mathbf{b} = (b_0\, b_1 \cdots b_{n-1})^T \in \mathbb{R}^n$, set

$$\mathrm{rcirc}(\mathbf{b}) = B = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{n-2} & b_{n-1} \\ b_1 & b_2 & b_3 & \dots & b_{n-1} & b_0 \\ b_2 & b_3 & b_4 & \dots & b_0 & b_1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ b_{n-2} & b_{n-1} & b_0 & \dots & b_{n-4} & b_{n-3} \\ b_{n-1} & b_0 & b_1 & \dots & b_{n-3} & b_{n-2} \end{pmatrix},$$

with $B \in \mathscr{L}_{n,1}$.

Observe that every matrix $B \in \mathscr{B}_{n,1}$ is symmetric, and the set $\mathscr{L}_{n,1}$ is a linear space over $\mathbb{R}$, but not an algebra. Note that, if B_1, $B_2 \in \mathscr{L}_{n,1}$, then $B_1 B_2$, $B_2 B_1 \in \mathscr{L}_{n,n-1}$ (see [20, Theorem 5.1.2]).

Now we give the next results.

Theorem 5.5. (see [8]) *The following inclusion holds:*

$$\mathscr{B}_n \subset \mathscr{L}_{n,1}.$$

Theorem 5.6. (see [8]) *One has*

$$\mathscr{B}_n \subset \mathscr{K}_{n,1}.$$

Theorem 5.7. (see [8]) *Let* $B =\mathrm{rcirc}(\mathbf{b}) \in \mathscr{B}_n$. *Then, the eigenvalues* $\lambda_j^{(B)}$ *of* B, $j = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$, *can be expressed as*

$$\lambda_j^{(B)} = \mathbf{b}^T \mathbf{u}^{(j)}. \tag{27}$$

Moreover, for $j = 1, 2, \dots \lfloor \frac{n-1}{2} \rfloor$, *we get*

$$\lambda_{n-j}^{(B)} = -\lambda_j^{(B)}.$$

Furthermore, it is $\lambda_0^{(B)} = 0$, *and* $\lambda_{n/2}^{(B)} = 0$ *if* n *is even.*

For the general computation of the eigenvalues of reverse circulant matrices, see, e.g., [11, §1.3 and Theorem 1.4.1], [46, Lemma 4.1].

Now we give the following

Theorem 5.8. (see [8]) *The following result holds:*

$$\mathscr{B}_n = \mathscr{J}_{n,1}.$$

Theorem 5.9. (see [8]) *The next result holds:*

$$\mathscr{D}_n = \mathscr{J}_{n,n-1}.$$

Theorem 5.10. (see [8]) *The next result holds:*

$$\mathscr{E}_n = \mathscr{P}_n = \mathscr{L}_{n,n-1} \cap \mathscr{L}_{n,1},$$

where

$$\mathscr{P}_n = \begin{cases} \left\{ C \in \mathbb{R}^{n\times n} \colon \text{there are } k_1, k_2 \text{ with } c_{i,j} = \left\{ \begin{array}{ll} k_1 & \text{if } i+j \text{ is even} \\ k_2 & \text{if } i+j \text{ is odd} \end{array} \right\} \right. & \text{if } n \text{ is even}, \\ \{ C \in \mathbb{R}^{n\times n} \colon \text{there is } k \text{ with } c_{i,j} = k \text{ for all } i,j = 0,1,\dots,n-1 \} & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 5.11. (see [8]) *The following result holds:*

$$\mathscr{K}_{n,1} = \mathscr{B}_n \oplus \mathscr{E}_n.$$

We note that

$$\mathscr{F}_n = \{ A \in \mathscr{L}_{n,1} : \text{ there is an asymmetric } \mathbf{a} \in \mathbb{R}^n \text{ with } A = \mathrm{rcirc}(\mathbf{a}) \} \tag{28}$$

(see also [5]). Now we prove the following

Proposition 5.12. *It is*

$$\mathscr{A}_n = \{ A \in \mathscr{L}_{n,n-1} : \textit{ there is an asymmetric } \mathbf{a} \in \mathbb{R}^n \textit{ with } A = \mathrm{circ}(\mathbf{a}) \}.$$

Proof. We begin with the inclusion $\supset$. Let $A \in \mathscr{A}_n$, $A =$circ$(\mathbf{a})$, with $\mathbf{a}$ asymmetric. Since $A \in \mathscr{L}_{n,n-1}$, its eigenvectors are given by

$$\mathbf{w}^{(j)} = (1 \ \omega_n^j \ \omega_n^{2j} \ \cdots \ \omega_n^{(n-1)j})^T,$$

and the eigenvalues of A are expressed by

$$\lambda_j = \mathbf{a}^T \mathbf{w}^{(j)}, \qquad j = 0,1,\dots,n-1.$$

Note that

$$\mathbf{w}^{(j)} = \mathbf{u}^{(j)} + \mathrm{i}\,\mathbf{v}^{(j)}, \tag{29}$$

if $j = 0,1,\dots,\lceil \frac{n-1}{2} \rceil$, and

$$\mathbf{w}^{(n-j)} = \mathbf{u}^{(j)} - \mathrm{i}\,\mathbf{v}^{(j)}, \tag{30}$$

if $j = 0,1,\dots \lfloor \frac{n-1}{2} \rfloor$. From (29) and (30) it follows that

$$\lambda_j = \mathbf{a}^T(\mathbf{u}^{(j)} + \mathrm{i}\,\mathbf{v}^{(j)}) = \mathrm{i}\,\mathbf{a}^T \mathbf{v}^{(j)} \in \mathrm{i}\,\mathbb{R}$$

for $j = 0,1,\dots,\lceil \frac{n-1}{2} \rceil$, and

$$\lambda_{n-j} = \mathbf{a}^T(\mathbf{u}^{(j)} - \mathrm{i}\,\mathbf{v}^{(j)}) = -\mathrm{i}\,\mathbf{a}^T \mathbf{v}^{(j)} = -\lambda_i \in \mathrm{i}\,\mathbb{R}$$

for $j = 0,1,\dots,\lfloor \frac{n-1}{2} \rfloor$.

Now we turn to the converse inclusion. Suppose that $A = F_n \Lambda F_n^*$, where $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in (\mathrm{i}\,\mathbb{R})^n$ and $\boldsymbol{\lambda}$ is asymmetric. The element $a_{k,l}$ is given by

$$\begin{aligned} a_{k,l} &= \frac{1}{n}\sum_{j=0}^{n-1} \omega_n^{kj}\,\lambda_j\,\overline{\omega_n^{lj}} = \frac{1}{n}\sum_{j=1}^{\lfloor (n-1)/2\rfloor} \lambda_j(\omega_n^{kj}\,\overline{\omega_n^{lj}} - \omega_n^{k(n-j)}\,\overline{\omega_n^{l(n-j)}}) = \\ &= \frac{1}{n}\sum_{j=1}^{\lfloor (n-1)/2\rfloor} \lambda_j(\omega_n^{(k-l)j} - \overline{\omega_n^{(k-l)(n-j)}}) = \\ &= \frac{2\mathrm{i}}{n}\sum_{j=1}^{\lfloor (n-1)/2\rfloor} \lambda_j \sin\left(\frac{2\pi j(k-l)}{n}\right). \end{aligned}$$

For $l = 0,\, 1, \ldots, n-1$, we get

$$a_{0,l} = -\frac{2\mathrm{i}}{n}\sum_{j=1}^{\lfloor (n-1)/2\rfloor} \lambda_j \sin\left(\frac{2\pi j l}{n}\right) \in \mathbb{R}.$$

Now we claim that the first row of A is asymmetric. Indeed, we have

$$\begin{aligned} a_{0,n-l} &= -\frac{2\mathrm{i}}{n}\sum_{j=1}^{\lfloor (n-1)/2\rfloor} \lambda_j \sin\left(\frac{2\pi j(n-l)}{n}\right) = \\ &= \frac{2\mathrm{i}}{n}\sum_{j=1}^{\lfloor (n-1)/2\rfloor} \lambda_j \sin\left(\frac{2\pi j l}{n}\right) = -a_{0,l}, \end{aligned}$$

getting the claim. □

From Proposition 5.12 it follows that

$$\mathscr{L}_{n,n-1} = \mathscr{C}_n \oplus \mathscr{A}_n. \tag{31}$$

Moreover, note that $\mathscr{B}_n \oplus \mathscr{F}_n \oplus \mathscr{E}_n = \mathscr{L}_{n,1}$. Since $\mathscr{E}_n = \mathscr{L}_{n,n-1}$, then $\mathscr{C}_n \oplus \mathscr{A}_n = \mathscr{L}_{n,n-1}$. Hence, we obtain

$$\mathscr{V}_n = \mathscr{C}_n \oplus \mathscr{B}_n \oplus \mathscr{F}_n \oplus \mathscr{A}_n = \mathscr{L}_{n,1} \cup \mathscr{L}_{n,n-1}.$$

6 Multiplication between β-matrices

It is not difficult to see that $\mathscr{V}_n$ is closed under the operations of sum between matrices. Now we recall that the eigenvalues $\lambda_j^{(C)}$ of $C = \mathrm{circ}(\mathbf{c}) \in \mathscr{C}_n$, $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, are given by

$$\lambda_j^{(C)} = \mathbf{c}^T \mathbf{u}^{(j)}.$$

Moreover, for $j = 1,\, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, we have

$$\lambda_{n-j}^{(C)} = \lambda_j^{(C)}.$$

Furthermore, the eigenvalues $\lambda_j^{(B)}$ of $B = \text{rcirc}(\mathbf{b}) \in \mathscr{B}_n$, $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, can be expressed as

$$\lambda_j^{(B)} = \mathbf{b}^T \mathbf{u}^{(j)},$$

and for $j = 1, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, we have

$$\lambda_{n-j}^{(B)} = -\lambda_j^{(B)}.$$

Now we give the following

Proposition 6.1. *Let* $\mathbf{a} = (a_0\, a_1 \cdots a_{n-1})^T$, $\mathbf{b} = (b_0\, b_1 \cdots b_{n-1})^T \in \mathbb{R}^n$ *be such that* $\mathbf{a}$ *is symmetric and* $\mathbf{b}$ *is asymmetric. Then,* $\mathbf{a}^T\, \mathbf{b} = 0$*.*

Proof. First of all, we observe that $b_0 = b_{n/2} = 0$. So, we have

$$\begin{aligned} \mathbf{a}^T\, \mathbf{b} &= \sum_{j=0}^{n-1} a_j b_j = \sum_{j=1}^{n-1} a_j b_j = \sum_{j=1}^{n/2-1} a_j b_j + a_{n/2} b_{n/2} + \sum_{j=n/2+1}^{n} a_j b_j = \\ &= \sum_{j=1}^{n/2-1} a_j b_j + \sum_{j=1}^{n/2-1} a_{n-j} b_{n-j} = \sum_{j=1}^{n/2-1} a_j b_j - \sum_{j=1}^{n/2-1} a_j b_j = 0. \end{aligned}$$

□

Proposition 6.2. *The eigenvalues* $\lambda_j^{(F)}$ *of* $F = \text{rcirc}(\mathbf{f}) \in \mathscr{F}_n$, $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, *are given by*

$$\lambda_j^{(F)} = \mathbf{f}^T \mathbf{v}^{(j)},$$

and for $j = 1, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, *we get*

$$\lambda_{n-j}^{(F)} = -\lambda_j^{(F)}.$$

Proof. We consider the following set of eigenvectors, whose first component is 1.

$$\begin{aligned} &\mathbf{u}^{(j)} + \mathbf{v}^{(j)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil; \\ &\mathbf{u}^{(j)} - \mathbf{v}^{(j)}, \quad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor. \end{aligned}$$

Hence, by Proposition 6.1, we obtain

$$\begin{aligned} \lambda_j^{(F)} &= \mathbf{f}^T (\mathbf{u}^{(j)} + \mathbf{v}^{(j)}) = \mathbf{f}^T\, \mathbf{v}^{(j)}, \qquad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil; \\ \lambda_{n-j}^{(F)} &= \mathbf{f}^T (\mathbf{u}^{(j)} - \mathbf{v}^{(j)}) = -\mathbf{f}^T\, \mathbf{v}^{(j)}, \quad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor. \end{aligned}$$

□

Proposition 6.3. *The eigenvalues $\lambda_j^{(A)}$ of A =circ$(\mathbf{a}) \in \mathscr{A}_n$, $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, are given by*

$$\lambda_j^{(A)} = \mathrm{i}\, \mathbf{a}^T \mathbf{v}^{(j)}, \tag{32}$$

and for $j = 1,\, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, we get

$$\lambda_{n-j}^{(A)} = -\lambda_j^{(A)}.$$

Proof. We consider the following set of eigenvectors, whose first component is 1.

$$\mathbf{u}^{(j)} + \mathrm{i}\, \mathbf{v}^{(j)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil;$$

$$\mathbf{u}^{(j)} - \mathrm{i}\, \mathbf{v}^{(j)}, \quad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor.$$

Hence, by Proposition 6.1, we obtain

$$\lambda_j^{(A)} = \mathbf{a}^T (\mathbf{u}^{(j)} + \mathrm{i}\, \mathbf{v}^{(j)}) = \mathrm{i}\, \mathbf{a}^T \mathbf{v}^{(j)}, \qquad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil;$$

$$\lambda_{n-j}^{(A)} = \mathbf{a}^T (\mathbf{u}^{(j)} - \mathrm{i}\, \mathbf{v}^{(j)}) = -\mathrm{i}\, \mathbf{a}^T \mathbf{v}^{(j)}, \quad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor.$$

□

Now we recall the following

Proposition 6.4. (see also [8]) *Given two γ-matrices G_1, G_2, we get:*

6.4.1) *If G_1, $G_2 \in \mathscr{C}_n$, then $G_1 G_2 \in \mathscr{C}_n$;*

6.4.2) *If G_1, $G_2 \in \mathscr{B}_n$, then $G_1 G_2 \in \mathscr{C}_n$;*

6.4.3) *If $G_1 \in \mathscr{C}_n$ and $G_2 \in \mathscr{B}_n$, then $G_1 G_2 = G_2 G_1 \in \mathscr{B}_n$.*

It is not difficult to see that, given $C \in \mathscr{C}_n$ and $V \in \mathscr{V}_n$, the eigenvalues of CV are equal to those of VC and are given by

$$\lambda_j^{(CV)} = \lambda_j^{(VC)} = \lambda_j^{(C)} \lambda_j^{(V)}, \quad j = 0, 1, \ldots, n-1.$$

Now we prove the following

Theorem 6.5. *Let $B \in \mathscr{B}_n$, $B =$*rcirc$(\mathbf{b})$, *and $F \in \mathscr{F}_n$, $F =$*rcirc$(\mathbf{b})$. *Then, $BF \in \mathscr{A}_n$ and the eigenvalues of BF are expressed by*

$$\lambda_j^{(BF)} = \mathrm{i}\, \lambda_j^{(B)} \lambda_j^{(F)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil;$$

$$\lambda_{n-j}^{(BF)} = -\lambda_j^{(BF)}, \qquad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor.$$

Proof. Let $A = BF$. Since $B, F \in \mathscr{L}_{n,1}$, then $A \in \mathscr{L}_{n,n-1}$ (see, e.g., [20]). So, to prove that $A \in \mathscr{A}_n$ it is enough to show that the first row of the matrix A is asymmetric, that is $a_{0,n-j} = -a_{0,j}$, $j = 0, 1, \dots \lceil \frac{n-1}{2} \rceil$. Indeed, if n is odd, we get

$$\begin{aligned} a_{0,n-j} &= b_0 f_{n-j} + \sum_{l=1}^{(n-1)/2} b_l \left(f_{(n-j+l) \pmod n} + f_{(2n-j-l) \pmod n} \right) = \\ &= -b_0 f_j - \sum_{l=1}^{(n-1)/2} b_l \left(f_{(j+l) \pmod n} + f_{(n+j-l) \pmod n} \right) = -a_{0,j}, \end{aligned}$$

and when n is even, we have

$$\begin{aligned} a_{0,n-j} &= b_0 f_{n-j} + b_{n/2} f_{(n/2-j) \pmod n} + \\ &+ \sum_{l=1}^{n/2-1} b_l \left(f_{(n-j+l) \pmod n} + f_{(2n-j-l) \pmod n} \right) = \\ &= -b_0 f_j - b_{n/2} f_{(n/2-j) \pmod n} - \\ &- \sum_{l=1}^{n/2-1} b_l \left(f_{(j+l) \pmod n} + f_{(n+j-l) \pmod n} \right) = -a_{0,j}. \end{aligned}$$

Thus, $A \in \mathscr{A}_n$.

We consider the following set of eigenvectors, whose first component is 1:

$$\mathbf{u}^{(j)} + \mathrm{i}\, \mathbf{v}^{(j)}, \quad j = 0, 1, \dots, \lceil \tfrac{n-1}{2} \rceil.$$

Hence, by Proposition 6.1, we obtain

$$\begin{aligned} \lambda_j^{(A)} &= \mathbf{b}^T F(\mathbf{u}^{(j)} + \mathrm{i}\, \mathbf{v}^{(j)}) = \left(\frac{1-\mathrm{i}}{2} \right) \mathbf{b}^T F(\mathbf{u}^{(j)} - \mathbf{v}^{(j)} + \mathrm{i}(\mathbf{u}^{(j)} + \mathbf{v}^{(j)})) = \\ &= \left(\frac{1-\mathrm{i}}{2} \right) \mathbf{b}^T (-\lambda_j^{(F)} (\mathbf{u}^{(j)} - \mathbf{v}^{(j)}) + \mathrm{i}\, \lambda_j^{(F)} (\mathbf{u}^{(j)} + \mathbf{v}^{(j)})) = \\ &= \left(\frac{1-\mathrm{i}}{2} \right) (-\lambda_j^{(F)} \lambda_j^{(B)} + \mathrm{i}\, \lambda_j^{(F)} \lambda_j^{(B)}) = \mathrm{i}\, \lambda_j^{(B)} \lambda_j^{(F)} \end{aligned}$$

for $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$;

$$\lambda_{n-j}^{(A)} = -\lambda_j^{(A)} = -\mathrm{i}\, \lambda_j^{(B)} \lambda_j^{(F)}$$

for $j = 1, 2, \dots \lfloor \frac{n-1}{2} \rfloor$, since the eigenvalues of $A \in \mathscr{A}_n$ are asymmetric. □

Now we demonstrate the following

Theorem 6.6. *Let $B \in \mathscr{B}_n$, $B =$*rcirc($\mathbf{b}$)*, and $F \in \mathscr{F}_n$, $F =$*rcirc($\mathbf{f}$). *Then, $FB \in \mathscr{A}_n$ and the eigenvalues of FB are expressed by*

$$\lambda_j^{(FB)} = -\mathrm{i}\, \lambda_j^{(B)} \lambda_j^{(F)}, \quad j = 0, 1, \dots, \lceil \tfrac{n-1}{2} \rceil;$$

$$\lambda_{n-j}^{(FB)} = -\lambda_j^{(FB)}, \qquad j = 1, 2, \dots \lfloor \tfrac{n-1}{2} \rfloor.$$

Proof. Let $A = FB$. As $B, F \in \mathscr{L}_{n,1}$, then $A \in \mathscr{L}_{n,n-1}$ (see, e.g., [20]). So, to prove that $A \in \mathscr{A}_n$ it is sufficient to show that the first row of the matrix A is asymmetric, that is $a_{0,n-j} = -a_{0,j}$, $j = 0, 1, \dots \lceil \frac{n-1}{2} \rceil$. Indeed, we have

$$\begin{aligned} a_{0,n-j} &= \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} f_l \left(b_{(n-j+l) \pmod n} - b_{(2n-j-l) \pmod n} \right) = \\ &= -\sum_{l=1}^{\lfloor (n-1)/2 \rfloor} f_l \left(b_{(j+l) \pmod n} - b_{(n+j-l) \pmod n} \right) = -a_{0,j} \end{aligned}$$

for $j = 1, 2, \dots \lfloor \frac{n-1}{2} \rfloor$. Therefore, $A \in \mathscr{A}_n$.

We consider the following set of eigenvectors, whose first component is 1:

$$\mathbf{u}^{(j)} + \mathrm{i}\, \mathbf{v}^{(j)}, \quad j = 0, 1, \dots, \lceil \tfrac{n-1}{2} \rceil.$$

Hence, by Proposition 6.1, we obtain

$$\begin{aligned} \lambda_j^{(A)} &= \mathbf{f}^T B (\mathbf{u}^{(j)} + \mathrm{i}\, \mathbf{v}^{(j)}) = \mathbf{f}^T (\lambda_j^{(B)} \mathbf{u}^{(j)} - \mathrm{i}\, \lambda_j^{(B)} \mathbf{v}^{(j)}) = \\ &= -\mathrm{i}\, \lambda_j^{(B)} (\mathbf{f}^T \mathbf{v}^{(j)}) = -\mathrm{i}\, \lambda_j^{(F)} \lambda_j^{(B)} \end{aligned}$$

for $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$;

$$\lambda_{n-j}^{(A)} = -\lambda_j^{(A)} = \mathrm{i}\, \lambda_j^{(F)} \lambda_j^{(B)}$$

for $j = 1, 2, \dots \lfloor \frac{n-1}{2} \rfloor$, because the eigenvalues of $A \in \mathscr{A}_n$ are asymmetric. □

Observe that, given $B \in \mathscr{B}_n$ and $F \in \mathscr{F}_n$, we get that $\lambda_j^{(FB)} = -\lambda_j^{(BF)}$. Therefore, $FB = -BF$.

Now we prove the following

Theorem 6.7. *Let $A \in \mathscr{A}_n$, $A = \mathrm{circ}(\mathbf{a})$ and $B \in \mathscr{B}_n$, $B = \mathrm{rcirc}(\mathbf{b})$. Then, $AB \in \mathscr{F}_n$ and the eigenvalues of AB are expressed by*

$$\lambda_j^{(AB)} = -\mathrm{i}\, \lambda_j^{(A)} \lambda_j^{(B)}, \quad j = 0, 1, \dots, \lceil \tfrac{n-1}{2} \rceil;$$

$$\lambda_{n-j}^{(AB)} = -\lambda_j^{(AB)}, \qquad j = 1, 2, \dots \lfloor \tfrac{n-1}{2} \rfloor.$$

Proof. Let $F = AB$. Since $A \in \mathscr{L}_{n,n-1}$ and $B \in \mathscr{L}_{n,1}$, then $F \in \mathscr{L}_{n,1}$ (see, e.g., [20]). So, to prove that $F \in \mathscr{F}_n$ it is enough to show that the first row of the matrix F is asymmetric, that is $f_{0,n-j} = -f_{0,j}$, $j = 0, 1, \dots \lceil \frac{n-1}{2} \rceil$. Indeed, we have

$$\begin{aligned} f_{0,n-j} &= \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} a_l \left(b_{(n-j+l) \pmod n} - b_{(2n-j-l) \pmod n} \right) = \\ &= -\sum_{l=1}^{\lfloor (n-1)/2 \rfloor} a_l \left(b_{(j+l) \pmod n} - b_{(n+j-l) \pmod n} \right) = -f_{0,j} \end{aligned}$$

for $j = 1, 2, \dots \lfloor \frac{n-1}{2} \rfloor$. Therefore, $F \in \mathscr{F}_n$.

We consider the following set of eigenvectors, whose first component is 1:

$$\mathbf{u}^{(j)} + \mathbf{v}^{(j)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil.$$

Hence, by Proposition 6.1, we obtain

$$\begin{aligned}
\lambda_j^{(F)} &= \mathbf{a}^T B(\mathbf{u}^{(j)} + \mathbf{v}^{(j)}) = \mathbf{a}^T (\lambda_j^{(B)} \mathbf{u}^{(j)} - \lambda_j^{(B)} \mathbf{v}^{(j)}) = \\
&= -\lambda_j^{(B)} (\mathbf{a}^T \mathbf{v}^{(j)}) = \mathrm{i}\, \lambda_j^{(A)} \lambda_j^{(B)}
\end{aligned}$$

for $j = 0, 1, \ldots, \lceil \frac{n-1}{2} \rceil$;

$$\lambda_{n-j}^{(F)} = -\lambda_j^{(F)} = -\mathrm{i}\, \lambda_j^{(A)} \lambda_j^{(B)}$$

for $j = 1, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, because the eigenvalues of $F \in \mathscr{F}_n$ are asymmetric. □

Theorem 6.8. *Let $B \in \mathscr{B}_n$, $B =$rcirc$(\mathbf{b})$, and $A \in \mathscr{A}_n$, $A =$circ$(\mathbf{b})$. Then, $BA \in \mathscr{F}_n$ and the eigenvalues of BA are given by*

$$\begin{aligned}
\lambda_j^{(BA)} &= -\mathrm{i}\, \lambda_j^{(B)} \lambda_j^{(A)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil; \\
\lambda_{n-j}^{(BA)} &= -\lambda_j^{(BA)}, \qquad\quad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor.
\end{aligned}$$

Proof. Let $F = BA$. Since $A \in \mathscr{L}_{n,n-1}$ and $B \in \mathscr{L}_{n,1}$, then $F \in \mathscr{L}_{n,1}$ (see, e.g., [20]). So, to prove that $F \in \mathscr{F}_n$ it is enough to show that the first row of the matrix F is asymmetric, namely $f_{0,n-j} = -f_{0,j}$, $j = 0, 1, \ldots \lceil \frac{n-1}{2} \rceil$. Indeed, if n is odd, we get

$$\begin{aligned}
f_{0,n-j} &= b_0\, a_{n-j} + \sum_{l=1}^{(n-1)/2} b_l \, (a_{(l-n+j) \pmod n} + a_{(l-j) \pmod n}) = \\
&= -b_0\, a_j - \sum_{l=1}^{(n-1)/2} b_l \, (a_{(j-l) \pmod n} + a_{(-j-l) \pmod n}) = -f_{0,j},
\end{aligned}$$

and when n is even, we have

$$\begin{aligned}
f_{0,n-j} &= b_0\, a_{n-j} + b_{n/2}\, a_{(n/2-j) \pmod n} + \\
&+ \sum_{l=1}^{n/2-1} b_l \, (a_{(l-n+j) \pmod n} + a_{(l-j) \pmod n}) = \\
&= -b_0\, a_j - b_{n/2}\, a_{(n/2-j) \pmod n} - \\
&- \sum_{l=1}^{n/2-1} b_l \, (a_{(j-l) \pmod n} + a_{(-j-l) \pmod n}) = -f_{0,j}.
\end{aligned}$$

Thus, $F \in \mathscr{F}_n$.

We consider the following set of eigenvectors, whose first component is 1:

$$\mathbf{u}^{(j)} + \mathbf{v}^{(j)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil.$$

By Proposition 6.1, we have

$$\begin{aligned}\lambda_j^{(F)} &= \mathbf{b}^T A(\mathbf{u}^{(j)}+\mathbf{v}^{(j)}) = \mathbf{b}^T A\left(\left(\frac{1-\mathrm{i}}{2}\right)(\mathbf{u}^{(j)}+\mathrm{i}\mathbf{v}^{(j)})+\left(\frac{1+\mathrm{i}}{2}\right)(\mathbf{u}^{(j)}-\mathrm{i}\mathbf{v}^{(j)})\right)= \\ &= \mathbf{b}^T\lambda_j^{(F)}\left(\frac{1-\mathrm{i}}{2}\right)(\mathbf{u}^{(j)}+\mathrm{i}\mathbf{v}^{(j)})-\mathbf{b}^T\lambda_j^{(F)}\left(\frac{1+\mathrm{i}}{2}\right)(\mathbf{u}^{(j)}-\mathrm{i}\mathbf{v}^{(j)})= \\ &= \left(\frac{1-\mathrm{i}}{2}\right)\lambda_j^{(B)}\lambda_j^{(A)}-\left(\frac{1+\mathrm{i}}{2}\right)\lambda_j^{(B)}\lambda_j^{(A)} = -\mathrm{i}\,\lambda_j^{(B)}\lambda_j^{(A)}\end{aligned}$$

for $j = 0,1,\dots,\lceil\frac{n-1}{2}\rceil$;

$$\lambda_{n-j}^{(F)} = -\lambda_j^{(F)} = \mathrm{i}\lambda_j^{(B)}\lambda_j^{(A)}$$

for $j = 1,2,\dots\lfloor\frac{n-1}{2}\rfloor$, since the eigenvalues of $F \in \mathscr{F}_n$ are asymmetric. □

Note that, given $A \in \mathscr{A}_n$ and $B \in \mathscr{B}_n$, we have that $\lambda_j^{(AB)} = -\lambda_j^{(BA)}$. Hence, $AB = -BA$. Now we give the following

Theorem 6.9. *Let $A \in \mathscr{A}_n$, $A =$circ$(\mathbf{a})$ and $F \in \mathscr{F}_n$, $F =$rcirc$(\mathbf{b})$. Then, $AF \in \mathscr{B}_n$ and the eigenvalues of AF are expressed by*

$$\lambda_j^{(AF)} = -\mathrm{i}\,\lambda_j^{(A)}\lambda_j^{(F)}, \quad j = 0,1,\dots,\lceil\tfrac{n-1}{2}\rceil;$$

$$\lambda_{n-j}^{(AF)} = -\lambda_j^{(AF)}, \qquad j = 1,2,\dots\lfloor\tfrac{n-1}{2}\rfloor.$$

Proof. Let $B = AF$. Since $A \in \mathscr{L}_{n,n-1}$ and $F \in \mathscr{L}_{n,1}$, then $B \in \mathscr{L}_{n,1}$ (see, e.g., [20]). Thus, to prove that $B \in \mathscr{B}_n$ it is sufficient to demonstrate that the first row of the matrix B is asymmetric, that is $b_{0,n-j} = -b_{0,j}$, $j = 0,1,\dots\lceil\frac{n-1}{2}\rceil$. Indeed, it is

$$\begin{aligned}b_{0,n-j} &= \sum_{l=1}^{\lfloor(n-1)/2\rfloor} a_l\left(f_{(n-j+l) \pmod n} - f_{(2n-j-l) \pmod n}\right) = \\ &= -\sum_{l=1}^{\lfloor(n-1)/2\rfloor} a_l\left(f_{(j+l) \pmod n} - f_{(n+j-l) \pmod n}\right) = -b_{0,j}\end{aligned} \tag{33}$$

for $j = 1,2,\dots\lfloor\frac{n-1}{2}\rfloor$. Therefore, $B \in \mathscr{B}_n$.

We consider the following set of eigenvectors, whose first component is 1:

$$\mathbf{u}^{(j)}, \quad j = 0,1,\dots,\lceil\tfrac{n-1}{2}\rceil.$$

Hence, by Proposition 6.1, we obtain

$$\begin{aligned}\lambda_j^{(B)} &= \mathbf{a}^T F\,\mathbf{u}^{(j)} = \frac{1}{2}\mathbf{a}^T F(\mathbf{u}^{(j)}+\mathbf{v}^{(j)}) - \frac{1}{2}\mathbf{a}^T F\,(\mathbf{u}^{(j)}-\mathbf{v}^{(j)}) = \\ &= \frac{1}{2}\mathbf{a}^T\lambda_j^{(F)}(\mathbf{u}^{(j)}+\mathbf{v}^{(j)}) - \frac{1}{2}\mathbf{a}^T\lambda_j^{(F)}(\mathbf{u}^{(j)}-\mathbf{v}^{(j)}) = -\mathrm{i}\,\lambda_j^{(A)}\lambda_j^{(F)}\end{aligned}$$

for $j = 0,1,\dots,\lceil\frac{n-1}{2}\rceil$;

$$\lambda_{n-j}^{(B)} = -\lambda_j^{(B)} = \mathrm{i}\,\lambda_j^{(A)}\lambda_j^{(F)}$$

for $j = 1,2,\dots\lfloor\frac{n-1}{2}\rfloor$, because the eigenvalues of $B \in \mathscr{B}_n$ are asymmetric. □

Now we prove the following

Theorem 6.10. *Let $A \in \mathscr{A}_n$, $A =$circ$(\mathbf{a})$ and $F \in \mathscr{F}_n$, $F =$rcirc$(\mathbf{b})$. Then, $FA \in \mathscr{B}_n$ and the eigenvalues of FA are given by*

$$\lambda_j^{(FA)} = -\mathrm{i}\,\lambda_j^{(F)}\,\lambda_j^{(A)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil;$$

$$\lambda_{n-j}^{(FA)} = -\lambda_j^{(FA)}, \qquad j = 1, 2, \ldots \lfloor \tfrac{n-1}{2} \rfloor.$$

Proof. Let $B = FA$. Since $A \in \mathscr{L}_{n,n-1}$ and $F \in \mathscr{L}_{n,1}$, then $B \in \mathscr{L}_{n,1}$ (see, e.g., [20]). Thus, to prove that $B \in \mathscr{B}_n$ it is sufficient to demonstrate that the first row of the matrix B is asymmetric, that is $b_{0,n-j} = -b_{0,j}$, $j = 0, 1, \ldots \lceil \frac{n-1}{2} \rceil$. Indeed, we get

$$\begin{aligned} b_{0,n-j} &= \sum_{l=1}^{\lfloor (n-1)/2 \rfloor} f_l\,(a_{(l-n+j) \pmod n} - a_{(l-j) \pmod n}) = \\ &= -\sum_{l=1}^{\lfloor (n-1)/2 \rfloor} f_l\,(a_{(j-l) \pmod n} - a_{(-j-l) \pmod n}) = -b_{0,j} \end{aligned} \tag{34}$$

for $j = 1, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$. Hence, $B \in \mathscr{B}_n$.

We consider the following set of eigenvectors, whose first component is 1:

$$\mathbf{u}^{(j)}, \quad j = 0, 1, \ldots, \lceil \tfrac{n-1}{2} \rceil.$$

Hence, by Proposition 6.1, we obtain

$$\begin{aligned} \lambda_j^{(B)} &= \mathbf{f}^T A\,\mathbf{u}^{(j)} = \frac{1}{2}\mathbf{f}^T A(\mathbf{u}^{(j)} + \mathrm{i}\,\mathbf{v}^{(j)}) + \frac{1}{2}\mathbf{f}^T A\,(\mathbf{u}^{(j)} - \mathrm{i}\,\mathbf{v}^{(j)}) = \\ &= \frac{1}{2}\mathbf{f}^T \lambda_j^{(A)}\,(\mathbf{u}^{(j)} + \mathrm{i}\,\mathbf{v}^{(j)}) + \frac{1}{2}\mathbf{f}^T \lambda_j^{(A)}\,(\mathbf{u}^{(j)} - \mathrm{i}\,\mathbf{v}^{(j)}) = \mathrm{i}\,\lambda_j^{(F)}\,\lambda_j^{(A)} \end{aligned}$$

for $j = 0, 1, \ldots, \lceil \frac{n-1}{2} \rceil$;

$$\lambda_{n-j}^{(B)} = -\lambda_j^{(B)} = -\mathrm{i}\,\lambda_j^{(F)}\,\lambda_j^{(A)}$$

for $j = 1, 2, \ldots \lfloor \frac{n-1}{2} \rfloor$, since the eigenvalues of $B \in \mathscr{B}_n$ are asymmetric. □

Observe that, if $A \in \mathscr{A}_n$ and $F \in \mathscr{F}_n$, then $\lambda_j^{(AF)} = -\lambda_j^{(FA)}$. Hence, $AF = -FA$.

Moreover note that, if B_1, $B_2 \in \mathscr{B}_n$, F_1, $F_2 \in \mathscr{F}_n$, A_1, $A_2 \in \mathscr{A}_n$, then $B_1 B_2$, $F_1 F_2$, $A_1 A_2 \in \mathscr{C}_n$.

7 Invertible β-matrices

In this section we present some results about invertibility of β-matrices. We prove the following

Theorem 7.1. *Given $V_1 \in \mathscr{V}_n$, $V_1 = C_1 + B_1 + F_1 + A_1$, with $C_1 \in \mathscr{C}_n$, $B_1 \in \mathscr{B}_n$, $F_1 \in \mathscr{F}_n$, $A_1 \in \mathscr{A}_n$, set $\sigma_j^{(A_1)} = -\mathrm{i}\,\lambda_j^{(A_1)}$, $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$. If the matrices*

$$\Theta_j = \begin{pmatrix} \lambda_j^{(C_1)} & \lambda_j^{(B_1)} & \lambda_j^{(F_1)} & -\sigma_j^{(A_1)} \\ \lambda_j^{(B_1)} & \lambda_j^{(C_1)} & \sigma_j^{(A_1)} & -\lambda_j^{(F_1)} \\ \lambda_j^{(F_1)} & -\sigma_j^{(A_1)} & \lambda_j^{(C_1)} & -\lambda_j^{(B_1)} \\ \sigma_j^{(A_1)} & -\lambda_j^{(F_1)} & \lambda_j^{(B_1)} & \lambda_j^{(C_1)} \end{pmatrix} \in \mathbb{R}^{4\times 4},$$

$j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$, are invertible, then there exists $V_2 \in \mathscr{V}_n$ such that $V_1 V_2 = I_n$.

Proof. First of all note that, if $V_2 \in \mathscr{V}_n$, then $V_2 = C_2 + B_2 + F_2 + A_2$, with $C_2 \in \mathscr{C}_n$, $B_2 \in \mathscr{B}_n$, $F_2 \in \mathscr{F}_n$, $A_2 \in \mathscr{A}_n$.

Observe that $V_1 V_2 = C_3 + B_3 + F_3 + A_3$, where

$$\begin{aligned} C_3 &= C_1C_2 + B_1B_2 + F_1F_2 + A_1A_2 \in \mathscr{C}_n, \\ B_3 &= C_1B_2 + B_1C_2 + F_1A_2 + A_1F_2 \in \mathscr{B}_n, \\ F_3 &= C_1F_2 + F_1C_2 + B_1A_2 + A_1B_2 \in \mathscr{F}_n, \\ A_3 &= C_1A_2 + A_1C_2 + B_1F_2 + F_1B_2 \in \mathscr{A}_n. \end{aligned}$$

By imposing $C_3 = I_n$, we get

$$\lambda_j^{(C_1)}\lambda_j^{(C_2)} + \lambda_j^{(B_1)}\lambda_j^{(B_2)} + \lambda_j^{(F_1)}\lambda_j^{(F_2)} + \lambda_j^{(A_1)}\lambda_j^{(A_2)} = 1$$

for $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$.

Moreover, by imposing $B_3 = O_n$, by virtue of Theorems 6.9 and 6.10 it follows that

$$\lambda_j^{(B_1)}\lambda_j^{(C_2)} + \lambda_j^{(C_1)}\lambda_j^{(B_2)} - \mathrm{i}\,\lambda_j^{(A_1)}\lambda_j^{(F_2)} + \mathrm{i}\,\lambda_j^{(F_1)}\lambda_j^{(A_2)} = 0$$

for $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$.

Furthermore, we impose $F_3 = O_n$. Then, from Theorems 6.7 and 6.8, it follows that

$$\lambda_j^{(F_1)}\lambda_j^{(C_2)} + \mathrm{i}\,\lambda_j^{(A_1)}\lambda_j^{(B_2)} + \lambda_j^{(C_1)}\lambda_j^{(F_2)} + \mathrm{i}\,\lambda_j^{(B_1)}\lambda_j^{(A_2)} = 0$$

for $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$.

Finally, by imposing $A_3 = O_n$, from Theorems 6.5 and 6.6 we obtain

$$\lambda_j^{(A_1)}\lambda_j^{(C_2)} - \mathrm{i}\,\lambda_j^{(F_1)}\lambda_j^{(B_2)} + \mathrm{i}\,\lambda_j^{(B_1)}\lambda_j^{(F_2)} + \lambda_j^{(C_1)}\lambda_j^{(A_2)} = 0$$

for $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$.

Now, put $\sigma_j^{(A_2)} = -\mathrm{i}\,\lambda_j^{(A_2)}$, $j = 0, 1, \dots, \lceil \frac{n-1}{2} \rceil$, $\boldsymbol{\vartheta}_j{}^T = (\lambda_j^{(C_2)}\ \lambda_j^{(B_2)}\ \lambda_j^{(F_2)}\ \sigma_j^{(A_2)})$. Since Θ_j is invertible, then the system $\Theta_j\,\boldsymbol{\vartheta}_j = (1\,0\,0\,0)^T$ has a unique solution. This ends the proof. □

Thus, it is not difficult to show that in most cases it is possible to compute the inverse of a β-matrix by means of DFFT and Hartley-type transforms.

8 Toeplitz matrix preconditioning

For each $n \in \mathbb{N}$, let us consider the following class:

$$\mathscr{T}_n = \{T_n \in \mathbb{R}^{n\times n} : T_n = (t_{k,j})_{k,j}, t_{k,j} = t_{|k-j|},\ k,j \in \{0,1,\dots,n-1\}\ \}. \tag{35}$$

Observe that the class defined in (35) coincides with the family of all real symmetric Toeplitz matrices.

Now we consider the following problem.
Given $T_n \in \mathscr{T}_n$, find

$$V_n(T_n) = \min_{V\in\mathscr{V}_n} \|V - T_n\|_F,$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

It is not difficult to see that, since T_n is symmetric, then we can assume that $V_n(T_n)$ is symmetric. Therefore, $V_n(T_n) = C_n(T_n) + B_n(T_n) + F_n(T_n)$, where $C_n(T_n) \in \mathscr{C}_n$, $B_n(T_n) \in \mathscr{B}_n$, and $F_n(T_n) \in \mathscr{F}_n$.

Theorem 8.1. *Let $\widehat{\mathscr{G}_n} = \mathscr{S}_n + \mathscr{H}_{n,1}$. Given $T_n \in \mathscr{T}_n$, one has*

$$G_n(T_n) = C_n(T_n) + B_n(T_n) = \min_{G\in\widehat{\mathscr{G}_n}} \|G - T_n\|_F = \min_{G\in\mathscr{G}_n} \|G - T_n\|_F, \tag{36}$$

where $C_n(T_n)$ =circ($\mathbf{c}$), with

$$c_j = \frac{(n-j)\,t_j + j\,t_{n-j}}{n}, \quad j \in \{1,2,\dots,n-1\};$$

$$c_0 = t_0,$$

and $B_n(T_n)$ =rcirc($\mathbf{b}$), where: for n even and $j \in \{1,2,\dots,n-1\} \setminus \{n/2\}$,

$$\begin{aligned} b_j &= \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j - t_{n-j}) + 4\sum_{k=1}^{(j-3)/2} \frac{2k+1}{n}(t_{2k+1} - t_{n-2k-1}) + \right.\\ &+ \left. 4\sum_{k=1}^{(n-j-3)/2} \frac{2k+1}{n}(t_{2k+1} - t_{n-2k-1})\right), \quad j \text{ odd;} \end{aligned}$$

$$\begin{aligned} b_j &= \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j - t_{n-j}) + 4\sum_{k=1}^{j/2-1} \frac{2k}{n}(t_{2k} - t_{n-2k}) + \right.\\ &+ \left. 4\sum_{k=1}^{(n-j)/2-1} \frac{2k}{n}(t_{2k} - t_{n-2k})\right), \quad j \text{ even;} \end{aligned}$$

for n even,

$$b_0 = \frac{2}{n}\left(\sum_{k=1}^{n/2-1} \frac{2k}{n}(t_{2k} - t_{n-2k})\right), \tag{37}$$

$$b_{n/2} \;=\; \frac{4}{n}\left(\sum_{k=1}^{n/4-1}\frac{2k}{n}(t_{2k}-t_{n-2k})\right); \tag{38}$$

for n odd and $j \in \{1,2,\ldots,n-1\}$,

$$\begin{aligned} b_j \;=\; & \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j-t_{n-j})+4\sum_{k=0}^{(j-3)/2}\frac{2k+1}{n}(t_{2k+1}-t_{n-2k-1})+\right.\\ +\; & \left. 4\sum_{k=1}^{(n-j)/2-1}\frac{2k}{n}(t_{2k}-t_{n-2k})\right), \quad j \text{ odd;} \end{aligned} \tag{39}$$

$$\begin{aligned} b_j \;=\; & \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j-t_{n-j})+4\sum_{k=1}^{j/2-1}\frac{2k}{n}(t_{2k}-t_{n-2k})+\right.\\ +\; & \left. 4\sum_{k=0}^{(n-j-3)/2}\frac{2k+1}{n}(t_{2k+1}-t_{n-2k-1})\right), \quad j \text{ even;} \end{aligned} \tag{40}$$

for n odd,

$$b_0 \;=\; \frac{2}{n}\left(\sum_{k=0}^{(n-3)/2}\frac{2k+1}{n}(t_{2k+1}-t_{n-2k-1})\right). \tag{41}$$

Proof. Let us define

$$\phi(\mathbf{c},\mathbf{b}) = \|T_n-\mathrm{circ}(\mathbf{c})-\mathrm{circ}(\mathbf{b})\|_F^2$$

for any two symmetric vectors $\mathbf{c}$, $\mathbf{b}\in\mathbb{R}^n$. If $j \in \{1,2,\ldots,n-1\}$, then we get

$$\frac{\partial\phi(\mathbf{c},\mathbf{b})}{\partial c_j} = -4(n-j)t_j-4jt_{n-j}+4\sum_{j=0}^{n-1}b_j+4nc_j. \tag{42}$$

Furthermore, one has

$$\frac{\partial\phi(\mathbf{c},\mathbf{b})}{\partial c_0} = -2nt_0+2\sum_{j=0}^{n-1}b_j+2nc_0. \tag{43}$$

If n is even and j is odd, $j\in\{1,\ldots,n-1\}$, then, since $c_{n-j}=c_j$, we have

$$\begin{aligned} \frac{\partial\phi(\mathbf{c},\mathbf{b})}{\partial b_j} \;=\; & -2\left(2t_j+4\sum_{k=0}^{(j-3)/2}t_{2k+1}+2t_{n-j}+4\sum_{k=0}^{(n-j-3)/2}t_{2k+1}-\right.\\ & \left.-4\sum_{k=0}^{n/4-1}c_{2k+1}-2nb_j\right) =\\ =\; & -2\left(2(t_j-c_j)+4\sum_{k=0}^{(j-3)/2}(t_{2k+1}-c_{2k+1})+2(t_{n-j}-c_{n-j})+\right.\\ & \left.+4\sum_{k=0}^{(n-j-3)/2}(t_{2k+1}-c_{2k+1})-2nb_j\right). \end{aligned} \tag{44}$$

If both n and j are even, $j \in \{1,2,\ldots,n-1\} \setminus \{n/2\}$, then, by arguing analogously as in the previous case, we deduce

$$\frac{\partial \phi(\mathbf{c},\mathbf{b})}{\partial b_j} = -2\Bigg(2(t_j - c_j) + 4\sum_{k=1}^{j/2-1}(t_{2k} - c_{2k}) + 2(t_{n-j} - c_{n-j}) + \\ +4(t_0 - c_0) + 4\sum_{k=1}^{(n-j)/2-1}(t_{2k} - c_{2k}) - 2n\,b_j\Bigg). \quad (45)$$

Moreover, if n is even, then one has

$$\frac{\partial \phi(\mathbf{c},\mathbf{b})}{\partial b_0} = -2\left(t_0 - c_0 + 2\sum_{k=1}^{n/2-1}(t_{2k} - c_{2k}) - n\,b_0\right), \quad (46)$$

getting (37). Furthermore, for n even, we have

$$\frac{\partial \phi(\mathbf{c},\mathbf{b})}{\partial b_{n/2}} = -2\left(2(t_{n/2} - c_{n/2}) + 4\sum_{k=1}^{n/4-1}(t_{2k} - c_{2k}) - nb_{n/2}\right). \quad (47)$$

Now, if both n and j are odd, $j \in \{0,1,\ldots,n-1\}$, then, taking into account (55), we obtain

$$\frac{\partial \phi(\mathbf{c},\mathbf{b})}{\partial b_j} = -2\Bigg(2\,(t_j - c_j) + 4\sum_{k=0}^{(j-3)/2}(t_{2k+1} - c_{2k+1}) + 2\,(t_{n-j} - c_{n-j}) + \\ +4\sum_{k=1}^{(n-j)/2-1}(t_{2k} - c_{2k}) + 2\,(t_0 - c_0) - 2n\,b_j\Bigg). \quad (48)$$

If n is odd and j is even, $j \in \{0,1,\ldots,n-1\}$, then we have

$$\frac{\partial \phi(\mathbf{c},\mathbf{b})}{\partial b_j} = -2\Bigg(2\,(t_j - c_j) + 4\sum_{k=1}^{j/2-1}(t_{2k} - c_{2k}) + 2\,(t_{n-j} - c_{n-j}) + \quad (49) \\ +\,4\sum_{k=1}^{(n-j-3)/2}(t_{2k+1} - c_{2k+1}) + 2\,(t_0 - c_0) - 2n\,b_j\Bigg).$$

Finally, for n odd, one has

$$\frac{\partial \phi(\mathbf{c},\mathbf{b})}{\partial b_0} = -2\left(t_0 - c_0 + 2\sum_{k=0}^{(n-3)/2}(t_{2k+1} - c_{2k+1}) - n\,b_0\right). \quad (50)$$

It is not difficult to see that the function ϕ is convex. By [22, Theorem 2.2], ϕ has exactly one point of minimum. From this it follows that ϕ admits exactly one stationary point. Now we claim that this point satisfies

$$\sum_{k=0}^{n/4-1} b_{2k+1} = 0 \quad (51)$$

and

$$b_0 + 2 \sum_{k=1}^{n/4-1} b_{2k} + b_{n/2} = 0 \tag{52}$$

when n is even, and

$$b_0 + 2 \sum_{j=1}^{(n-1)/2} b_j = 0 \tag{53}$$

if n is odd, that is $B_n(T_n) \in \mathscr{B}_n$. From (51)-(53) and (42)-(43) we get (55)-(55). Furthermore, from (55)-(55) and (44)-(50) we obtain (55)-(41). Finally, (51)-(52) follow from (55)-(38), while (53) is a consequence of (39)-(41). □

Theorem 8.2. *Given $T_n \in \mathscr{T}_n$, one has*

$$V_n(T_n) = C_n(T_n) + B_n(T_n) + F_n(T_n) = \min_{V \in \mathscr{V}_n} \|V - T_n\|_F, \tag{54}$$

where $C_n(T_n) =$circ$(\mathbf{c})$, with

$$c_j = \frac{(n-j)\,t_j + j\,t_{n-j}}{n}, \quad j \in \{1, 2, \ldots, n-1\};$$

$$c_0 = t_0,$$

and $B_n(T_n) =$rcirc$(\mathbf{b})$, where: for n even and $j \in \{1, 2, \ldots, n-1\} \setminus \{n/2\}$,

$$\begin{aligned} b_j &= \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j - t_{n-j}) + 4\sum_{k=1}^{(j-3)/2} \frac{2k+1}{n}(t_{2k+1} - t_{n-2k-1}) + \right. \\ &+ \left. 4\sum_{k=1}^{(n-j-3)/2} \frac{2k+1}{n}(t_{2k+1} - t_{n-2k-1})\right), \quad j \text{ odd}; \end{aligned} \tag{55}$$

$$\begin{aligned} b_j &= \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j - t_{n-j}) + 4\sum_{k=1}^{j/2-1} \frac{2k}{n}(t_{2k} - t_{n-2k}) + \right. \\ &+ \left. 4\sum_{k=1}^{(n-j)/2-1} \frac{2k}{n}(t_{2k} - t_{n-2k})\right), \quad j \text{ even}; \end{aligned} \tag{56}$$

for n even,

$$b_0 = \frac{2}{n}\left(\sum_{k=1}^{n/2-1} \frac{2k}{n}(t_{2k} - t_{n-2k})\right),$$

$$b_{n/2} = \frac{4}{n}\left(\sum_{k=1}^{n/4-1} \frac{2k}{n}(t_{2k} - t_{n-2k})\right);$$

for n odd and $j \in \{1,2,\ldots,n-1\}$,

$$\begin{aligned} b_j &= \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j-t_{n-j})+4\sum_{k=0}^{(j-3)/2}\frac{2k+1}{n}(t_{2k+1}-t_{n-2k-1})+\right. \\ &+ \left. 4\sum_{k=1}^{(n-j)/2-1}\frac{2k}{n}(t_{2k}-t_{n-2k})\right), \quad j \text{ odd;} \end{aligned} \tag{57}$$

$$\begin{aligned} b_j &= \frac{1}{2n}\left(\frac{4j-2n}{n}(t_j-t_{n-j})+4\sum_{k=1}^{j/2-1}\frac{2k}{n}(t_{2k}-t_{n-2k})+\right. \\ &+ \left. 4\sum_{k=0}^{(n-j-3)/2}\frac{2k+1}{n}(t_{2k+1}-t_{n-2k-1})\right), \quad j \text{ even;} \end{aligned} \tag{58}$$

for n odd,

$$b_0 = \frac{2}{n}\left(\sum_{k=0}^{(n-3)/2}\frac{2k+1}{n}(t_{2k+1}-t_{n-2k-1})\right); \tag{59}$$

$$f_j = \frac{t_j-t_{n-j}}{n}, \quad j \in \{1,2,\ldots,n-1\}; \tag{60}$$

$$f_0 = 0. \tag{61}$$

Proof. Set

$$\widetilde{\phi}(\mathbf{c},\mathbf{b},\mathbf{f}) = \|T_n - \text{circ}(\mathbf{c}) - \text{rcirc}(\mathbf{b}) - \text{rcirc}(\mathbf{f})\|_F^2$$

for each symmetric vector $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$ and for every asymmetric vector $\mathbf{f} \in \mathbb{R}^n$. By proceeding analogously as in (42)-(50) and taking into account the asymmetry of $\mathbf{f}$, we get that the derivatives

$$\frac{\partial\widetilde{\phi}(\mathbf{c},\mathbf{b},\mathbf{f})}{\partial c_j}, \quad \frac{\partial\widetilde{\phi}(\mathbf{c},\mathbf{b},\mathbf{f})}{\partial b_j}$$

have the same expressions as the respective derivatives

$$\frac{\partial\phi(\mathbf{c},\mathbf{b})}{\partial c_j}, \quad \frac{\partial\phi(\mathbf{c},\mathbf{b})}{\partial b_j}$$

in (42)-(50), $j = 0, 1, \ldots, n-1$. Furthermore, for any $n \in \mathbb{N}$ and $j \in \{1,2,\ldots,n-1\}$ we get

$$\frac{\partial\widetilde{\phi}(\mathbf{c},\mathbf{b},\mathbf{f})}{\partial f_j} = 4(n f_j - t_n + t_{n-j}). \tag{62}$$

Proceeding similarly as we dealt with the function ϕ in Theorem 8.1, it is not difficult to prove the convexity of the function $\widetilde{\phi}$. From this and [22, Theorem 2.2] again, it follows that $\widetilde{\phi}$ has exactly one point of minimum, and hence $\widetilde{\phi}$ admits exactly one stationary point. By arguing analogously as in Theorem 8.1, it is possible to show that the same conditions as in (51)-(53) are satisfied, and the assertion of the theorem follows. □

Now we show how the approximation found in β-matrices allows to obtain also preconditioned linear systems with eigenvalues clustered around 1. For every $n \in \mathbb{N}$, set

$$\begin{aligned} \widehat{\mathscr{T}_n} &= \{t \in \mathscr{T}_n : \text{ there is a function } f(z) = \sum_{j=-\infty}^{+\infty} t_j z^j, \\ & \qquad \text{with } z \in \mathbb{C}, |z| = 1, \text{ and such that } \sum_{j=-\infty}^{+\infty} |t_j| < +\infty\}. \end{aligned} \tag{63}$$

Observe that any function defined by a power series as in the first line of (63) is real-valued, and the set of such functions satisfying the condition

$$\sum_{j=-\infty}^{+\infty} |t_j| < +\infty$$

is called *Wiener class* (see, e.g., [5], [16, §3]).

Given a function f belonging to the Wiener class and a matrix $T_n \in \widehat{\mathscr{T}_n}$, $T_n(f) = (t_{k,j})_{k,j}$: $t_{k,j} = t_{|k-j|}$, $k, j \in \{0, 1, \ldots, n-1\}$, and $f(z) = \sum_{j=-\infty}^{+\infty} t_j z^j$, then we say that $T_n(f)$ is *generated by* f.

We will often use the following property of absolutely convergent series (see, e.g., [5, 15]).

Lemma 8.3. *Let* $\sum_{j=1}^{\infty} t_j$ *be an absolutely convergent series. Then, we get*

$$\lim_{n\to+\infty} \left[\frac{1}{n} \left(\sum_{k=1}^{n} k\,|t_k| + \sum_{k=\lceil (n+1)/2 \rceil}^{n} (n-k)\,|t_k| \right) \right] = 0.$$

Proof. Let $S = \sum_{j=1}^{\infty} |t_j|$. Choose arbitrarily $\varepsilon > 0$. By hypothesis, there is a positive integer n_0 with

$$\sum_{k=n_0+1}^{\infty} |t_k| \le \frac{\varepsilon}{4}. \tag{64}$$

Let $n_1 = \max \left\{ \frac{2\,n_0\,S}{\varepsilon}, 2\,n_0 \right\}$. Taking into account (64), for every $n > n_1$ it is

$$\begin{aligned} 0 &\le \frac{1}{n} \left(\sum_{k=1}^{n} k\,|t_k| + \sum_{k=\lceil (n+1)/2 \rceil}^{n} (n-k)\,|t_k| \right) = \\ &= \frac{1}{n} \sum_{k=1}^{n_0} k\,|t_k| + \frac{1}{n} \sum_{k=n_0+1}^{n} k\,|t_k| + \frac{1}{n} \sum_{k=\lceil (n+1)/2 \rceil}^{n} (n-k)\,|t_k| \le \\ &\le \frac{1}{n_1} n_0 \sum_{k=1}^{n_0} |t_k| + 2 \sum_{k=n_0+1}^{n} |t_k| \le \frac{\varepsilon}{2\,n_0\,S} n_0\,S + 2\,\frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

So, the assertion follows. □

Theorem 8.4. *For $n \in \mathbb{N}$, given $T_n(f) \in \widehat{\mathscr{T}_n}$, let $C_n(f) = C_n(T_n(f))$, $B_n(f) = B_n(T_n(f))$, $F_n(f) = F_n(T_n(f))$ be as in Theorem* 8.2, *and set $V_n(f) = C_n(f) + B_n(f) + F_n(f)$. Then, the following statements hold.*

8.4.1) *For every $\varepsilon > 0$ there is a positive integer n_0, such that for each $n \geq n_0$ and for every eigenvalue $\lambda_j^{(V_n(f))}$ of $V_n(f)$, it is*

$$\lambda_j^{(V_n(f))} \in [f_{\min} - \varepsilon, f_{\max} + \varepsilon], \quad j \in \{0, 1, \ldots, n-1\}, \tag{65}$$

where $f_{\min}$ and $f_{\max}$ denote the minimum and the maximum value of f, respectively.

8.4.2) *For every $\varepsilon > 0$ there are k, $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$ the number of eigenvalues $\lambda_j^{((V_n(f))^{-1} T_n(f))}$ of $V_n^{-1}(f)\, T_n(f)$ such that $|\lambda_j^{((V_n(f))^{-1} T_n(f))} - 1| > \varepsilon$ is less than k, namely the spectrum of $(V_n(f))^{-1}\, T_n(f)$ is clustered around* 1.

Proof. We begin with proving 8.4.1). Let $G_n(f) = C_n(f) + B_n(f)$. Choose arbitrarily $\varepsilon > 0$. We denote by $\lambda_j^{(C_n(f))}$ (resp., $\lambda_j^{(B_n(f))}$, $\lambda_j^{(F_n(f))}$, $\lambda_j^{(G_n(f))}$) the generic j-th eigenvalue of $C_n(f)$ (resp., $B_n(f)$, $F_n(f)$, $G_n(f)$) in the order given by Theorem 5.3 (resp., Theorem 5.7, Proposition 6.2). First, we claim that

$$\lambda_j^{(G_n(f))} \in [f_{\min} - \varepsilon/2, f_{\max} + \varepsilon/2], \quad j \in \{0, 1, \ldots, n-1\}. \tag{66}$$

To prove (66) it is enough to show that this property holds (in correspondence with $\varepsilon/4$) for each $\lambda_j^{(C_n(f))}$, $j = 0, 1, \ldots, n-1$, and that

$$\lambda_j^{(B_n(f))} \in [-\varepsilon/4, \varepsilon/4] \text{ for every } n \geq n_0 \text{ and } j \in \{0, 1, \ldots, n-1\}. \tag{67}$$

Indeed, since $C_n(f), B_n(f) \in \mathscr{G}_n$, we have

$$\lambda_j^{(G_n(f))} = \lambda_j^{(C_n(f))} + \lambda_j^{(B_n(f))} \text{ for all } j \in \{0, 1, \ldots, n-1\},$$

getting the claim.

Now we consider the case n odd. For every $j \in \{0, 1, \ldots, n-1\}$, since $c_j = c_{n-j}$ and thanks to (55), one has

$$\begin{aligned}
\left|\lambda_j^{(C_n(f))}\right| &= \left|\sum_{h=0}^{n-1} c_h \cos(2\pi h j)\right| = \left|c_0 + 2\sum_{h=1}^{(n-1)/2} c_h \cos(2\pi h j)\right| = \\
&= \Bigg| t_0 + 2\sum_{h=1}^{(n-1)/2} t_h \cos(2\pi h j) - \\
&- \sum_{h=1}^{(n-1)/2} \frac{h}{n} t_h \cos(2\pi h j) + \sum_{h=1}^{(n-1)/2} \frac{h}{n} t_{n-h} \cos(2\pi h j)\Bigg| \leq \\
&\leq \sum_{h=-(n-1)/2}^{(n-1)/2} |t_h| \left(e^{\mathrm{i}\frac{2\pi j}{n}}\right)^h + \sum_{h=1}^{(n-1)/2} \frac{h}{n}|t_h| + \sum_{h=1}^{(n-1)/2} \frac{h}{n}|t_{n-h}| \leq \\
&\leq \sum_{h=-\infty}^{+\infty} |t_h| \left(e^{\mathrm{i}\frac{2\pi j}{n}}\right)^h + \sum_{h=1}^{(n-1)/2} \frac{h}{n}|t_h| + \sum_{h=(n+1)/2}^{n-1} \frac{n-h}{n}|t_h|.
\end{aligned} \tag{68}$$

Choose arbitrarily $\varepsilon > 0$. Note that the first addend of the last term in (68) tends to $f\left(e^{\mathrm{i}\frac{2\pi j}{n}}\right)$ as n tends to $+\infty$, and hence, without loss of generality, we can suppose that it belongs to the interval $[f_{\min} - \varepsilon/12, f_{\max} + \varepsilon/12]$ for n sufficiently large. By Lemma 8.3, it is

$$\lim_{n\to+\infty}\left(\sum_{h=1}^{(n-1)/2}\frac{h}{n}|t_h| + \sum_{h=(n+1)/2}^{n-1}\frac{n-h}{n}|t_h|\right) = 0.$$

When n is even, we get

$$\begin{aligned}
\left|\lambda_j^{(C_n(f))}\right| &= \left|\sum_{h=0}^{n-1} c_h \cos(2\pi h j)\right| = \left|c_0 + 2\sum_{h=1}^{n/2-1} c_h\cos(2\pi h j) + (-1)^{n/2} c_{n/2}\right| = \\
&= \Bigg| t_0 + 2\sum_{h=1}^{n/2-1}\cos(2\pi h j)\, t_h + (-1)^{n/2} t_{n/2} - \\
&- \sum_{h=1}^{n/2-1}\frac{h}{n} t_h \cos(2\pi h j) + \sum_{h=1}^{n/2-1}\frac{h}{n} t_{n-h}\cos(2\pi h j)\Bigg| \le \\
&\le \sum_{h=-n/2+1}^{n/2} |t_h| \left(e^{\mathrm{i}\frac{2\pi j}{n}}\right)^h + \sum_{h=1}^{n/2-1}\frac{h}{n}|t_h| + \sum_{h=1}^{n/2-1}\frac{h}{n}|t_{n-h}| \\
&\le \sum_{h=-\infty}^{+\infty} |t_h| \left(e^{\mathrm{i}\frac{2\pi j}{n}}\right)^h + \sum_{h=1}^{n/2-1}\frac{h}{n}|t_h| + \sum_{h=n/2+1}^{n-1}\frac{n-h}{n}|t_h|.
\end{aligned}$$

Thus, it is possible to repeat the same argument used in the previous case, getting 8.4.1).

Now we turn to 8.4.2). From Theorem 5.7 we obtain

$$\lambda_j^{(B_n(f))} = \begin{cases} \displaystyle\sum_{h=0}^{n-1} b_h \cos\left(\frac{2\pi j h}{n}\right) & \text{if } j \le n/2, \\[2ex] \displaystyle -\sum_{h=0}^{n-1} b_h \cos\left(\frac{2\pi (n-j) h}{n}\right) & \text{if } j > n/2. \end{cases}$$

So, without loss of generality, it is enough to prove 8.4.2) for $j \le n/2$.

We first consider the case when n is even. We get

$$\begin{aligned}
\left|\lambda_j^{(B_n(f))}\right| &\le \left|\sum_{h=0}^{n-1} b_h \cos\left(\frac{2\pi j h}{n}\right)\right| \le \sum_{h=0}^{n-1}|b_h| = \\
&= |b_0| + \sum_{h=1}^{n/4-1}|b_{2h}| + \sum_{h=n/4+1}^{n/2-1}|b_{2h}| + |b_{n/2}| + \sum_{h=0}^{n/2-1}|b_{2h+1}| = \qquad (69) \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

So, in order to obtain 8.4.2), it is enough to prove that each addend of the last line of (69) tends to 0 as n tends to $+\infty$. We get:

$$\begin{aligned} I_1 &= |b_0| \le \sum_{k=1}^{n/2-1} \frac{4k}{n^2}|t_{2k}| + \sum_{k=1}^{n/2-1} \frac{4k}{n^2}|t_{n-2k}| = \\ &= \sum_{k=1}^{n/2-1} \frac{4k}{n^2}|t_{2k}| + \sum_{k=1}^{n/2-1} \frac{2n-4k}{n^2}|t_{2k}| \le \frac{4n-8}{n^2}\sum_{h=1}^{\infty}|t_h| \le \frac{4S}{n}, \end{aligned} \tag{70}$$

where $S = \sum_{h=1}^{\infty} |t_h|$. From (70) it follows that $I_1 = |b_0|$ tends to 0 as n tends to $+\infty$. Analogously it is possible to check that $I_4 = |b_{n/2}|$ tends to 0 as n tends to $+\infty$.

Now we estimate the term $I_2 + I_3$. We first observe that

$$\begin{aligned} &\frac{1}{n^2}\sum_{h=1}^{n/4-1}(4h-n)(|t_{2h}|+|t_{n-2h}|) + \frac{1}{n^2}\sum_{h=n/4+1}^{n/2-1}(4h-n)(|t_{2h}|+|t_{n-2h}|) \le \\ \le\ &\frac{1}{n^2}\sum_{h=1}^{n/2-1}(4h-n)(|t_{2h}|+|t_{n-2h}|) \le \\ \le\ &\frac{1}{n^2}\sum_{h=1}^{n/2-1}4h|t_{2h}| + \frac{1}{n^2}\sum_{h=1}^{n/2-1}(4h-n)|t_{n-2h}| = \\ =\ &\frac{1}{n^2}\sum_{h=1}^{n/2-1}4h|t_{2h}| + \frac{1}{n^2}\sum_{h=1}^{n/2-1}(2n-4h)|t_{2h}|. \end{aligned} \tag{71}$$

Arguing analogously as in (70), it is possible to see that the quantities at the first hand of (71) tend to 0 as n tends to $+\infty$.

Furthermore, we have

$$\frac{2}{n}\sum_{h=1}^{n/2-1}\left(\sum_{k=1}^{h-1}\frac{2k}{n}|t_{2k}|\right) = \sum_{k=1}^{n/2-2}\frac{4k}{n^2}\left(\frac{n}{2}-2-k\right)|t_{2k}| \le \sum_{k=1}^{n/2-2}\frac{2k}{n}|t_{2k}|, \tag{72}$$

$$\frac{2}{n}\sum_{h=1}^{n/2-1}\left(\sum_{k=1}^{h-1}\frac{2k}{n}|t_{n-2k}|\right) = \sum_{k=1}^{n/2-2}\frac{4k}{n^2}\left(\frac{n}{2}-2-k\right)|t_{n-2k}| \le \sum_{k=2}^{n/2-1}\frac{2k}{n}|t_{2k}|, \tag{73}$$

$$\frac{2}{n}\sum_{h=1}^{n/2-1}\left(\sum_{k=1}^{n/2-h-1}\frac{2k}{n}|t_{2k}|\right) \le \sum_{k=1}^{n/2-2}\frac{2k}{n}|t_{2k}|, \tag{74}$$

and

$$\begin{aligned} &\frac{2}{n}\sum_{h=1}^{n/2-1}\left(\sum_{k=1}^{n/2-h-1}\frac{2k}{n}|t_{n-2k}|\right) \le \sum_{k=1}^{n/2-2}\frac{4k}{n^2}\left(\frac{n-2k-2}{2}\right)|t_{n-2k}| = \\ =\ &\sum_{k=2}^{n/2-1}\frac{(n-2k)(2k-2)}{n^2}|t_{2k}| \le \sum_{k=2}^{n/2-1}\frac{2k}{n}|t_{2k}|. \end{aligned} \tag{75}$$

Summing up (71)-(75), from (56) we obtain

$$\begin{aligned} I_2+I_3 &= \sum_{h=1}^{n/4-1}|b_{2h}|+\sum_{h=n/4+1}^{n/2-1}|b_{2h}| \le \frac{1}{n^2}\sum_{h=1}^{n/2-1}4h|t_{2h}|+\frac{1}{n^2}\sum_{h=1}^{n/2-1}4h|t_{n-2h}|+ \\ &+ \sum_{k=1}^{n/2-2}\frac{4k}{n}|t_{2k}|+\sum_{k=2}^{n/2-1}\frac{4k}{n}|t_{2k}|. \end{aligned} \tag{76}$$

Thus, taking into account Lemma 8.3, it is possible to check that the terms at the right hand of (76) tend to 0 as n tends to $+\infty$.

Now we estimate the term I_5. One has

$$\begin{aligned} \frac{2}{n}\sum_{h=0}^{n/2-1}\left(\sum_{k=0}^{h-1}\frac{2k+1}{n}|t_{2k+1}|\right) &= \sum_{k=0}^{n/2}\frac{2k+1}{n}|t_{2k+1}| = \\ = \sum_{k=0}^{n/2}\frac{2k}{n}|t_{2k+1}|+\sum_{k=1}^{n/2}\frac{1}{n}|t_{2k+1}| &= J_1+J_2. \end{aligned}$$

Thanks to Lemma 8.3, it is possible to check that J_1 tends to 0 as n tends to $+\infty$. Moreover, we have

$$0 \le J_2 \le \frac{S}{n}, \tag{77}$$

and hence I_4 tends to 0 as n tends to $+\infty$. Analogously as in the previous case, it is possible to prove that

$$\begin{aligned} I_5 &= \sum_{k=0}^{n/2-1}|b_{2k+1}| \le \frac{1}{n^2}\sum_{k=0}^{n/2-1}2(2k+1)|t_{2k+1}|+ \\ &+ \frac{2}{n}\sum_{k=1}^{n/2-2}\left(\frac{n}{2}-1-k\right)\left(\frac{2k+1}{n}\right)|t_{n-2k-1}| \le \\ &\le \sum_{k=1}^{n/2-2}\frac{2(2k+1)}{n}|t_{2k+1}|+\sum_{k=2}^{n/2-1}\frac{2(2k+1)}{n}|t_{2k+1}|. \end{aligned} \tag{78}$$

By virtue of Lemma 8.3 and (77), we get that I_5 tends to 0 as n tends to $+\infty$. Therefore, all addends of the right hand of (69) tend to 0 as n tends to $+\infty$. Thus, (67) follows from (69), (70), (76) and (78).

When n is odd, it is possible to proceed analogously as in previous case. This proves (66).

Now we claim that the eigenvalues of $F_n(f)$ lie between $-\varepsilon/2$ and $\varepsilon/2$ for n large enough. We have:

$$|\lambda_j^{F_n(f)}| = \left|\sum_{k=0}^{n-1} f_j \sin\left(\frac{2\pi k j}{n}\right)\right| \le \sum_{k=0}^{n-1}|f_j| \le \frac{1}{n}\sum_{k=0}^{n-1}|t_j|+\frac{1}{n}\sum_{k=0}^{n-1}|t_{n-j}|. \tag{79}$$

Since f belongs to the Wiener class, we get the claim.

Moreover, we observe that

$$G_n(f)+F_n(f) = Q_n\,(\Lambda^{(G_n(f))} + Y_n\,\Lambda^{(F_n(f))}\,Y_n^T)\,Q_n^T,$$

where

$$\begin{aligned}
\Lambda^{(G_n(f))} &= \boldsymbol{\lambda}^{(G_n(f))} = (\lambda_0^{(G_n(f))}\,\lambda_1^{(G_n(f))}\cdots\lambda_{n-1}^{(G_n(f))})^T,\\
\Lambda^{(F_n(f))} &= \boldsymbol{\lambda}^{(F_n(f))} = (\lambda_0^{(F_n(f))}\,\lambda_1^{(F_n(f))}\cdots\lambda_{n-1}^{(F_n(f))})^T.
\end{aligned}$$

Thus, the matrix $G_n(f)+F_n(f)$ is similar to $\Lambda^{(G_n(f))} + Y_n\,\Lambda^{(F_n(f))}\,Y_n^T$. Note that

$$Y_n\,\Lambda^{(F_n(f))}\,Y_n^T = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda_0^{(F_n(f))} \\ 0 & \dots & 0 & \lambda_1^{(F_n(f))} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \lambda_2^{(F_n(f))} & 0 & \dots & 0 \\ \lambda_1^{(F_n(f))} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Therefore, 8.4.1) follows from the Gerschgorin theorem (see, e.g., [30]).

Now we turn to 8.4.2), that is we prove that the spectrum of $(V_n(f))^{-1}\,T_n(f)$ is clustered around 1. Since $(V_n(f))^{-1}(T_n(f)-V_n(f)) = (V_n(f))^{-1}T_n(f) - I_n$, where I_n is the identity matrix, it is enough to check that the eigenvalues of $(V_n(f))^{-1}(T_n(f)-V_n(f))$ are clustered around 0.

Choose arbitrarily $\varepsilon > 0$. Since f belongs to the Wiener class, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$\sum_{j=n_0+1}^{\infty} |t_j| \le \varepsilon.$$

Proceeding similarly as in the proof of [5, Theorem 3 (ii)], we get

$$T_n(f)-V_n(f) = T_n(f)-C_n(f)-B_n(f)-F_n(f) = W_n^{(n_0)} + Z_n^{(n_0)} + E_n^{(n_0)},$$

where $W_n^{(n_0)}$, $Z_n^{(n_0)}$, $E_n^{(n_0)}$ are suitable matrices such that $W_n^{(n_0)}$ and $Z_n^{(n_0)}$ agree with the $(n-n_0)\times(n-n_0)$ leading principal submatrices of $T_n(f)-C_n(f)$ and $B_n(f)+F_n(f)$, respectively. We have:

$$\begin{aligned}
\mathrm{rank}(E_n^{(n_0)}) &\le 2\,n_0;\\
\|W_n^{(n_0)}\|_1 &\le \frac{2}{n}\sum_{k=1}^{n-n_0-1} k\,|t_{n-k}-t_k| \le \frac{2}{n}\sum_{k=1}^{n_0} k\,|t_k| + 4\sum_{k=n_0+1}^{\infty} |t_k|; \qquad (80)\\
\|Z_n^{(n_0)}\|_1 &\le \sum_{h=0}^{n-1} (|b_h|+|f_h|),
\end{aligned}$$

where the symbol $\|\cdot\|_1$ denotes the 1-norm of the involved matrix. Let $n_1 > n_0$ be a positive integer with

$$\frac{1}{n_1}\sum_{k=1}^{n_0} k\,|t_k| \le \varepsilon \quad\text{and}\quad \sum_{h=0}^{n-1}(|b_h|+|f_h|) \le \varepsilon. \qquad (81)$$

Note that such an n_1 does exist, thanks to Lemma 8.3 and since all terms of (69) and (79) tend to 0 as n tends to $+\infty$. From (80) and (81) it follows that

$$\|W_n^{(n_0)} + W_n^{(n_0)}\|_1 \leq \|W_n^{(n_0)}\|_1 + \|Z_n^{(n_0)}\|_1 \leq 8\varepsilon. \tag{82}$$

From (82) and the Cauchy interlace theorem (see, e.g., [48]) we deduce that the eigenvalues of $T_n(f) - V_n(f)$ are clustered around 0, with the exception of at most $k = 2\,n_0$ of them. By the Courant-Fisher minimax characterization of the matrix $(V_n(f))^{-1}\,(T_n(f) - V_n(f))$ (see, e.g., [48]), we obtain

$$\lambda_j^{(V_n(f))^{-1}\,(T_n(f)-V_n(f))} \leq \frac{\lambda_j^{(T_n(f)-V_n(f))}}{f_{\min}} \tag{83}$$

for n large enough. From (83) we deduce that the spectrum of $(V_n(f))^{-1}\,(T_n(f) - V_n(f))$ is clustered around 0, namely for every $\varepsilon > 0$ there are $k, n_1 \in \mathbb{N}$ with the property that for each $\varepsilon > 0$ the number of eigenvalues $\lambda_j^{(V_n(f))^{-1}\,T_n(f)}$ such that $\left|\lambda_j^{(V_n(f))^{-1}\,T_n(f)} - 1\right| > \varepsilon$ is at most equal to k. □

Note that a similar result can be obtained by approximating $G_n(f) = C_n(f) + B_n(f)$ (see [8]).

9 Experimental results

In order to test the goodness of the proposed approximations, we have proceeded as follows: fixed the dimension n and the range of values which the involved Toeplitz matrices can assume, we have created 10000 different instances of Toeplitz symmetric matrices T_n, whose values have been randomly and uniformly chosen in the interior of the prefixed range. Moreover, we have computed the approximation $C_n(T_n)$ given in [17], the approximation $H_n(T_n)$ presented in [5] and the approximations $G_n(T_n)$ and $V_n(T_n)$ given in (36) and (54), respectively. Furthermore, we have computed the mean error in terms of difference between the matrix T_n and the preconditioning matrix evaluated with respect to the Frobenius norm. In Table 1 the considered range is $[0,1]$. In this case, as expected, $V_n(T_n)$ turns to be the best approximation, while $G_n(T_n)$ is the second best approximation in mean. In Table 2, the considered interval is $[-1,1]$, and the obtained results are analogous to the previous ones. In Table 3, to generate the first row of the Toeplitz symmetric matrix, we have proceeded as follows. We have taken the value of the first entry equal to 1. To determinate the value of the i-th entry, we have multiplied the value of the $i-1$-th entry by a random constant chosen uniformly in $[0.9,1]$. Such a choice allows to better simulate the Toeplitz matrices present in the blur operators. The behavior of the errors is similar to that of the prevous cases. Moreover, from Tables 1-3 it is possible to see that, for large numbers, the approximations $C_n(T_n)$ and $H_n(T_n)$ give similar results, while the approximations $G_n(T_n)$ and $V_n(T_n)$. Furthermore, as seen in Table 4, for large numbers the approximation $G_n(T_n)$ is always better than the approximation $H_n(T_n)$. Since the multiplication of $V_n(T_n)$ by a vector needs three fast discrete transforms, while the multiplication of $V_n(T_n)$ by a vector requires only one fast discrete transform. Thus we deduce that, for n very large, $G_n(T_n)$ is the better solution in terms both of approximation and in computational costs.

	$\|T_n - C_n(T_n)\|_F$	$\|T_n - H_n(T_n)\|_F$	$\|T_n - G_n(T_n)\|_F$	$\|T_n - V_n(T_n)\|_F$
$n = 20$	3.1389	3.1156	3.0770	3.0532
$n = 25$	4.1076	4.0885	3.9591	3.9392
$n = 30$	4.8062	4.7903	4.7369	4.7207
$n = 35$	5.7528	5.7390	5.5989	5.5847
$n = 40$	6.4536	6.4416	6.3811	6.3689
$n = 45$	7.4243	7.4135	7.2649	7.2538
$n = 50$	8.1211	8.1114	8.0471	8.0373
$n = 100$	16.46786	16.46293	16.38939	16.38444
$n = 1000$	166.48101	166.48051	166.39821	166.39771

Table 1: Mean error obtained by the various approximations with respect to 10000 instances of randomly generated Toeplitz matrices T_n with entries in $[0,1]$.

	$\|T_n - C_n(T_n)\|_F$	$\|T_n - H_n(T_n)\|_F$	$\|T_n - G_n(T_n)\|_F$	$\|T_n - V_n(T_n)\|_F$
$n = 5$	0.73470	0.65623	0.65593	0.56618
$n = 10$	1.44816	1.40540	1.40531	1.36116
$n = 15$	2.42566	2.39475	2.28953	2.25668
$n = 20$	6.2564	6.2098	6.1313	6.0838
$n = 25$	8.2016	8.1633	7.8982	7.8584
$n = 30$	9.6160	9.5842	9.4776	9.4453
$n = 35$	11.517	11.489	11.210	11.182
$n = 40$	12.915	12.891	12.771	12.747
$n = 45$	14.835	14.813	14.521	14.499
$n = 50$	16.292	16.272	16.141	16.121
$n = 100$	32.92819	32.91833	32.76966	32.75976
$n = 1000$	332.72496	332.72396	332.56154	332.56054

Table 2: Mean error obtained by the various approximations with respect to 10000 instances of randomly generated Toeplitz matrices T_n with entries in $[-1,1]$.

	$\|T_n - C_n(T_n)\|_F$	$\|T_n - H_n(T_n)\|_F$	$\|T_n - G_n(T_n)\|_F$	$\|T_n - V_n(T_n)\|_F$
$n = 5$	0.18725	0.16362	0.18190	0.15743
$n = 10$	0.71534	0.68775	0.67302	0.64363
$n = 15$	1.43778	1.41100	1.33331	1.30439
$n = 20$	2.28601	2.26095	2.10745	2.08025
$n = 25$	3.17788	3.15482	2.92053	2.89542
$n = 30$	4.07270	4.05158	3.73644	3.71341
$n = 35$	4.95798	4.93865	4.54353	4.52243
$n = 40$	5.79877	5.78109	5.31037	5.29105
$n = 45$	6.59117	6.57494	6.03320	6.01547
$n = 50$	7.30809	7.29317	6.68763	6.67133
$n = 100$	11.56697	11.55943	10.60308	10.59485
$n = 1000$	13.68293	13.68225	13.43137	13.43068

Table 3: Mean error obtained by the various approximations with respect to 10000 instances of randomly generated Toeplitz matrices T_n with entries in $[0,1]$ in decreasing way.

	$range = [-1,1]$	$range = [-1,1], decreasing$
$n = 5$	4994	0
$n = 10$	5019	9992
$n = 15$	8989	10000
$n = 20$	8727	10000
$n = 25$	9794	10000
$n = 30$	9765	10000
$n = 35$	9973	10000
$n = 40$	9943	10000
$n = 45$	9993	10000
$n = 50$	9990	10000
$n = 100$	10000	10000
$n = 1000$	10000	10000

Table 4: Number of times in which the first proposed approximation gives better results than that in [5] with respect to 10000 instances of randomly generated Toeplitz matrices T_n with entries in $[0,1]$ in decreasing way.

References

[1] Andrecut, M.: Applications of left circulant matrices in signal and image processing, Modern Physics Letters B, **22**, 231–241 (2008)

[2] Badeau, R., Boyer, R.: Fast multilinear singular value decomposition for structured tensors. SIAM J. Matrix Anal. Appl. **30**, 1008–1021 (2008)

[3] Bedini, L., Gerace, I., Salerno, E., Tonazzini, A.: Models and Algorithms for Edge-Preserving Image Reconstruction. Adv. Imaging Electron Physics **97**, 86–189 (1996)

[4] Bedini, L., Gerace, I., Tonazzini, A.: A Deterministic Algorithm for Reconstruction Images with Interacting Discontinuities. Computer Vision Graphics and Image Processing: Graphical Models Images Processing **56**, 109–123 (1994)

[5] Bini, D., Favati, P.: On a matrix algebra related to the discrete Hartley transform. SIAM J. Matrix Anal. Appl. **14**, 500–507 (1993)

[6] Blake, A.: Comparison of the Efficiency of Deterministic and Stochastic Algorithms for Visual Reconstruction. IEEE Trans. Pattern Anal. Machine Intell. **11**, 2–12 (1989)

[7] Blake, A., Zisserman, A.: Visual Reconstruction. MIT Press, Cambridge, MA (1987)

[8] Boccuto, A., Gerace, I., Giorgetti, V., Greco, F.: Gamma–matrices: a new class of simultaneously diagonalizable matrices. https://arxiv.org/abs/2107.05890 (2021)

[9] Boccuto, A., Gerace, I., Martinelli, F.: Half-quadratic image restoration with a non-parallelism constraint, J. Math. Imaging Vis. **59**, 270–295 (2017)

[10] Boccuto, A., Gerace, I., Pucci, P.: Convex Approximation Technique for Interacting Line Elements Deblurring: A New Approach. J. Math. Imaging Vis. **44**, 168–184 (2012)

[11] Bose, A., Saha, K.: Random circulant matrices. CRC Press, Taylor & Francis Group, Boca Raton-London-New York (2019)

[12] Bouman, C., Sauer, K.: A Generalized Gaussian Image Model for Edge-Preserving MAP Estimation. IEEE Trans. Image Process. **2** (3), 296–310 (1993)

[13] Brézis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York-Dordrecht-Heidelberg-London (2011)

[14] Carrasquinha, E., Amado, C., Pires, A. M., Oliveira, L.: Image reconstruction based on circulant matrices. Signal Processing: Image Communication **63**, 72–80 (2018)

[15] Chan, R.: The spectrum of a family of circulant preconditioned Toeplitz systems. SIAM J. Numer. Anal. **26**, 503–506 (1989)

[16] Chan, R. H., Jin, X.-Q., Yeung, M.-C.: The Spectra of Super-Optimal Circulant Preconditioned Toeplitz Systems. SIAM J. Numer. Anal. **28**, 871–879 (1991)

[17] Chan, R. H., Strang, G.: Toeplitz equations by conjugate gradients with circulant preconditioner. SIAM J. Sci. Stat. Comput. **10**, 104–119 (1989)

[18] Charbonnier, P., Blanc-Féraud, L., Aubert, G., Barlaud, M.: Deterministic Edge-Preserving Regularization in Computed Imaging. IEEE Trans. Image Process. **6**, 298–311 (1997)

[19] Codenotti, B., Gerace, I., Vigna, S.: Hardness results and spectral techniques for combinatorial problems on circulant graphs. Linear Algebra Appl. **285**, 123–142 (1998)

[20] Davis, P. J.: Circulant matrices. John Wiley & Sons, New York (1979)

[21] Demoment, G.: Image Reconstruction and Restoration: Overview of Common Estimation Structures and Problems. IEEE Trans. Acoust., Speech, and Signal Processing **37**, 2024–2036 (1989)

[22] Di Fiore, C., Zellini, P.: Matrix algebras in optimal preconditioning. Linear Algebra Appl. **335**, 1–54 (2001)

[23] Ding, W., Qi, L., Wei, Y.: Fast Hankel tensor-vector product and applications to exponential data fitting. Numerical Linear Algebra Appl. **22**, 814–832 (2015)

[24] Discepoli, M., Gerace, I., Mariani, R., Remigi, A.: A Spectral Technique to Solve the Chromatic Number Problem in Circulant Graphs. In: Computational Science and its Applications - International Conference on Computational Science and Its Applications 2004, Lecture Notes in Computer Sciences **3045**, 745-754 (2004)

[25] Donatelli, M., Serra–Capizzano, S.: Antireflective boundary conditions for deblurring problems. J. Electr. Comput. Eng., Art. ID 241467, 18 pp. (2010)

[26] Evans, D. J., Okolie, S. O.: Circulant matrix methods for the numerical solution of partial differential equations by FFT convolutions J. Computational Appl. Math. **8**, 238–241 (1982)

[27] Geman, D., Reynolds, G.: Constrained restoration and the recovery of discontinuities. IEEE Trans. Pattern Analysis and Machine Intelligence **14** (3), 367–383 (1992)

[28] Gilmour, A. E.: Circulant matrix methods for the numerical solution of partial differential equations by FFT convolutions. Appl. Math. Modelling **12**, 44–50 (1988)

[29] Gerace, I., Greco, F.: The Travelling Salesman Problem in symmetric circulant matrices with two stripes. Mathematical Ctructures in Computer Science, **18**, Special Issue 1: In memory of Sauro Tulipani, 165–175 (2008)

[30] Gerschgorin, S.: Über die Abgrenzung der Eigenwerte einer Matrix. Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk **6**, 749–754 (1931)

[31] Greco, F., Gerace, I.: The Traveling Salesman Problem in Circulant Weighted Graphs With Two Stripes. Electronic Notes in Theoretical Computer Science **169**, 99–109 (2007)

[32] Gutekunst, S. C., Williamson, D. P.: Characterizing the Integrality Gap of the Subtour LP for the Circulant Traveling Salesman Problem. SIAM J. Discrete Math. **33**, 2452–2478 (2019)

[33] Gutekunst, S. C., Jin, B., Williamson, D. P.: The Two-Stripe Symmetric Circulant TSP is in P. https://people.orie.cornell.edu/dpw/2stripe.pdf (2021)

[34] Györi, I., Horváth, L.: Utilization of Circulant Matrix Theory in Periodic Autonomous Difference Equations. Int. J. Difference Equations **9**, 163–185 (2014)

[35] Györi, I., Horváth, L.: Existence of periodic solutions in a linear higher order system of difference equations. Computers and Math. Appl. **66**, 2239–2250 (2013)

[36] Hadamard, J.: Lectures on Cauchy's Problem in Linear Partial Differential Equations. New Haven, Yale Univ. Press, Yale (1923)

[37] Henriques, J. F.: Circulant structures in computer vision. Ph. D. thesis., Department of Electrical and Computer Engineering, Faculty of Science and Technology, Coimbra (2015)

[38] Lei, Y. J., Xu, W. R., Lu, Y., Niu, Y. R., Gu, X. M.: On the symmetric doubly stochastic inverse eigenvalue problem. Linear Algebra Appl. **445**, 181–205 (2014)

[39] Mobahi, H., Fisher, J. W.: A Theoretical Analysis of Optimization by Gaussian Continuation. In: W.-K. Wong and D. Lowd (Eds.), Twenty–Ninth Conference on Artificial Intelligence of the Association for the Advancement of Artificial Intelligence (AAAI), Proceedings. Austin, Texas, USA, January 25-30, 2015, pp. 1205–1211 (2015)

[40] Nikolova, M.: Markovian reconstruction using a GNC approach. IEEE Trans. Image Processing **8** (9), 1204–1220 (1999)

[41] Nikolova, M., Ng, M. K., Tam C.–P.: On ℓ_1 Data Fitting and Concave Regularization for Image Recovery. SIAM J. Sci. Comput., **35** (1), A397–A430 (2013)

[42] Nikolova, M., Ng, M. K., Zhang, S., Ching, W.-K.: Efficient Reconstruction of Piecewise Constant Images Using Nonsmooth Nonconvex Minimization. SIAM J. Imaging Sci. **1** (1), 2–25 (2008)

[43] Papy, J. M., De Lauauer, L., Van Huffel, S.: Exponential data fitting using multilinear algebra: The single-channel and the multi-channel case. Numerical Linear Algebra Appl. **12**, 809–826 (2005)

[44] Qi, L.: Hankel tensors: Associated Hankel matrices and Vandermonde decomposition. Commun. Math. Sci. **13**, 113–125 (2015)

[45] Robini, M. C., Magnin, I. E.: Optimization by Stochastic Continuation. SIAM J. Imaging Sci. **3** (4), 1096–1121 (2010)

[46] Serra-Capizzano, S., Sesana, D.: A note on the eigenvalues of g-circulants (and of g-Toeplitz, g-Hankel matrices). Calcolo **51**, 639–659 (2014)

[47] Tee, G. J.: Eigenvectors of block circulant and alternating circulant matrices. New Zealand J. Math. **36**, 195–211 (2007)

[48] Wilkinson, J.: The algebraic eigenvalue problem. Clarendon Press, Oxford (1965)