An Algebra of Infinity[1]

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Correction of Dirac Delta Function is presented. An algebra of infinity, i.e., Alpha Algebra, is introduced. An algebra of infinitesimality, a.k.a., Omega Algebra, is illustrated.

Prologue[3]

Hello everyone, thank you for your kind and generous readership //:-D This is a research paper, but I will keep it as entertaining as possible. We'll present several new ideas about mathematical infinity and infinitesimality in this paper. Please enjoy-

I. Correction of Dirac Delta Function

1. Rectangle Function

The following method is something this author learned from a webpage in stackexchange.com a while ago. It is about a definition of a rectangle function R(x), using analytic geometry:

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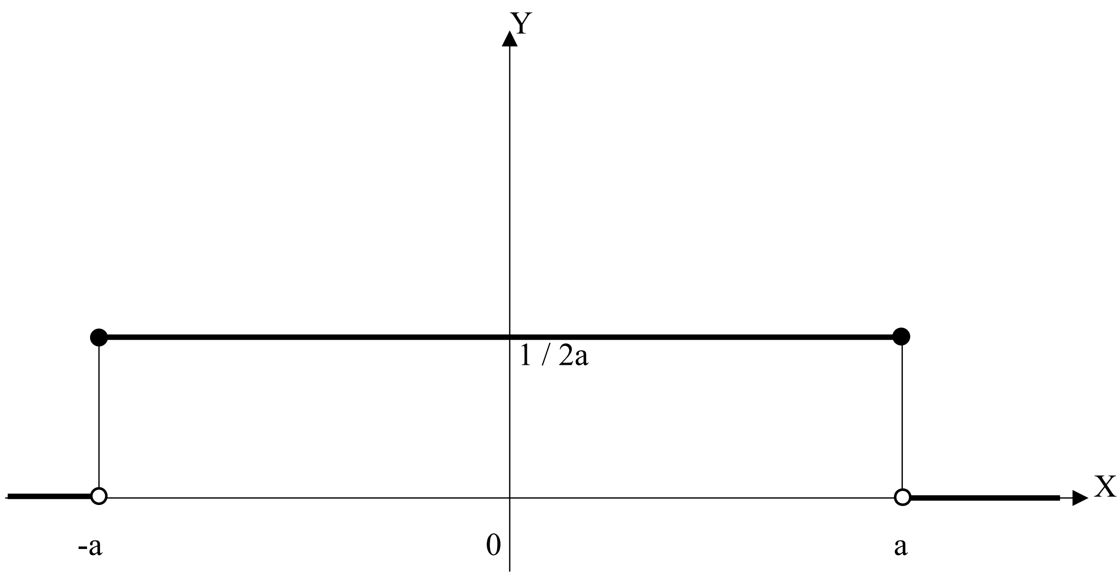
[1] This paper is dedicated to the author's family members and friends who also played parental figures, eternal inspirers, continuing educators, and spiritual mentors to him in being there for him when no one else was, who corrected him when he was wrong, and who taught him life lessons and everlasting wisdoms. He also especially thanks the wonderful people that he met online in social media like in twitter, Instagram, linkedIn, facebook, blogger, youtube, vimeo, and online databases like bexpress of Berkeley Law, vixra, social science research network, etc. He also thanks anonymous writers and scholars who contribute to stackexchange.com, Wikipedia.com, and other websites. Started being written on 1/19/2022. He also thanks his former and current employers, colleagues at work. He also specially thanks his ex-girlfriends and ex-dates and female friends, who kindly granted him their precious feminine time and helped him to survive in this harsh and cold and lonely world otherwise… He also thank his brotherly male friends too for their brotherly love- He's a secular Christian, politically independent, and a private academic. And yes, he's 100% heterosexual, an openly straight man. He is currently running for the US Senate as an independent Alaskan //:-)

[2] A lawyer by trade, a scientist by hobby, a humanologist by mission, a U.S. Army veteran by record, a former computer programmer, a prior PhD candidate in computational biology (withdrawn after 2 years without a degree), a former actor/writer/director/indie-filmmaker/background-music-composer. Born in the USA, 1978. Grew up in Seoul, South Korea as a child and a teenager. Returned to America as a college student. Still growing up in America as a person //!-)

[3] For the sketch summary of this paper, see https://huhnkielee.blogspot.com/2021/10/disproof-of-dirac-delta-function.html .

Assume $a > 0$.

$$R(x) = 0 \quad \text{if} \quad |x| > a$$
$$= 1/2a \quad \text{if} \quad |x| \le a$$



That is, R(x) is a rectangle whose bottom side is between '-a' and 'a' on the x-axis. So the rectangle has the width of '2a'. Its height is defined as '1/2a'.

Now, let us define the area function A(a), representing the area of the rectangle above:

$$A(a) = 2a * (1 / 2a) = 1$$

Next, we find the limit of the area as 'a' approaches to zero:

$$\lim_{a \to 0} A(a) = \lim_{a \to 0} 1 = 1$$

That is, the R(x) rectangle is becoming very thin and very tall, as the rectangle's width shrinks and its height elongates. But, the speed of horizontal shrinking and the speed of vertical elongation are identical. That is why the limit of the area still converges to one.

2. Dirac Delta Function as It Is Defined[4]

Next, let us see the definition of Dirac's Delta function:

$$D(x) = 0 \quad \text{if } x \neq 0$$
$$= \infty \quad \text{if } x = 0$$

And, D(x) must satisfy the following condition:

$$\int_{-\infty}^{+\infty} D(x)\, dx = 1$$

Now, let's get the integration of D(x) from the first and second part of the definition above:

$$\int_{-\infty}^{+\infty} D(x)\, dx = \int_{-\infty}^{-0} D(x)\, dx + \int_{-0}^{+0} D(x)\, dx + \int_{+0}^{+\infty} D(x)\, dx$$
$$= 0 \quad + \quad \infty \quad + \quad 0 = \infty \neq 1$$

We reached a contradiction and therefore the proof of the nonexistence of Dirac Delta Function is complete.

Note that "1" is a constant, a fixed number. +∞ is not a constant, not a fixed number, but a variable that perpetually increases at a certain speed. Thus, +∞ cannot be equal to 1. Thus, there cannot exist such D(x). Q.E.D.

[4] See https://en.wikipedia.org/wiki/Dirac_delta_function .

3. Explanation of Mr. Dirac's Errors

Now, let's discuss where Mr. Paul Dirac erred.

The error that Mr. Paul Dirac committed is that he equalized infinitesimality with zero. Zero is a constant, a fixed number that does not change over time. In contrast, infinitesimality, such as "1 / ∞", is a variable that decreases perpetually in its magnitude. The erroneous part of the definition of Dirac's Delta function is:

$$D(x) = +\infty \qquad if\ x = 0$$

Mr. Paul Dirac used the variable ∞ and the constant 0 in the same equation and that was the error. This author argues that any mathematical theory based on Dirac's Delta function is a faulty one, because Dirac's Delta function is a mathematically wrong function. Not many mathematicians have the bravery to challenge the accuracy of a colossal, historic Nobel Laureate like Mr. Paul Dirac, because doing so might jeopardize their professional reputations. This is why the falsehood of Dirac's Delta function has persisted in mathematics for a century. It is time to correct the errors, ladies and gentlemen.[5]

4. A Correction of This Author's Error

This author made an error above. Let's correct the mistake:

$$\int_{-\infty}^{+\infty} D(x)\,dx = \int_{-\infty}^{-0} D(x)\,dx + \int_{-0}^{+0} D(x)\,dx + \int_{+0}^{+\infty} D(x)\,dx = 0 + (\infty * 1/\infty) + 0 = 1$$

Basically, if the speed of infinity is equal to the speed of infinitesimality, then the product of such infinity and infinitesimality can be equal to one.

[5] This author also disproved Mr. Albert Einstein's special and general relativity theories, if you are interested: https://vixra.org/abs/2009.0211 https://vixra.org/abs/2010.0192 .

5. <u>Corrected Dirac Delta Function</u>

Let us institute the following rule about functions:

If an is a constant, the output should be a constant too.

The contrapositive of the propositional statement above would be:

If an output is a variable, the input should be a variable too.

Let's analyze the middle term of the right side of the previous equation:

$$\int_{-0}^{+0} D(x)\,dx = \int_{-0}^{0} D(x)\,dx + \int_{0}^{+0} D(x)\,dx = \left(\frac{\infty}{2} * \frac{1}{\infty}\right) + \left(\frac{\infty}{2} * \frac{1}{\infty}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

Then, the correct way to define Dirac Delta function is as follows:

(1) *If x is a constant,*

D(x) = 0 *if* $x \neq 0$

= undefined *if* $x = 0$

(2) *If x is a variable,*

D(+0) $= \frac{\infty}{2}$ *[+0 is the positive infinitesimality with speed 1/x]*

D(-0) $= \frac{\infty}{2}$ *[-0 is the negative infinitesimality with speed 1/x]*

(3) *And, D(x) must satisfy the following condition:*

$$\int_{-\infty}^{+\infty} D(x)\, dx = 1$$

Again, an infinite or infinitesimal number is not a constant, but a variable. We will discuss the calculation between two infinite or infinitesimal numbers later in this paper.

II. Relative Number Density Theory of Infinity

1. A Preliminary: Additive Set Theory[6]

We'll define operations between a set and a number that we'll call 'set-number operation'. Let '○' be a variable that represents an operator, like +, -, *, or /. We will define operations between a set 'A' and a number 'x' as follows:

$$A \circ x = \{ m \circ x \mid m \in A \}$$

For examples,

$$\{3, 5, 10\} + 1 = \{4, 6, 11\}$$

$$2 / \{2, 1, 7\} = \{1, 0.5, 2/7\}$$

Note that:

$$| A \circ x | = | A |$$

That is, the number of members of a set does not change after an application of operations with a number.

[6] See https://en.wikipedia.org/wiki/Freeman_Dyson ; https://en.wikipedia.org/wiki/Additive_number_theory ; https://en.wikipedia.org/wiki/Sumset ; https://en.wikipedia.org/wiki/Minkowski_addition .

Then, let's think about the next situation of applying several operators in sequence.

$A \circ x = B$

$|B| = |A|$

$(A \circ x) \circ y = C$

$B \circ y = C$

$|C| = |B \circ y| = |B| = |A|$

As we can see, no matter how many times we conduct operation between a set and a number, the resulting set's size is equal to the original input set's size. Let's call this 'cardinality conservation theorem.'

2. Unit Infinity

Mr. Georg Cantor[7] defined the size of an infinite set, in terms of one-to-one correspondence. We do not think that is a right way to define the size of an infinite set and thus we will hereby propose an alternative way to define it.

Let's think about the set of all natural numbers, a.k.a., the set of all positive integers, N:

$$N = \{1, 2, 3, \ldots\} = \bigcup_{m=1}^{\infty} \{m\}$$

We can also define N recursively:

(1) $1 \in N$
(2) $x \in N \rightarrow x + 1 \in N$

Now, let us define alpha as a unit infinity, which has the size of N:

$$\alpha \equiv |N|$$

[7] See https://en.wikipedia.org/wiki/Cardinal_number .

3. Relative Size of Other Infinite Sets

Then, we can express the size of the set of all integers, I, in terms of alpha, like so:

$$I = (-1*N) \cup \{ 0 \} \cup N$$

$$| I | = \alpha + 1 + \alpha = 2 \alpha + 1$$

Next, let's think about the size of the set all rational numbers, Q, in terms of alpha. A rational number is defined as follows:

$$x \in Q \rightarrow \text{ there exist } a \text{ and } b\text{, such that } x = a / b\text{, and } a \in N,\ b \in N$$

Let's think about the set of all the rational numbers larger than 0 and less than or equal to 1:

$$Q(0, 1) = \{1/1, 1/2, 1/3, \ldots\} = 1 / N$$

As we can see, the size of the set above is the same as that of the natural number set:

$$| Q(0, 1) | = | 1 / N | = | N | = \alpha$$

Now, let's think about all the rational numbers larger than 1 and less than or equal to 2:

$$Q(1, 2) = \{1 + 1/1, 1 + 1/2, 1 + 1/3, \ldots\} = 1 + \{1/1, 1/2, 1/3, \ldots\} = 1 + (1 / N)$$

Again, the size of the set above is equal to the size of the natural number set, N:

$$| Q(1, 2) | = | 1 + (1 / N) | = | N | = \alpha$$

Then, let's express the size of all positive rational numbers, in terms of alpha:

$$Q(0, +\infty) = Q(0, 1) \cup Q(1, 2) \cup Q(2, 3) \cup Q(3, 4) \cup \ldots$$

As you can see, the size of the set above is as follows:

$$| Q(0, +\infty) | = \alpha + \alpha + \alpha + \alpha + \ldots = \sum_{i=1}^{i=\alpha} \alpha = \alpha^2$$

Next, the size of the set of all the negative rational numbers would be:

$$| Q(-\infty, 0) | = | -1 * Q(0, +\infty) | = | Q(0, +\infty) | = \alpha^2$$

Finally, the size of the set of all rational numbers, Q, will be:

$$Q = Q(-\infty, 0) \cup \{0\} \cup Q(0, +\infty)$$

$$|Q| = \alpha^2 \quad + \quad 1 \quad + \quad \alpha^2$$
$$= 2\,\alpha^2 + 1$$

What would be the size of all irrational numbers? We will leave that question to our distinguished readers or future generation mathematicians //:-)

III. Introduction to Alpha Algebra

1. Axiomatic Definitions

The following properties of our infinite algebra are definitional and axiomatic, inspired by big-O notation convention in computer science and L'Hopital's Rule in calculus. [8]

(1) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \rightarrow \quad f(\alpha) \gg g(\alpha)$

(2) $1 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty \quad \rightarrow \quad f(\alpha) > g(\alpha)$

(3) $1 = \lim_{x\to\infty} \frac{f(x)}{g(x)} \quad \rightarrow \quad f(\alpha) = g(\alpha)$

(4) $0 < \lim_{x\to\infty} \frac{f(x)}{g(x)} < 1 \quad \rightarrow \quad f(\alpha) < g(\alpha)$

(5) $0 = \lim_{x\to\infty} \frac{f(x)}{g(x)} \quad \rightarrow \quad f(\alpha) \ll g(\alpha)$

(6) $f(\alpha) \gg g(\alpha) \quad \rightarrow \quad f(\alpha) + g(\alpha) = f(\alpha)$ [Absorption Rule]

(7) $f(\alpha) \ll g(\alpha) \quad \rightarrow \quad f(\alpha) + g(\alpha) = g(\alpha)$ [Absorption Rule]

2. Motivational Instances

Let's make one important example:

$\alpha + 1 = \alpha$

This is a philosophically profound statement. Let's see what it means for us:

[8] See https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule ; https://en.wikipedia.org/wiki/Big_O_notation .

$N = \{1, 2, 3, 4, \ldots \}$

$|N| = \alpha$

$I^{0+} = \{0\} \cup N = \{0, 1, 2, 3, \ldots\}$ [Set of Non-negative Ingeters]

$I^{0+} = \{0, 1, 2, 3, \ldots\} = \{1\text{-}1, 2\text{-}1, 3\text{-}1, 4\text{-}1, \ldots \} = N - 1$ [Set-Number Operation]

$|\mathrm{N}| = |\mathrm{N} - 1| = \alpha$ [Cardinality Conservation Theorem]

$| \{0\} \cup N | = \alpha + 1$

$| \{0\} \cup N | = | I^{0+} | = \alpha$

$\therefore \quad \alpha = \alpha + 1$

Of course,

$0 \neq 1$

In our brand-new infinity algebra, you can't cancel alpha's out from both sides of the equation.[9] It's not the same as the algebra that we're used to. Then what's the point of inventing a new algebra? It's for convenience's sake, as we will see later in the paper. At least it'll serve as a good, efficient notational convention //:-)

[9] In computer programming languages, "x = x + 1" is called an increment, and '=' is called an assignment operator, which looks similar to what we do here, but they're entirely different concepts however. See https://en.wikipedia.org/wiki/Assignment_operator_(C%2B%2B) .

3. Illustrative Examples

The purposes of alpha algebra are two-folds. First, we want to simplify the calculation of limits and big-O notations. Second, we want to distinguish different kinds of infinite or infinitesimal numbers. How a computer programming algorithm scales according to increasing input size[10] is analogous to the 'speed' of a mathematical function.

For instance, the following two functions have the same slope of 2:

$$f(x) = 2y + 1$$

$$g(x) = 2y - 3$$

'Speed' of a function is different from the tangential slope of the function at a given point, however. It's about relative speed, like when we compare two functions, we want to see which function eventually wins out, upon an input that is so humongous that it's like infinity.

Let's look at the traditional way of finding a limit of the following function:

$$\lim_{x\to\infty} (2x^3 + \sin x - \ln x) / (x^2 - 1) = \infty$$

But, the thing is, the ending result of 'infinity' is too broad a term. Now let's use our new method of alpha algebra to find out what 'kind' of infinity the result of the above is:

$$\lim_{x\to\infty} (2x^3 + \sin x - \ln x) / (x^2 - 1) = (2\alpha^3 + \sin \alpha - \ln \alpha) / (\alpha^2 - 1)$$

$$= 2\alpha^3 / \alpha^2 \qquad \text{[Absorption Rule]}$$

$$= 2\alpha$$

The absorption rule in alpha algebra is basically a progeny of L'Hopital's rule. Using the power of the absorption rule, we can easily see the following simplification in succession, all in our heads, given we're familiar with traditional algebra:

[10] See https://en.wikipedia.org/wiki/Computational_complexity_theory .

$f(\alpha) = 2^{\alpha} + \alpha^{3} - \alpha^{0.5} + \ln \alpha + 1/\alpha - 10 = 2^{\alpha}$

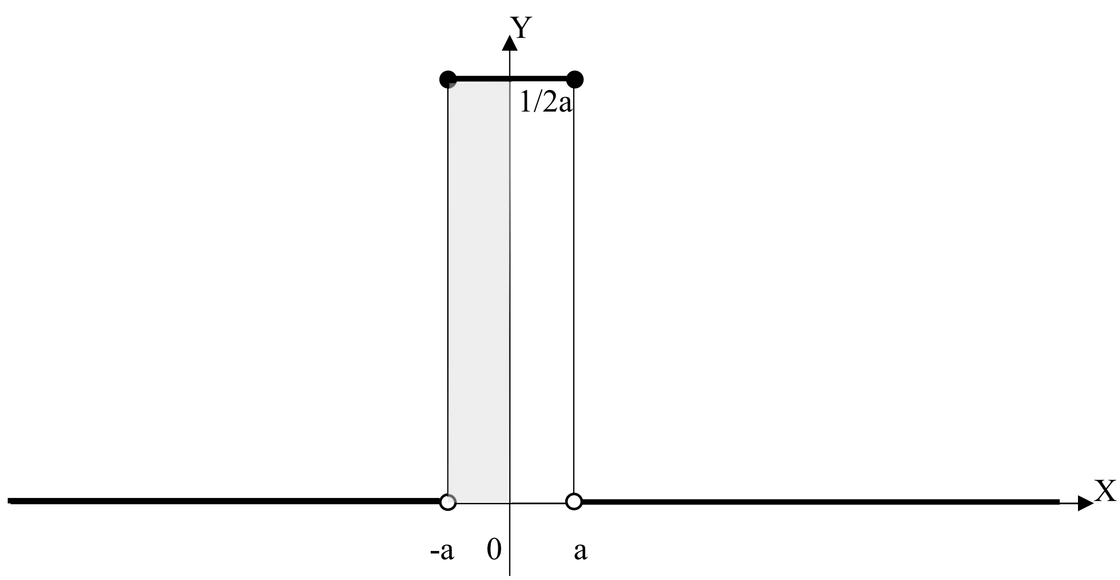
Except the absorption situation, alpha behaves like any other variables:

$\alpha + 2\,\alpha = 3\,\alpha$

$\alpha^{2} / \alpha^{4} = 1 / \alpha^{2}$

4. <u>Dirac Delta Revisited</u>

Let us draw the diagram again:



In the above diagram, let us divide our rectangle into two parts, vertically along the y-axis. We have the left half and the right half. If our distinguished readers kindly recall, we are horizontally shrinking, compressing the rectangle so that it becomes tall and thin. As we are shortening the horizontal width and elongating the vertical height of the rectangle, the area of the rectangle stays the same as 1:

$1 = (a + a) * (1/2a)$

Also, the area of the entire rectangle is the same as the summation of the areas of the left half and the right half:

$$A(a) = 1 = a * 1/2a \quad + \quad a * 1/2a$$

5. Unit Infinitesimality[11]

Earlier in this paper, we defined alpha as a unit infinity. In this section, we will define a unit infinitesimality and call it 'omega':

$$\Omega \equiv 1 / \alpha$$

Continuing from the previous section, let us calculate the limit as 'a' approaches zero:

$$\lim_{a \to 0} A(a) = \lim_{a \to 0} \left(a * \frac{1}{2a} \quad + \quad a * \frac{1}{2a} \right) = \Omega * \frac{1}{2\Omega} + \Omega * \frac{1}{2\Omega}$$

$$= \Omega * \frac{\alpha}{2} + \Omega * \frac{\alpha}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

[11] See https://www.biblehub.com/revelation/22-13.htm . In the Book of Revelation, God says, "I am Alpha and Omega."

6. <u>Breaking the Mathematical Taboo: Dividing by Zero</u>[12]

In elementary schools, we learned that we should never divide by zero, because if we do so, it will break the equation:

$$2 * 0 = 3 * 0$$

$$2 * 0 / 0 = 3 * 0 / 0$$

$2 = 3$ [incorrect equation]

$2 \neq 3$ [correct inequality[13]]

$2 < 3$ [correct inequality]

In our brand-new-new world of alpha-omega algebra, we have a solution to this problem:

$$2\,\Omega \neq 3\,\Omega$$

$$2\,\Omega > 3\,\Omega$$

As we can see, omega is not zero, but it is infinitesimality. Zero is a constant, a fixed number. Infinitesimality is a variable, which changes and decreases forever, nearing zero. Two times omega is approaching zero more slowly than three times omega. Because three times omega is approaching zero faster than two time omega, we can say three time omega is 'smaller' than two times omega.

[12] German philosopher, Friedrich Nietzsche once said he is a philosopher with a hammer, meaning he's an iconoclast. We're kinda playing the role of mathematicians with a hammer, metaphotically speaking, to break a new ground in mathematics //!-) This is an ah-ha moment, a good time of mathematical epiphany. See https://en.m.wikipedia.org/wiki/Ecstasy_of_Saint_Teresa .

[13] See https://en.wikipedia.org/wiki/Inequality_(mathematics) .

7. <u>Introduction to Omega Algebra</u>

Omega algebra is like a mirror image of alpha algebra. Let's compare the alpha-omega algebra with traditional algebra that we're familiar with:

$2 < 3$

$-2 > -3$

$1/2 > 1/3$

$2 \alpha < 3 \alpha$

$2 \Omega > 3 \Omega$

Then, what would the absorption rule would look like in the omega world?

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It will be an antiparallel image of alpha algebra's absorption rule, so:

$\alpha + \alpha^2 = \alpha^2$

$\Omega + \Omega^2 = \Omega$

Let's find some illustrative example in polynomials, to assist ourselves in understanding the unfamiliar, uncharted territory of omega algebra:

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{1}{x^2}\right) = \frac{1}{\Omega}+\frac{1}{\Omega^2} = \alpha + \alpha^2 = \alpha^2$$

$$\lim_{x\to \infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = \frac{1}{\alpha}+\frac{1}{\alpha^2} = \alpha^{-1} + \alpha^{-2} = \alpha^{-1}$$

$$= \Omega \quad + \quad \Omega^2 \quad = \Omega$$

In the second equation above, we used the absorption rule in alpha algebra:

$\alpha^{-1} >> \alpha^{-2}$

$\therefore \alpha^{-1} + \alpha^{-2} = \alpha^{-1}$

$\therefore \Omega + \Omega^2 = \Omega$

Q.E.D.

Epilogue[14]

Hello everyone, thank you for your kind and generous readership //:-D We hope you enjoyed the show. Our next article to write and publish will be titled, "Generalized Algebraic Number System". There, we'll introduce some interesting new concepts in algebra.[15]

Thank you for your time and see you later, kind and generous ladies and gentlemen //:-)

[14] This paper was started being written on 1/19/2022. It was finished being written on 1/21/2022 //:-)
[15] See https://en.wikipedia.org/wiki/The_Road_Not_Taken .