GRAPH THINNESS, A LOWER BOUND AND COMPLEXITY

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ABSTRACT. The *thinness* of a simple graph $G = (V, E)$ is the smallest integer k for which there exist a total order $(V, <)$ and a partition of V into k classes $(V_1, ..., V_k)$ such that, for all $u, v, w \in V$ with $u < v < w$, if u, v belong to the same class and $\{u, w\} \in E$, then $\{v, w\} \in E$. We prove that

- there are n-vertex graphs of thinness $n - o(n)$, which answers a question of Bonomo-Braberman, Gonzalez, Oliveira, Sampaio, and Szwarcfiter,
- the computation of thinness is NP-hard, which is a solution to a long standing open problem posed by Mannino and Oriolo.

1. INTRODUCTION

The notion of a k-thin graph was introduced by Mannino, Oriolo, Ricci, Chandran [14, 15] and motivated by applications to frequency assignment problems. One particular result in [15] showed that the maximum weight stable set problem can be solved in polynomial time, provided that the input graph is given with the corresponding ordering and partition of its vertices, as in the definition of a k-thin graph, and where k should be bounded by a constant fixed in advance. We remark that G is 1-thin if and only if G is an interval graph, so the result of [15] came as a generalization of the earlier polynomial time solution for the maximum weight stable set problem on interval graphs [11, 13]. Nowadays, the polynomial time solutions, in cases when the inputs are restricted to the k-thin form with bounded k, are known for the maximum weight stable set [15], list matrix partition, rainbow domination [2], capacitated graph coloring [7], and several other problems [1, 6].

Problem 1 (GRAPH THINNESS).

Given: A simple graph $G = (V, E)$, a positive integer k.

Question: Do there exist

- *a total ordering $<$ of the vertex set V and*
- *a partition of V into k disjoint classes $s = (s_1, \dots, s_k)$*

such that, for all $u, v, w \in V$ with

$$(u < v < w) \quad \text{AND} \quad (\{u, w\} \in E) \quad \text{AND} \quad (u, v \in s_i \text{ for some } i), \tag{1.1}$$

one always has $\{v, w\} \in E$?

Definition 2. If (G, k) is a yes-instance in Problem 1, then the graph G is called *k-thin*. The *thinness* of G is the smallest integer τ for which G is τ-thin.

The aim of this paper is to prove the following.

Theorem 3. *The problem* GRAPH THINNESS *is NP-complete.*

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Theorem 4. *There exist graphs with n vertices and thinness $n - o(n)$.*

The above mentioned algorithmic applications motivate the study of the algorithmic complexity of k-thin graph recognition. As said above, the case $k = 1$ corresponds to interval graphs, which can be detected in polynomial time [8]. For general k, the question of the determination of the algorithmic complexity of detecting k-thin graphs was posed by Mannino and Oriolo [14] in 2002, and, until now, it remained open despite further extensive research [1, 2, 3, 4, 5, 6, 9, 10, 16].

In what follows, the pair $(s, <)$ as in Problem 1 is to be called a *certificate* of the *k-thinness* of G, and, since the validity of such a certificate can be checked in polynomial time, Problem 1 belongs to NP. Therefore, the remaining part of Theorem 3 is the NP-hardness, and we prove it with the use of the following gadget.

Problem 5 (GRAPH THINNESS WITH A GIVEN PARTITION)**.**
Given: A simple graph $G = (V, E)$, an integer k, a partition $s = (s_1, \ldots, s_k)$ of V.
Question: Does there exist a total ordering $<$ of the set V such that, for all $u, v, w \in V$ satisfying the conditions (1.1)*, one always has $\{v, w\} \in E$?*

Definition 6. If (G, k, s) is a yes-instance in Problem 5, then the partition s is said to *allow* a certificate of the k-thinness of G.

The complexity of Problem 5 is known.

Theorem 7 (Bonomo, de Estrada [2])**.** *Problem 5 is NP-complete.*

Several other results on the complexity of graph thinness are as follows. Bonomo and de Estrada [2] give a polynomial time algorithm to determine the optimal value of k in the definition of the k-thinness provided that the ordering $<$ is fixed. Also, they prove [2, 6] that the NP-completeness of the k'-thinness recognition implies the NP-completeness of the k-thinness recognition, for any fixed k', k with $k' \leqslant k$. Bonomo-Braberman, Gonzalez, Oliveira, Sampaio, and Szwarcfiter [6] extend this result to a related notion of *proper thinness* and several other graph invariants. Sampaio, Oliveira, and Szwarcfiter [16] construct a polynomial time algorithm for a notion similar to the 2-thinness but with respect to the so called *precedence proper thinness*. Bonomo-Braberman, Oliveira, Sampaio, and Szwarcfiter [3, 4, 5] show that the *precedence proper k-thinness* is NP-complete for general k and polynomial time solvable when k is fixed. In a more recent preprint, Bonomo-Braberman and Brito [1] present a polynomial time algorithm for the situation when, additionally to the partition $(V_1, \ldots, V_k)$ in the definition of the k-thinness, one is given the restriction of the ordering $<$ to every set in the partition.

Remark 8. Throughout our paper, all graphs are assumed to be simple.

The paper has the following structure. In the forthcoming Section 2, we recall some relevant notation and prove several results needed in our discussion. In Section 3, we present the polynomial reduction from GRAPH THINNESS WITH A GIVEN PARTITION to GRAPH THINNESS, and, in view of Theorem 7, this implies the validity of Theorem 3. In Section 4, we switch to Theorem 4 and discuss its motivation, and we prove this theorem with a probabilistic argument. The remaining Section 5 collects several further remarks on our arguments and possible further work.

2. An auxiliary construction

We begin with two caveats on the use of some standard notation.

Remark 9. A *clique* of a graph $G = (V, E)$ is a subset of V that induces a complete subgraph of G. In particular, for any $u \in V$, the sets $\varnothing$ and $\{u\}$ are cliques of G.

Remark 10. We write $U \subset V$ for two sets U, V if every element of U is contained in V. In particular, we can write $U \subset V$ even if $U = V$.

We proceed with several techniques needed in our reduction.

Definition 11. Assume $G = (V, E)$ is a simple graph, and let $U \subset V$. We define $\mathcal{B}(G, U)$ as the graph

- with the vertex set $V \cup \{\alpha, \beta\}$, where $\alpha, \beta \notin V$ are new vertex labels,
- with all edges in E, and, apart from these, with an edge from α to every vertex in $V \setminus U$, and with an edge from β to every vertex in $V \setminus U$.

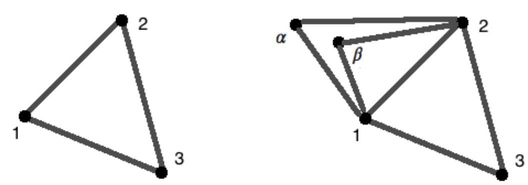

Figure 1. An example of G and $\mathcal{B}(G, \{3\})$ as in Definition 11.

Our next result may look similar to Lemma 16 in [2], which states that

$$\text{thinness}(G \vee 2K_1) = \text{thinness}(G) + 1$$

if G is not a clique, where $G \vee 2K_1$ appears as $\mathcal{B}(G, \varnothing)$ in Definition 11.

Lemma 12. *Let G, U be as in Definition 11. Suppose that, for some integer k, a partition $s = (s_1, \ldots, s_k)$ allows a k-thinness certificate of $\mathcal{B}(G, U)$. Then for some*

- *label $i \in \{1, \ldots, k\}$ and*
- *subset $C \subset V \setminus U$ that is a clique of G,*

no vertex in $V \setminus (U \cup C)$ belongs to the class s_i.

Proof. Let $<$ be an ordering of $V \cup \{\alpha, \beta\}$ that is compatible with the partition s, so that $(s, <)$ is a certificate of the k-thinness of $\mathcal{B}(G, U)$. Since the labels α, β can be swapped without changing the graph $\mathcal{B}(G, U)$, we can assume without loss of generality that $\alpha < \beta$, and then we take a label i such that $\alpha \in s_i$.

Step 1. Let $w \in V \setminus U$ be a vertex with $w < \alpha$. If $w \in s_i$, then we have

$$w < \alpha < \beta \text{ and } w, \alpha \in s_i. \tag{2.1}$$

However, Definition 11 implies that w and β are adjacent in $\mathcal{B}(G, U)$, but at the same time α, β are not adjacent. Therefore, we arrived at a contradiction with the fact that $(s, <)$ is a k-thinness certificate, and hence we cannot have $w \in s_i$.

Step 2. Now let $w', w'' \in V \setminus U$ be two vertices such that

$$\alpha < w' < w'' \text{ and } w' \in s_i. \tag{2.2}$$

Also, by Definition 11, the vertices α, w'' are adjacent in $\mathcal{B}(G,U)$, and, since $(s,<)$ is a k-thinness certificate, we get $\{w', w''\} \in E$.

In Step 1, we showed that every vertex

$$w \in s_i \cap (V \setminus U) \tag{2.3}$$

should satisfy $\alpha < w$. Using Step 2, we see that, if there are two such vertices w', w'', then they should be adjacent in E. In other words, the set of all vertices w as in (2.3) should be a clique of G. □

Now we explain how to extend a k-thinness certificate of G to $\mathcal{B}(G,U)$.

Lemma 13. *Let G, U be as in Definition 11. Suppose that, for some integer k, a partition $s = (s_1, \ldots, s_k)$ and ordering $<$ certify the k-thinness of G, where*

$$s_1 = U. \tag{2.4}$$

Also, we define $\sigma_1 = s_1 \cup \{\alpha, \beta\}$ and extend the ordering $<$ by adding the relations

$$v < \alpha,\ v < \beta,\ \alpha < \beta$$

for all $v \in V$. Then the partition $s' = (\sigma_1, s_2, \ldots, s_k)$ and the extended ordering $<$ are a k-thinness certificate of $\mathcal{B}(G,U)$.

Proof. In order to apply the definition of the k-thinness, we take $u, v, w \in V \cup \{\alpha, \beta\}$ such that $u < v < w$ and u, w are adjacent in $\mathcal{B}(G,U)$, and

$$u, v \text{ belong to the same class of the partition } s'. \tag{2.5}$$

We need to check that v, w are adjacent in $\mathcal{B}(G,U)$.

Step 1. If $u, v, w \in V$, the conclusion follows because the purported certificate of the k-thinness of $\mathcal{B}(G,U)$ extends the initial certificate for G.

Step 2. If exactly one of the vertices u, v, w is in $\{\alpha, \beta\}$, then we assume without loss of generality that $w = \beta$. By Definition 11, since β and u are adjacent, we get $u \notin U$, and, according to the condition (2.4), this implies $u \notin s_1$. Now we apply (2.5) to get $v \notin s_1$, and then we use (2.4) to get $v \notin U$, from which, by Definition 11, we get a desired conclusion that v and β are adjacent in $\mathcal{B}(G,U)$.

Step 3. If both α, β appear among u, v, w, then $v = \alpha$, $w = \beta$. Similarly to Step 2, we get $u \notin s_1$ and hence $u \notin \sigma_1$. Since $v = \alpha \in \sigma_1$ by the assumptions of the lemma, the condition (2.5) violates, so there is nothing to prove in Step 3.

Since Steps 1, 2, 3 cover all possibilities, the proof is complete. □

3. The reduction

The following is the main construction in our reduction.

Definition 14. Assume that $G = (V,E)$ is a simple graph, k is a positive integer, and $s = (s_1, \ldots, s_k)$ is a partition of V. We define the graph $\mathcal{G}(G,k,s)$ as

$$\mathcal{B}(\mathcal{B}(\ldots \mathcal{B}(G, s_1), \ldots), s_{k-1}), s_k),$$

that is, in other words, $\mathcal{G}(G,k,s)$ is the k-fold application of the construction in Definition 11, in which the i-th application is

$$\mathcal{G}_i := \mathcal{B}\left(\mathcal{G}_{i-1}, s_i\right), \tag{3.1}$$

where $\mathcal{G}_0 = G$, and $\mathcal{G}_{i-1}$ is the graph obtained at the $(i-1)$-st iteration.

For yes-instances of Problem 5, the desired outcome is straightforward.

Lemma 15. *If (G, k, s) is a yes-instance of Problem 5, then $\mathcal{G}(G, k, s)$ is k-thin.*

Proof. Let α_i, β_i be the vertices added to the graph at the i-th application of Definition 11. Then, for every $t \in \{0, \ldots, k\}$, the partition

$$(s_1 \cup \{\alpha_1, \beta_1\}, \ldots, s_t \cup \{\alpha_t, \beta_t\}, s_{t+1}, \ldots, s_k) \tag{3.2}$$

allows a k-thinness certificate of $\mathcal{G}_t$ because, in fact, for $t = 0$, this is true as (G, k, s) is a yes-instance, and, for $t > 0$, this follows from Lemma 13 by the induction. In particular, the $t = k$ version of (3.2) certifies the k-thinness of $\mathcal{G}_k = \mathcal{G}(G, k, s)$. $\square$

A converse direction of Lemma 15 requires some further work.

Definition 16. If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are simple graphs with $V_1 \cap V_2 = \varnothing$, then we define $G_1 \oplus G_2$ as the graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Definition 17. If $s = (s_1, \ldots, s_k)$ is a partition of a set S, and P is a subset of S, then the partition

$$(s_1 \cap P, \ldots, s_k \cap P)$$

is called the *restriction* of s on P.

Lemma 18. *Let $G^1, \ldots, G^{k+1}$ be non-empty simple graphs. We consider the graph*

$$G = G^1 \oplus \ldots \oplus G^{k+1}$$

and a partition $s = (s_1, \ldots, s_k)$ of the vertex set of G. If, for every $q \in \{1, \ldots, k+1\}$,

- (N1) *the graph G^q is not $(k-1)$-thin,*
- (N2) *the triple (G^q, k, ψ^q) is a no-instance for Problem 5, where ψ^q is the restriction of s on the vertex set of G^q,*

then the graph $\mathcal{G}(G, k, s)$ is not k-thin.

Proof. We argue by contradiction, so we assume that $\mathcal{G}(G, k, s)$ is k-thin. Therefore, some partition $\sigma = (\sigma_1, \ldots, \sigma_k)$ allows a k-thinness certificate of $\mathcal{G}(G, k, s)$, and then, for all $t \in \{0, \ldots, k\}$ and $j \in \{1, \ldots, k\}$, we define

$$\sigma_{jt} := \sigma_j \cap V_t,$$

where V_t is the vertex set of the graph $\mathcal{G}_t$ as in Definition 14. Since the k-thinness certificates remain valid at the restrictions to induced subgraphs, the partition

$$(\sigma_{1t}, \ldots, \sigma_{kt})$$

certifies the k-thinness of $\mathcal{G}_t$. We recall that

$$\mathcal{G}_t = \mathcal{B}(\mathcal{G}_{t-1}, s_t) \tag{3.3}$$

by the condition (3.1). An application of Lemma 12 to the graph (3.3) allows us to find a clique $\mathcal{C}_t$ in $\mathcal{G}_{t-1}$ such that the set $V_t \setminus (s_t \cup \mathcal{C}_t)$ lies in the union of at most $(k-1)$ classes in $(\sigma_{1\,t}, \ldots, \sigma_{k\,t})$. A restriction of the latter statement to the vertices of G gives us a clique C_t of G and an index r_t such that

$$V \setminus (s_t \cup C_t) \text{ is a subset of } V \setminus \sigma_{0\,r_t} \tag{3.4}$$

for all $t \in \{1, \ldots, k\}$. Since we had $G = G^1 \oplus \ldots \oplus G^{k+1}$ by the initial assumption, there exists an index $q \in \{1, \ldots, k+1\}$ for which

$$G^q \text{ does not intersect } C_1 \cup \ldots \cup C_k. \tag{3.5}$$

Now we take the partition $\psi^q = (\psi_{q1}, \dots, \psi_{qk})$ as in (N2), that is, ψ^q is the restriction of s on the vertex set U^q of G^q, which means that we have

$$\psi_{qj} = s_j \cap U^q \tag{3.6}$$

for all $j \in \{1, \dots, k\}$. Also, we define another partition $\tau = (\tau_1, \dots, \tau_k)$ of U^q as the restriction of σ, that is, we get

$$\tau_j = \sigma_{0j} \cap U^q \tag{3.7}$$

for all $j \in \{1, \dots, k\}$. Now we use the conditions (3.6) and (3.7) to restrict the formula (3.4) to the set U^q. In view of (3.5), we get that

$$U^q \setminus \psi_{qt} \text{ is a subset of } U^q \setminus \tau_{r_t}$$

for all $t \in \{1, \dots, k\}$, and hence

$$\tau_{r_t} \text{ is a subset of } \psi_{qt} \tag{3.8}$$

for all $t \in \{1, \dots, k\}$. If $t \to r_t$ is a permutation, then (3.8) implies $\tau_{r_t} = \psi_{qt}$ for all t, which shows that τ is a permutation of ψ^q, and hence ψ^q allows a k-thinness certificate of G^q. This contradicts to the condition (N2), and hence, in fact, the mapping $t \to r_t$ cannot be injective. Therefore, there exists a label h such that

$$h = r_{t_1} = r_{t_2} \text{ for some } t_1 \neq t_2,$$

and then τ_h is a subset of both ψ_{qt_1} and ψ_{qt_2}. Since ψ^q is a partition, this implies that τ_h is an empty set, so the graph G^q admits a k-thinness certificate in which the empty sets appears in the corresponding partition of the vertices. This means that G^q is $(k-1)$-thin, so we obtain a contradiction to (N1) and complete the proof. $\square$

We need to generalize the notation from Definition 16.

Definition 19. Let V_1, V_2 be disjoint sets. For some k, let $s^1 = (s_{11}, \dots, s_{1k})$, $s^2 = (s_{21}, \dots, s_{2k})$ be partitions of V_1, V_2, respectively. Then we define

$$s^1 \oplus s^2 = (s_{11} \cup s_{21}, \dots, s_{1k} \cup s_{2k}).$$

Definition 20. Let V_1, V_2 be disjoint sets, and let $<_1$ and $<_2$ be total orderings on V_1, V_2, respectively. Then a total ordering $<$ on $V_1 \cup V_2$ is denoted $<_1 \oplus <_2$ if

- $<_1$ is contained in $<$,
- $<_2$ is contained in $<$,
- one has $v_1 < v_2$, for all $v_1 \in V_1$, $v_2 \in V_2$.

Observation 21. Assume $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be simple graphs with disjoint vertex sets. Let k be an integer, and assume that

- $<_1$ is a total ordering of V_1,
- $<_2$ is a total ordering of V_2,
- s^1 is a partition of V_1 into k classes,
- s^2 is a partition of V_2 into k classes,

then the following are equivalent:

- $(s^1 \oplus s^2, <_1 \oplus <_2)$ is a k-thinness certificate of $G_1 \oplus G_2$,
- $(s^i, <_i)$ is a k-thinness certificate of G_i with both $i = 1, 2$.

We need one further auxiliary graph, which was previously considered in [2, 6].

Definition 22. We consider the graph H_k with the vertices

$$\{\alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$$

in which $*_i$ and $\star_j$ are adjacent if and only if $i \neq j$, for any $*$, $\star$ in $\{\alpha, \beta\}$.

Remark 23. According to Lemma 16 in [2], the thinness of H_k equals k.

Definition 24. We write $(\psi_k, <_k)$ for the k-thinness certificate of H_k defined by

- $\psi_k = (\{\alpha_1, \beta_1\}, \dots, \{\alpha_k, \beta_k\})$,
- $*_i < \star_j$ if $i < j$ and $*, \star \in \{\alpha, \beta\}$,
- $\alpha_i < \beta_i$, for any $i \in \{1, \dots, k\}$.

Now we are ready to proceed with the reduction.

Definition 25. Let (G, k, s) be an instance of Problem 5. We create

- the $k+1$ copies $G_1, \dots, G_{k+1}$ of G,
- the $k+1$ copies $H_{k1}, \dots, H_{k\,k+1}$ of H_k

so that the vertex sets of all these copies are pairwise disjoint. We define

$$\overline{G} = (G_1 \oplus H_{k1}) \oplus \dots \oplus (G_{k+1} \oplus H_{k\,k+1}),$$

and, assuming that s_j and ψ_{kj} denote the copies of the corresponding partitions of the vertex sets of G and H_k, we take

$$\overline{s} = (s_1 \oplus \psi_{k1}) \oplus \dots \oplus (s_{k+1} \oplus \psi_{k\,k+1}),$$

and then we define the graph

$$\Gamma(G, k, s) := \mathcal{G}(\overline{G}, k, \overline{s}),$$

where $\mathcal{G}$ stands for the construction in Definition 14.

Theorem 26. *An instance (G, k, s) of Problem 5 is a 'yes' if and only if the graph*

$$\Gamma(G, k, s) = \mathcal{G}(\overline{G}, k, \overline{s})$$

is k-thin.

Proof. If (G, k, s) is a yes-instance in Problem 5, then $(\overline{G}, k, \overline{s})$ is also a yes-instance by Observation 21. In this case, the graph $\Gamma(G, k, s)$ is k-thin by Lemma 15.

If (G, k, s) is a no-instance in Problem 5, then we apply Lemma 18 to the graph $\overline{G}$. Then the corresponding condition (N1) is true by Remark 23, and we get the validity of (N2) from Observation 21. Therefore, the assertion of Lemma 18 is applicable, and hence the graph $\Gamma(G, k, s)$ is not k-thin. □

Theorem 26 gives a polynomial reduction to GRAPH THINNESS from Problem 5, which is known to be NP-complete [2]. This implies Theorem 3.

4. Graphs with large thinness

As said above, the notion of thinness is being extensively studied for almost two decades, but there are still many open questions on the behavior of this function and its relations to other graph invariants [1, 2, 5, 6, 14, 15]. One particular natural problem concerns the largest value of the thinness of an n-vertex graph.

Problem 27 (Section 5 in [6])**.** *Is there an n-vertex graph G with thinness $> n/2$?*

This section is devoted to the proof of Theorem 4, which gives an affirmative solution to Problem 27, and, in fact, this theorem determines the largest value of the thinness in the asymptotic sense. As we will see, our proof is probabilistic.

Definition 28. A graph $G = (V, E)$ with $|V| = 3m$ is called *m-obstructive* if one can enumerate its vertices as $(u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_m)$ so that, for every i and j in $\{1, \ldots, m\}$, one has either $\{u_i, w_j\} \notin E$ or $\{v_i, w_j\} \in E$.

Lemma 29. *If m, n are positive integers with $m > 11 \ln n$, then there exists a graph with n vertices which has no m-obstrucive induced subgraphs.*

Proof. We consider the random graph $G = (V, E)$ with $|V| = n$ such that the edges of G appear independently with probability $1/2$ each. For every fixed non-repeating sequence $\alpha = (u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_m)$ of vertices in V, the probability that α certifies the m-obstruction is $(3/4)^{m^2}$ because there are m^2 independent choices of (i, j) as in Definition 28, and each of the corresponding events $\{u_i, w_j\} \notin E$ or $\{v_i, w_j\} \in E$ happens with probability $1/2$ (which implies that their union occurs with probability $3/4$ by the independence). Since there are a total of at most n^{3m} ways to choose α, the expected total number of all those choices of α which give the m-obstruction certificates is at most

$$n^{3m} \cdot (3/4)^{m^2} = \exp\left(3m \ln n - m^2 \ln(4/3)\right) < 1,$$

and hence some choices of G do not admit m-obstructions at all. $\square$

Now we are ready to complete the proof of Theorem 4.

Theorem 30. *For any positive integer n, there exists a graph with n vertices whose thinness is at least $n - 72 \ln n$.*

Proof. Using Lemma 29, we take a graph $G = (V, E)$ with $|V| = n$ such that

$$G \text{ has no induced } m\text{-obstructive subgraphs with any } m > 11 \ln n. \tag{4.1}$$

We are going to complete the proof by showing that the thinness of G is at least $n - 72 \ln n$ as desired. If this was not the case, there would exist an ordering $(V, <)$ and a partition of V into at most $n - 72 \ln n$ classes as in the definition of the thinness, and then, for some integer c satisfying

$$c \geqslant 72 \ln n/3 = 24 \ln n, \tag{4.2}$$

we should be able to find c disjoint pairs in each of which both vertices are in the same class (the bound (4.2) follows because the worst case scenario is when every class has either 1 or 3 vertices). We enumerate these pairs as follows:

$$(u_1, v_1),\ \ldots,\ (u_c, v_c) \ \text{ with } \ v_1 < v_2 < \ldots < v_c \ \text{ and } \ u_i < v_i \ \text{ for all } \ i.$$

By the thinness, for any $i \in \{1, \ldots, c\}$, it never occurs that

$$(\{u_i, x\} \in E) \ \text{ AND } \ (\{v_i, x\} \notin E) \ \text{ with } \ x \in \{v_{i+1}, \ldots, v_c\}. \tag{4.3}$$

Now we define $m = \lfloor c/2 \rfloor$ and note that the sequence

$$\alpha = (u_1, \ldots, u_m, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m})$$

induces an m-obstruction because the condition (4.3) never occurs. By (4.1), we get $m \leqslant 11 \ln n$ and hence $c \leqslant 22 \ln n + 1$. A comparison to (4.2) implies $24 \ln n \leqslant 22 \ln n + 1$, which is a contradiction unless we are in the trivial case $n = 1$. $\square$

5. Concluding remarks

We constructed a polynomial reduction from GRAPH THINNESS WITH A GIVEN PARTITION to GRAPH THINNESS. In view of the result in [2], this proves the NP-completeness of GRAPH THINNESS, but the complexity status of the

- recognition of the graphs of thinness 2,
- computataion of the proper thinness of a graph

and many other related problems remains open [1, 2, 3, 4, 5, 6, 9, 10, 16].

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