A basis for set theory without the use of an axiom.

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Abstract

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Introduction

Things of any kind have properties, and they can be selected on the basis of their properties. This leads to collections of different things. Everything in such a collection exist there exactly one times and has the selecting property, and everything that has the selecting property is present in the collection. If one uses this type of collection to form sets, then the contradictions known from naive set theory appear. The reason for these contradictions always seems to be the same. If a property is chosen in such a way that the set created with the help of this property also has this property, then the situation arises that the created set should contain all objects with the property, but cannot contain them all. To avoid these contradictions, only properties whose formed sets do not have these properties are allowed. The consequence of this is that a set can never contain itself. And this in turn leads to the fact that Russell's Antinomy[1] is excluded from the formation of sets.

The criterion for deciding the existence of a set.

Without the use of an axiom, we can arrange the following:

1. P(x) is true if and only if x has the property P.
2. The term [x|P(x)] contains x if and only if x has the property P.
3. The term [x|P(x)] represents a set if and only if the term [x|P(x)] has not the property P. The existence of such a set is indicated by writing {x|P(x)} for the tern [x|P(x)].
   See also other definitions of sets[2].

Since such a set $\{x|P(x)\}$ doesn't have the property P, the set itself cannot be contained. For the term $\{x|P(x)\}$ therefore always $\{x|P(x)\} \notin \{x|P(x)\}$ applies and $\neg P(\{x|P(x)\})$ is always true. This leads to a decision criterion for the existence of a set. If we use square brackets to represent an attempt to form a set and curly braces to indicate the existence of a set, the criterion for the existence of a set is like follows:

$\neg P([x|P(x)]) \iff ([x|P(x)] \notin [x|P(x)]) \iff \exists \{x|P(x)\}$.

Regarding the properties, a distinction can be made between the following categories:

1. The property is always true.
2. The property is always false.
3. There are objects with the property, but there are also objects without the property.

Regarding the first case, where the property is always true.
This means $P([x|P(x)])$ is true and because of the decision criterion $[x|P(x)]$ cannot be a set. Below are some examples where this is the case.

$P \equiv$ "is an object" leads to $P([x|P(x)])$ since $[x|P(x)]$ is an object. This means the set of all things doesn't exist.
$P \equiv$ "is a set" leads to $P([x|P(x)])$ since $[x|P(x)]$ should be a set. This means the set of all sets doesn't exist.
$P \equiv (x=x)$ leads to $P([x|P(x)])$ since $[x|P(x)]=[x|P(x)]$ is true. This means the set of all identities doesn't exist.
$P \equiv (x \notin x)$ leads to $P([x|P(x)])$ since x is in this context a set and therefore, because of the decision criterion for a set, $x \notin x$ is always true. This means a set created with Russell's Antinomy $x \notin x$ doesn't exist. The following conclusions clarify this once again.

- $\exists \{x|x\notin x\} \Longrightarrow \{x|x\notin x\} \notin \{x|x\notin x\} \Longrightarrow \neg \exists \{x|x\notin x\}$.
- $\neg \exists \{x|x\notin x\} \Longrightarrow \{x|x\notin x\} \in \{x|x\notin x\} \Longrightarrow \neg \exists \{x|x\notin x\}$

After formulating Russell's Antinomy, $x\notin x \Rightarrow x\in x$ and $x\in x \Rightarrow x\notin x$ were inferred, which is formally correct, and thereupon the entire so-called "naive set theory" was discarded. Since the decision criterion excludes Russell's Antinomy from the formation of sets, these conclusions are correct, but not relevant for set theory presented in this paper.

Regarding the second case, where the property is always false. This means $P([x|P(x)])$ is false and therefore $\neg P([x|P(x)])$ is true. The decision criterion now says that the set $\{x|P(x)\}$ exist. Because $P(x)$ is always false, this set cannot contain any element. It is the empty set $\emptyset$.

Regarding the third case, where the property is sometimes true. Here, if one can show that P([x|P(x)]) is false, then it's proven, that the set {x|P(x)} exists.
If one can show that P([x|P(x)]) is true, then it's proven, that the object-selection [x|P(x)] cannot be a set.

Conclusion.

Properties that are always true are excluded from the formation of sets. Therefore, each set has the property of not being contained within itself. Russell's Antinomy thus becomes a property that is always true for sets. As a result, it is also excluded from the formation of sets.
This leads to a basis for set theory without the use of an axiom.

References

[1] Weck, Peter and Image Created "Russell's Antinomy" https://mathimages.swarthmore.edu/index.php/Russell's Antinomy

[2] Stover, Christopher and Weisstein, Eric W. "Set." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Set.html