Analysis of the Collatz conjecture through the methods of sequence theory

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Abstract

A solution is proposed for the so-called Collatz conjecture, also known as the "$3n + 1$ problem". The idea of the proof involves representing the algorithm's operations through an infinite sequence as a formal object. Hypotheses regarding the convergence or divergence are considered within the framework of sequence theory corresponds to hypotheses about the output values of the algorithm.

Keywords

$3n + 1$ problem; Collatz conjecture; convergence of sequences

1. Formulation of the algorithm

The task of this paragraph is to formulate the most convenient expression for the operation of the algorithm on some arbitrary natural number initially given to the algorithm, denoted as n_0. We will denote algorithmic operations as follows:

$$n_1 \xrightarrow{3n+1} 3n_1 + 1,$$

$$n_2 \xrightarrow{n/2} \frac{1}{2}n_2; \qquad (1.1)$$

where n is a natural number at any iteration of the algorithm, $n \in \mathbb{N}[1;\infty)$; n_1 is the same, but indicating that the number is odd; n_2 is an even number. It is evident that the number expressed $3n_1 + 1$ is even, while $\frac{1}{2}n_2$ can be either even or odd.

Definition 1.1

Any initial odd natural number can be expressed by the following formula:

$$n_0 = a_0 2^{m_0}; \tag{1.2}$$

where a_0 is a positive rational number, $a_0 \in \mathbb{Q}\left(\frac{1}{2}; 1\right]$; m_0 is a natural number corresponding to the proposed formula, $m_0 \in \mathbb{N}[0; \infty)$. For example, $5 = \frac{5}{8} \cdot 2^3$, $6 = \frac{3}{4} \cdot 2^3$, $27 = \frac{27}{32} \cdot 2^5$, $1 = 1 \cdot 2^0$, etc.

In the case where the initial number is even, it can be represented as:

$$n_0 = a_0 2^{m_0 + k};$$

where k is a natural number corresponding to the number of initial operations $\frac{1}{2} n_2$ leading to an odd number expressed by the formula $a_0 2^{m_0}$. For simplicity, we omit this representation, assuming that an odd number is an input into the algorithm.

To uniquely define the transformation of a number into the form (1.2), let us define recursive functions $a_0(n)$ and $m_0(n, x)$; n is passed as an argument to the functions as n_0, with $x = 0$:

$$a_0(n) = \begin{cases} a_0(n/2) & \text{if } n > 1, \\ n & \text{if } n \le 1; \end{cases} \tag{1.3}$$

$$m_0(n, x) = \begin{cases} m_0(n/2, x+1) & \text{if } n > 1, \\ x & \text{if } n \le 1. \end{cases} \tag{1.4}$$

Let us consider the result of applying the operation $3n + 1$ to the initial number:

$$a_0 2^{m_0} \xrightarrow{3n+1} 3a_0 2^{m_0} + 1 = \frac{3}{4} a_0 2^{m_0+2} + 1 = 2^{m_0+2}\left[\frac{3}{4} a_0 + \frac{1}{2^{m_0+2}}\right]. \tag{1.5}$$

In the proposed formulation, the expression in square brackets does not exceed unit: $\left[\frac{3}{4} a_0 + \frac{1}{2^{m_0+2}}\right] \le 1$, which corresponds to the applied constraints for parameters: $a_0 \in \mathbb{Q}\left(\frac{1}{2}; 1\right]$, $m_0 \in \mathbb{N}[0; \infty)$.

Definition 1.2

The application of the operation $3n + 1$ is equivalently defined as the application of the algorithm f_{3n+1}:

$$f_{3n+1}(n) = \begin{cases} 3n + 1 \cdot & \text{if } n \equiv 1 \pmod 2, \\ n \cdot & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{1.6}$$

So, for any natural number n, $f_{3n+1}(n) = n_2$, where n_2 is an even natural number, which is evident. Here, we use the notation typical for Markov algorithms, but in the context of algebraic transformations over numerical variables; the dot here denotes the final operation of the algorithm scheme.

After implementing the operation (1.5), based on the algorithm definition, the next operation is division. We will consider it in the form of the following representation:

$$2^{m_0+2}\left[\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right] \xrightarrow{\frac{1}{2}n} 2^{m_0+1}\left[\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right]. \qquad (1.7)$$

For an arbitrary number, we cannot determine how many division operations will be needed to obtain an odd number. Therefore, we will say that the number two in the formula (1.7) acquires some power m_1. Let's provide an estimate for this number: $0 \le m_1 \le m_0 + 1$, and at any step of the algorithm: $0 \le m_{i+1} \le m_i + 1$.

Definition 1.3

Similarly, the application of this operation is defined as the application of the algorithm $f_{n/2}$:

$$f_{n/2}(n) = \begin{cases} n\cdot & \text{if } n \equiv 1 \pmod 2, \\ f_{n/2}(n/2) & \text{if } n \equiv 0 \pmod 2. \end{cases} \qquad (1.8)$$

The result of applying this algorithm to any natural number is an odd number: $f_{n/2}(n) = n_1$.

Definition 1.4

Let us introduce the function c, which, when given an arbitrary number, outputs an odd number obtained as a result of one iteration of the investigated algorithm, such that:

$$c(n_0) = 2^{m_1}\left[\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right]. \qquad (1.9)$$

We will use the commonly accepted notation for the corresponding recursive transformations:

$$c^1(n_0) = c(n_0),\, c^{k+1}(n_0) = c\left(c^k(n_0)\right). \qquad (1.10)$$

This function, as well as the corresponding algorithm, can be considered as the composition of previously defined algorithms: $c = f_{n/2} \circ f_{3n+1}$, which can be equivalently expressed as $c(n) = f_{n/2}\big(f_{3n+1}(n)\big)$.

Sequential application of the function c to a number will be considered as a mathematical sequence. Thus, the sequential application of k iterations of the investigated algorithm to the initial odd number n_0 will result in the corresponding sequence of odd numbers $\{c^k(n_0)\}$.

Note 1.1

We denote the Collatz algorithm by the symbol $\mathbb{C}$, in the sense of an object of the theory of Markov normal algorithms. Then the following formulation holds, related to the mentioned theory: $\mathbb{C}: n_0 \vdash c^1(n_0) \vdash c^2(n_0) \cdots \vdash c^k(n_0)$. It can be considered that the problem is to find the validity of $\mathbb{C}: n_0 \vDash 1$ for any n_0.

Example 1.1

For the initial number $n_0 = 19$, we obtain the following sequence: $c^1(19) = 29$, $c^2(19) = c^1(29) = 11$, $c^3(19) = c^2(29) = c^1(11) = 17$, $c^4(19) = 13$, $c^5(19) = 5$, $c^6(19) = 1$, $c^7(19) = 1$, and so on. Thus, starting from the sixth member of the sequence $\{c^k(19)\}$, all subsequent members are equal to the unit.

Definition 1.5

Let us introduce the function $C(n_0)$, which outputs the limit value of the sequence $\{c^k(n_0)\}$ as the number of Collatz algorithm iterations approaches infinity, such that:

$$C(n_0) = \lim_{k\to\infty} c^k(n_0). \tag{1.11}$$

Note 1.2

The expression $C(n_0) = \lim_{k\to\infty} c^k(n_0)$ in formula (1.11) does not in any way imply the assumption of the existence of the limit in the sense of the theory of limits. Indeed, if it were stated, "$C(n_0) = \lim_{k\to\infty} c^k(n_0) = g$, where $g \in \mathbb{N}$", that would be a mistake. However, Definition 1.5 does not contain such an assertion. So, by referring to the notation $C(n_0)$ or $\lim_{k\to\infty} c^k(n_0)$, we do not mislead ourselves regarding the accepted understanding of the existence of a natural number corresponding to the notation $C(n_0)$.

For the explicit expression of the function C, first consider the sequential application of iterations of the algorithm to an arbitrary number:

$$c^2(n_0) = 2^{m_2}\left[\left(\frac{3}{4}\right)^2 a_0 + \frac{3}{4}\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}}\right)\right];$$

and then:

$$c^3(n_0) = 2^{m_3}\left[\left(\frac{3}{4}\right)^3 a_0 + \left(\frac{3}{4}\right)^2\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}} + \left(\frac{3}{4}\right)^{-2}\frac{1}{2^{m_2+2}}\right)\right].$$

As a result, we express an arbitrary k-th term of the sequence:

$$c^k(n_0) = 2^{m_k}\begin{bmatrix}\left(\frac{3}{4}\right)^k a_0 + \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}} + \cdots\right. \\ \left. \dots + \left(\frac{3}{4}\right)^{-k+0}\frac{1}{2^{m_{k-2}+2}} + \left(\frac{3}{4}\right)^{-k+1}\frac{1}{2^{m_{k-1}+2}}\right)\end{bmatrix}. \tag{1.12}$$

This formula can be written as containing a number series:

$$c^k(n_0) = 2^{m_k}\left[\left(\frac{3}{4}\right)^k a_0 + \left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right]. \tag{1.13}$$

Example 1.2

The sixth term of the sequence from Example 1.1 using formula (1.13) can be expressed as follows:

$$c^6(19) = 2^3\left[\left(\frac{3}{4}\right)^6\frac{19}{32} + \left(\frac{3}{4}\right)^{6-1}\left(\frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{-2+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-3+1}\frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{-4+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-5+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-6+1}\frac{1}{2^{5+2}}\right)\right] = 1.$$

From this representation, it follows that the investigation of the Collatz algorithm is identical to the investigation of the possible values of the function $C(n_0)$, representing the limit of the sequence $\{c^k(n_0)\}$:

$$C(n_0) = \lim_{k\to\infty} 2^{m_k}\left[\left(\frac{3}{4}\right)^k a_0 + \left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right]. \tag{1.14}$$

The declarative form of the Collatz conjecture in the context of the proposed interpretation can be expressed as: $\forall n_0(C(n_0) = 1)$. Note that we are reasoning about the transformation of odd numbers. For even numbers, assuming the hypothesis for odds, the proof follows a straightforward scheme: $c^1(n_2) = n_0$, where n_2 is any even natural number, so $\forall n_0\big(c^k(n_0) = 1\big) \supset \forall n_2\big(c^{k+1}(n_2) = 1\big)$, and in the limit $\forall n_0(C(n_0) = 1) \supset \forall n_0(C(n_2) = 1)$. This is also justified by the fact that any even number can be constructively obtained from its corresponding odd number by multiplying it by two.

Let us define the value of the limit of the sequence expressed by formula (1.14) by stating the following theorem about the limit of a sequence: if all members of the sequence are equal to the same number, then the sequence converges to that number. This statement is in strict accordance with the definition of the concept of the limit of a sequence. Thus, the statement $C(n_0) = 1$ means that for the sequence defined as $\{c^k(n_0)\}$, there exist segments of this sequence where all members are equal to the unit. This reasoning allows us to consider the result of the Collatz algorithm within the formal definitions of the theory of sequences. Also, since the sequence $\{c^k(n_0)\}$ is defined on natural numbers, it cannot infinitely approach some natural number. Therefore, a necessary condition for the existence of the limit $C(n_0) = 1$ is the assumption of the value of one for all members of the sequence starting from some member.

The sequence of numbers $\{c^k(n_0)\}$ in light of the theorems on sequences can be viewed as the result of algebraic transformations of other sequences, i.e.:

$$\{c^k(n_0)\} = \{2^{m_k}\}\left[\left\{\left(\frac{3}{4}\right)^k a_0\right\} + \left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}\right]. \quad (1.15)$$

For ease of presentation, we will also use the notation:

$$\{c^k(n_0)\} = \{2^{m_k}\}\left[\left\{\left(\frac{3}{4}\right)^k a_0\right\} + \left\{\left(\frac{3}{4}\right)^{k-1} s_k\right\}\right], \quad (1.16)$$

where s_k is the partial sum of the series $S = \sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^{-i+1} \left(\frac{1}{2}\right)^{m_{i-1}+2}$.

Note 1.3

In the definition of the numerical series S, the parameter m_{i-1} is present. By this parameter, we mean the function determined by the number n_0 passed to the Collatz algorithm, which in turn determines the parameter m_{i-1} as a function: $m_{i-1} = f(n_0, i)$. We assume that this note eliminates formal objections regarding the definition of the numerical series; this series can be considered as functional.

Moreover, the series is considered specified since there is a unambiguous algorithm for determining $m_{i-1} = f(n_0, i)$ at each iteration through the considered Collatz algorithm.

For further reasoning, it will also be convenient to represent the expression of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s_k\right\}$, introducing the coefficient into the sum of the polynomial:

$$\left(\frac{3}{4}\right)^{k-1} s_k = \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}} = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{k-i} \frac{1}{2^{m_{i-1}+2}}. \tag{1.17}$$

Definition 1.6

By its definition, the statement $C(n_0) = 1$ is identical to $\mathbb{C}: n_0 \vDash 1$. However, to conform to the formal definition of a Markov algorithm, let us introduce an additional formal algorithm that halts the Collatz algorithm:

$$f_1(n) = \begin{cases} n \cdot & \text{if } n \neq 1, \\ \Lambda \cdot & \text{if } n = 1; \end{cases} \tag{1.18}$$

where Λ is a symbol not present in the left-hand side of the substitution formulas $\mathbb{C}$.

Define the Collatz algorithm as a composition: $\mathbb{C} = f_1 \circ c$. Then it holds that $\mathbb{C}: \Lambda \sqsupset$, allowing us to make assumptions regarding the Collatz algorithm defined as a Markov algorithm $\mathbb{C}(n_0) = \Lambda$. Since $\mathbb{C}: n_0 \vDash 1 \vdash \Lambda$, we consider $\mathbb{C}(n_0) = 1$ as an equivalent formulation.

Thus, we have related the statement $C(n_0) = 1$, referring to the theory of limits of sequences, to the formulation $\mathbb{C}(n_0) = 1$ in the theory of normal algorithms. Moreover, the first can be interpreted as a statement about the existence of an infinite sequence of units, $\{1\}$ as a segment of the sequence $\{c^k(n_0)\}$, while the latter statement asserts the finiteness of the sequence $\{c^k(n_0)\}$, terminating it at the unit.

In this work, we will use the language of sequence theory to investigate an object described as a certain algorithm. We consider the choice of the method of investigation to be well-founded.

Proposition 1.1

The refutation of the Collatz hypothesis will be the proof of either of the following two statements: (1) for some specific number n'_0, the sequence $\{c^k(n'_0)\}$ is bounded but does not converge to the unit, or $\exists n'_0(C(n'_0) \neq 1) \wedge (C(n'_0) \neq \infty)$; (2) for some specific number n''_0, the sequence $\{c^k(n''_0)\}$ diverges to infinity, or $\exists n''_0(C(n''_0) = \infty)$.

For the first statement, we can consider not just a single scenario; the sequence may either have a limit different from the unit, i.e., $C(n'_0) = g$, $g \neq 1$, or it may be divergent, i.e., not have a limit. The second statement necessarily implies the absence of a limit.

Refuting these statements will lead to the proof of the Collatz conjecture, i.e., $\neg\exists n'_0\big((C(n'_0) \neq 1) \wedge (C(n'_0) \neq \infty)\big) \wedge \neg\exists n''_0(C(n''_0) = \infty) \supset \forall n_0(C(n_0) = 1)$. This statement follows from the analysis of the interpretation domain.

Investigating a specific normal algorithm $\mathbb{C}$ will not allow us to distinguish between these statements related to limits; we will find that $\mathbb{C}$ is not applicable to both n'_0 and n''_0 without their distinction.

2. Investigating the assumption of the existence of a bounded sequence not converging to the unit

Consider the limit of a bounded sequence $\{c^k(n'_0)\}$ satisfying the statement (1) of Proposition 1.1. The hypothetical number n'_0 is defined using the general formula (1.2), so that $n'_0 = a'_0 2^{m'_0}$. In this case, the sequence $\{2^{m'_k}\}$ is bounded by definition of the hypothetical number n'_0, since the expression in square brackets cannot exceed one. Hence, we obtain the relation:

$$\{c^k(n'_0)\} < \{2^{m'_k}\} \leq \{2^M\}, \tag{2.1}$$

where M is a number defined as the maximum value of the parameter m'_k, i.e., $M = \max\limits_k(\{m'_k\})$. This allows us to consider the sequence $\{c^k(n'_0)\}$ as bounded by 2^M. We relate the maximum estimate of the parameter to its actual value by introducing an error: $\Delta_{(M)k} = M - m'_k$.

Then, for the partial sum of the numerical series s'_k, we obtain the following lower bound:

$$\underline{s'}_k = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \left(\frac{1}{2}\right)^{M+2}, \tag{2.2}$$

a lower bound because $m_{i-1} \leq M$ by definition, and $M + 2$ is a power of a number less than one, so that $s'_k \geq \underline{s'}_k$.

Notice that the obtained partial sum $\underline{s}'_k$ is a geometric progression; let us transform the expression into a more convenient form using the formula for the sum of a geometric progression:

$$\underline{s}'_k = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \left(\frac{1}{2}\right)^{M+2} = \left(\frac{1}{2}\right)^{M+2} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{i-1} = \left(\frac{1}{2}\right)^{M+2} \left(\frac{\left({}^{4}\!/_{3}\right)^{k}-1}{\left({}^{4}\!/_{3}\right)-1}\right) \tag{2.3}$$

Let us find the limit of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} \underline{s}'_k\right\}$:

$$\lim_{k\to\infty} \left({}^{4}\!/_{3}\right)^{-k+1} \underline{s}'_k = \left(\frac{1}{2}\right)^{M+2} \lim_{k\to\infty} \frac{\left({}^{4}\!/_{3}\right)^{k-k+1} - \left({}^{4}\!/_{3}\right)^{-k+1}}{\left({}^{4}\!/_{3}\right)-1} = \left(\frac{1}{2}\right)^{M+2} \lim_{k\to\infty} \left(4 - 3 \cdot \left({}^{3}\!/_{4}\right)^{k-1}\right) =$$
$$4\left(\frac{1}{2}\right)^{M+2} = \left(\frac{1}{2}\right)^{M}. \tag{2.4}$$

Similar reasoning is applied to the upper bound $\overline{s}'_k$. In this case, we assume that the parameter m'_k takes the minimum possible value, which is zero:

$$\overline{s}'_k = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \left(\frac{1}{2}\right)^{0+2} = \left(\frac{1}{2}\right)^{2} \left(\frac{\left({}^{4}\!/_{3}\right)^{k}-1}{\left({}^{4}\!/_{3}\right)-1}\right), \tag{2.6}$$

$$\lim_{k\to\infty} \left({}^{4}\!/_{3}\right)^{-k+1} \overline{s}'_k = \lim_{k\to\infty} \left({}^{4}\!/_{3}\right)^{-k+1} \left(\frac{1}{2}\right)^{2} \left(\frac{\left({}^{4}\!/_{3}\right)^{k}-1}{\left({}^{4}\!/_{3}\right)-1}\right) = \left(\frac{1}{2}\right)^{0} = 1. \tag{2.7}$$

It is evident that if the limit of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$ exists, it is contained within the interval $\mathbb{Q}\left[\left(\frac{1}{2}\right)^{M}; 1\right]$, defined on rational numbers.

The partial sum itself can be expressed with the following formula, assuming that $m'_i = M - \Delta_{(M)i}$:

$$s'_k = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m'_{i-1}+2}} = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{M-\Delta_{(M)i-1}+2}} = \left(\frac{1}{2}\right)^{M+2} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{i-1} 2^{\Delta_{(M)i-1}}. \tag{2.8}$$

The value taken by the parameter m'_k can be considered as the value of some function defined on natural numbers, i.e., $m'_k = f(n'_0, k)$. We do not have an explicit expression for this function since we are considering a number whose existence is only assumed; nevertheless, the sequence $\{m'_k\}$ should be considered specified, as there is a law for finding any of its members given n'_0. However, it is possible to use certain approximations for which the values of this parameter are constant: $m'_k = f(n'_0, k)$ is replaced by $m'_k(p) = f(n'_0) = const$. The approximation s'_k will be denoted as $s'_k(p)$, so that the parameter m'_k at any iteration is replaced by some approximation in the form of the number $p = \mathbb{N}[0, M]$. Obviously, in this representation, $s'_k(0) = \overline{s}'_k$ and $s'_k(M) = \underline{s}'_k$, i.e., correspond to the upper and lower bounds defined earlier.

It is not difficult to find the limit of the sequence of approximations $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k(p)\right\}$ by analogy with upper and lower bounds:

$$\lim_{k\to\infty}\left({}^4/_3\right)^{-k+1} s'_k(p) = \lim_{k\to\infty}\left({}^4/_3\right)^{-k+1}\left(\frac{1}{2}\right)^{p+2}\left(\frac{\left({}^4/_3\right)^k-1}{\left({}^4/_3\right)-1}\right) = \left(\frac{1}{2}\right)^p. \tag{2.9}$$

Let us find the ratio of consecutive terms for the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k(p)\right\}$:

$$q_{s'_k(p)} = \frac{\left(\frac{3}{4}\right)^{k-1} s'_k(p)}{\left(\frac{3}{4}\right)^{k-1-1} s'_{k-1}(p)} = \frac{3}{4}\frac{\left(\frac{1}{2}\right)^{p+2}\left(\frac{\left({}^4/_3\right)^k-1}{\left({}^4/_3\right)-1}\right)}{\left(\frac{1}{2}\right)^{p+2}\left(\frac{\left({}^4/_3\right)^{k-1}-1}{\left({}^4/_3\right)-1}\right)} = \frac{\left({}^4/_3\right)^{k-1}-\frac{3}{4}}{\left({}^4/_3\right)^{k-1}-1} = 1+\frac{1}{4\left(\left({}^4/_3\right)^{k-1}-1\right)}. \tag{2.10}$$

The obtained result shows that for all approximations, the ratio of consecutive terms is a single function depending on the number of terms in the sequence. However, in the limit, the sequence can be considered as a geometric progression:

$$\lim_{k\to\infty} q_{s'_k(p)} = 1. \tag{2.11}$$

It is worth noting that the obtained result applies to the function of approximations of the sequence. Now let us find the ratio of terms of the considered sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$:

$$q_{s'_k} = \frac{\left(\frac{3}{4}\right)^{k-1} s'_k}{\left(\frac{3}{4}\right)^{(k-1)-1} s'_{k-1}} = \frac{\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{k-i}\frac{1}{2^{m_{i-1}+2}}}{\sum_{i=1}^{k-1}\left(\frac{3}{4}\right)^{k-1-i}\frac{1}{2^{m_{i-1}+2}}} = 1+\frac{2^{\Delta(M)k-1}}{\sum_{i=1}^{k-1}2^{\Delta(M)i-1}}. \tag{2.12}$$

In other words, the k -th term of the sequence can be represented in terms of its ratio to the first term as follows:

$$\left(\frac{3}{4}\right)^{k-1} s'_k = \left(\left(\frac{3}{4}\right)^{1-1} s'_1\right)\prod_{i=2}^{k} q_{s'_k}. \tag{2.13}$$

The obtained ratio indicates the monotonicity of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$; it is non-decreasing according to (2.12). Considering its boundedness by the definition of the expression in square brackets, according to the Cauchy criterion, we conclude the necessary convergence of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$, i.e., it has a limit.

However, this does not contradict the hypothesis about the property of n'_0; we have only ruled out the scenario of the sequence formed by the number n'_0 having no limit. But now we can use the theorem on the sum of limits:

$$\lim_{k\to\infty} c^k(n'_0) = \lim_{k\to\infty} 2^{M-\Delta(M)k}\left[\left(\frac{3}{4}\right)^k a' + \left(\frac{3}{4}\right)^{k-1} s'_k\right] = \lim_{k\to\infty} 2^{M-\Delta(M)k}\left(\frac{3}{4}\right)^k a' +$$
$$\lim_{k\to\infty} 2^{M-\Delta(M)k}\left(\frac{3}{4}\right)^{k-1} s'_k = \lim_{k\to\infty} 2^{M-\Delta(M)k}\left(\frac{3}{4}\right)^{k-1} s'_k. \tag{2.14}$$

Note 2.1

In general, the convergence of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$ can also be justified by the convergence of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} \overline{s}'_k\right\}$ formed by the majorizing series since $\forall k(s'_k \le \overline{s}'_k)$.

We assume that establishing the convergence fact using formula (2.14) together with the limit function of series estimates in (2.9) may be sufficient to conclude the limit converging to the unit. Since the limit $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$ exists, this sequence can only converge to one of its estimates, i.e., $\exists p\left(\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1} s'_k = \lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1} s'_k(p)\right)$, as the set of values $\{n'_k\}$ is discrete and bounded by definition, making it finite. Then:

$$\lim_{k\to\infty} c^k(n'_0) = \lim_{k\to\infty} 2^{M-\Delta(M)k}\left[\left(\frac{3}{4}\right)^k a' + \left(\frac{3}{4}\right)^{k-1} s'_k\right] = \lim_{k\to\infty} 2^{M-\Delta(M)k}\left(\frac{3}{4}\right)^k a' +$$
$$\lim_{k\to\infty} 2^{M-\Delta(M)k}\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1} s'_k(p) = 0 + \lim_{k\to\infty} 2^{M-\Delta(M)k-p}, \tag{2.15}$$

demonstrating the convergence of n'_0 to the number two to some power. Since, by definition, the number n'_k must be odd, the permissible power of two is zero, i.e., $n'_k = 2^0$, which contradicts the definition of such a number according to Proposition 1.1.

Nevertheless, a more demonstrative approach, we believe, will be the subsequent line of reasoning.

Proposition 2.1

The property of the hypothetical number n'_0 can be interpreted as the presence of a cycle of length, or a period t in the sequence $\{c^k(n'_0)\}$, excluding the presence of the unit in the cycle. This property is defined by the nature of this number, which forms a bounded sequence. In other words, starting from some r, we have: $n'_0, c^1(n'_0), \cdots, c^r(n'_0), \cdots, c^{r+t}(n'_0), \cdots$; – where $c^r(n'_0) = c^{r+t}(n'_0), c^{r+1}(n'_0) = c^{r+t+1}(n'_0), \cdots$ and so on. Formally, we will not exclude the minimum cycle length $c^r(n'_0) = c^{r+1}(n'_0)$ when $c^k(n'_0) \neq 1$, i.e., when $t = 1$. In summary, the statement (1) of Proposition 1.1 can be replaced by a semantically equivalent form: $\exists t \exists r \forall k \left(c^{r+k}(n'_0) = c^{r+k+t}(n'_0) \right)$.

Definition 2.1

We formally refer to the partial sum of the series s'_k as the sum of the series S' itself. Consider that starting from the term $c^r(n'_0)$, the sequence forms a cycle, so that the remainder of the series starting from this term is denoted as S'_t. We denote a possible finite sequence of partial sums of the series S', which does not contain cycles, as S'^*. Based on this definition, the relationship is evident: $S' = S'^* + S'_t$.

Analyzing the limit of the sequence $\left\{ \left(\frac{3}{4}\right)^{k-1} s'_k \right\}$, we can discard any finite number of initial terms of the series:

$$\lim_{k\to\infty} 2^{M-\Delta_{(M)k}} \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{M-\Delta_{(M)i-1}+2}} = \lim_{k\to\infty} 2^{M-\Delta_{(M)k}} \left(\frac{3}{4}\right)^{k-1} \left(\sum_{i=1}^{r-1} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{M-\Delta_{(M)i-1}+2}} + \sum_{i=r}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{M-\Delta_{(M)i-1}+2}} \right) = \lim_{k\to\infty} 2^{M-\Delta_{(M)k}} \left(\frac{3}{4}\right)^{k-1} \sum_{i=r}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{M-\Delta_{(M)i-1}+2}}.$$

We can obtain the same result in the interpretation of the theory of normal algorithms. Let $c^r(n'_0)$ be the number from which the algorithm cycles, $\mathbb{C}: n'_0 \vDash c^r(n'_0) \vDash c^{r+t}(n'_0) \cdots$, $c^r(n'_0) = c^{r+t}(n'_0)$; so that we can discard a finite number of iterations leading from n'_0 to $c^r(n'_0)$ and immediately attribute the cyclic property to the number n'_0. The uniqueness of the cycle follows from the boundedness of the sequence defined on the discrete set of natural numbers formed by the number n'_0.

Intermediate reasoning on partial sums of a series

We find it necessary to resort to the remarkable formula concerning the sum of the terms of a geometric progression. A geometric progression is defined by the following recursive formula: $q^{k+1} = qq^k$. Usually, a constant coefficient is attached to the sum formula, so the sum of terms $\{aq^k\}$ is considered. In our simplified reasoning, we will take $a = 0$ and $q \neq 1$.

The presented formulas will be valid for all positive rational numbers except unit. We will express the formulas as finite sums. Formally, we write the formula for the sum of terms of a finite geometric progression:

$$q^0 + q^1 + q^2 + \cdots + q^{k-1} = \sum_{i=1}^{k} q^{i-1} = \frac{q^k - 1}{q - 1}.$$

As the denominator of the progression, we can represent the number in an arbitrary power t, so we get an equivalent expression:

$$(q^t)^0 + (q^t)^1 + (q^t)^2 + \cdots + (q^t)^{k-1} = \sum_{i=1}^{k} (q^t)^{(i-1)} = \frac{(q^t)^k - 1}{q^t - 1}.$$

When considering series, i.e., infinite sums, we can refer to the theorem stating that the convergence of a convergent positive series is not affected by the rearrangement of its terms, and the sum remains unchanged. Since we can also consider non-convergent limit sequences of partial sums, we assert that the commutative law with respect to the addition operation for the sequences under consideration is not violated when taking limits. However, the formulation will be carried out for finite partial sums.

Let us group the terms of the geometric progression sum in a specific way and perform rather detailed transformations:

$$\left(q^0 + q^{0+t} + q^{0+2t} + \cdots + q^{0+(k-1)t}\right) + \left(q^1 + q^{1+t} + \cdots + q^{1+(k-1)t}\right) + \cdots + \left(q^{(t-1)} + q^{(t-1)+t} + \cdots + q^{(t-1)+(k-1)t}\right) = \sum_{i=1}^{k} q^{t(i-1)} + \sum_{i=1}^{k} q^{t(i-1)+1} + \sum_{i=1}^{k} q^{t(i-1)+2} + \cdots + \sum_{i=1}^{k} q^{t(i-1)+(t-1)} =$$
$$\sum_{i=1}^{k} q^{t(i-1)} + q \sum_{i=1}^{k} q^{t(i-1)} + q^2 \sum_{i=1}^{k} q^{t(i-1)} + \cdots + q^{(t-1)} \sum_{i=1}^{k} q^{t(i-1)} =$$
$$\sum_{j=1}^{t} \sum_{i=1}^{k} q^{t(i-1)+(j-1)} = \sum_{i=1}^{k} q^{t(i-1)} \sum_{j=1}^{t} q^{(j-1)} = \frac{(q^t)^k - 1}{(q^t) - 1} + q \frac{(q^t)^k - 1}{(q^t) - 1} + q^2 \frac{(q^t)^k - 1}{(q^t) - 1} + \cdots +$$
$$q^{t-1} \frac{(q^t)^k - 1}{(q^t) - 1} = \frac{(q^t)^k - 1}{(q^t) - 1} \sum_{j=1}^{t} q^{(j-1)} = \frac{(q^t)^k - 1}{(q^t) - 1} \frac{q^t - 1}{q - 1}.$$

It is easy to verify that, in order to obtain this result, we initially took $n = k \cdot t; n, k, t \in \mathbb{N}$ terms of the simplest geometric progression for summation. Thus, we can write for any natural $k, t > 0$:

$$\sum_{i=1}^{kt} q^{i-1} = \frac{(q^t)^k - 1}{(q^t) - 1} \frac{q^t - 1}{q - 1}.$$

We will need these relationships to justify the transformations over the considered sequence of partial sums.

Return to the discussion of the limit of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s'_k\right\}$. We will take a cyclic segment of the sequence $\{s'_k\}$ and group identical terms. Since the length of the cycle is finite, we will obtain a finite number of series. Based on the concept of the cycle and its length t, we can consider the coefficients $2^{\Delta_{(M)(i-1)+nt}}$, where $n \in \mathbb{N}$, to be identical. However, since these coefficients are the same, we can simplify their notation, limiting ourselves to only a finite possible number: $2^{\Delta_{(M)(i-1)+nt}} = 2^{\Delta_{(M)(t)}}$. Formally, this can be justified by applying the rule of generalization: the statement $\left(2^{\Delta_{(M)}}(t) \equiv 2^{\Delta_{(M)}}(t)\right)$ is a tautology, $\left(\forall i \forall n 2^{\Delta_{(M)}}(t, i, n) \equiv 2^{\Delta_{(M)}}(t)\right)$ by the rule of generalization (Gen). In our interpretation, this statement is true because the cycle members, separated by the period of the cycle, are equal by definition: the number of distinct members of an infinite cyclic sequence is limited by the length of the cycle. Let us take a partial sum of a series that is a multiple of the period of $2^{\Delta_{(M)(i-1)}}$ and perform the transformations discussed earlier:

$$s'_k = \left(\frac{1}{2}\right)^{M+2} \sum_{i=1}^{kt} \left(\frac{4}{3}\right)^{i-1} 2^{\Delta_{(M)i-1}} = \left(\frac{1}{2}\right)^{M+2} \sum_{j=1}^{t} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{t(i-1)+(j-1)} 2^{\Delta_{(M)(i-1)j}} =$$
$$\left(\frac{1}{2}\right)^{M+2} \sum_{j=1}^{t} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{t(i-1)+(j-1)} 2^{\Delta_{(M)j}} = \left(\frac{1}{2}\right)^{M+2} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{t(i-1)} \sum_{j=1}^{t} \left(\frac{4}{3}\right)^{(j-1)} 2^{\Delta_{(M)j}} =$$
$$\left(\frac{1}{2}\right)^{M+2} \frac{\left({}^4/_3{}^t\right)^k - 1}{\left({}^4/_3{}^t\right) - 1} \sum_{j=1}^{t} \left(\frac{4}{3}\right)^{(j-1)} 2^{\Delta_{(M)j}}. \tag{2.16}$$

Then, write the expression for the limit of the sequence $\left(\frac{3}{4}\right)^{k-1} s'_k$:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} s'_k = \left(\frac{3}{4}\right)^{k-1} \left(\frac{1}{2}\right)^{M+2} \frac{\left({}^4/_3{}^t\right)^k - 1}{\left({}^4/_3{}^t\right) - 1} \sum_{j=1}^{t} \left(\frac{4}{3}\right)^{(j-1)} 2^{\Delta_{(M)j}} =$$
$$\frac{4}{3} \left(\frac{1}{2}\right)^{M+2} \frac{\left({}^4/_3{}^{t-1}\right)^k - {}^3/_4{}^k}{\left({}^4/_3{}^t\right) - 1} \sum_{j=1}^{t} \left(\frac{4}{3}\right)^{(j-1)} 2^{\Delta_{(M)j}}. \tag{2.17}$$

We obtained the limit value in the form of a function of the parameter t: $\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} s'_k = f(t)$. It seems obvious that the only acceptable value for t, ensuring the convergence of the limit, is $t = 1$. However, since we have previously established that the sequence must converge, we substitute this value to obtain the limit value:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} s'_k = \lim_{k\to\infty} \frac{4}{3} \left(\frac{1}{2}\right)^{M+2} \frac{\left({}^4/_3{}^{t-1}\right)^k - {}^3/_4{}^k}{\left({}^4/_3{}^t\right) - 1} \sum_{j=1}^{t} \left(\frac{4}{3}\right)^{(j-1)} 2^{\Delta_{(M)j}} \Bigg|_{t=1} =$$
$$\lim_{k\to\infty} \frac{4}{3} \left(\frac{1}{2}\right)^{M+2} \frac{(1)^k - {}^3/_4{}^k}{{}^1/_3} \sum_{j=1}^{1} \left(\frac{4}{3}\right)^{(j-1)} 2^{\Delta_{(M)j}} = \lim_{k\to\infty} \left(\frac{1}{2}\right)^{M} \left((1)^k - {}^3/_4{}^k\right) 2^{\Delta_{(M)1}} = \left(\frac{1}{2}\right)^{M} 2^{\Delta_{(M)1}}. \tag{2.18}$$

Here, $\Delta_{(M)1}$ signifies the following: we assume that we do not know the specific number to which the deviation from the maximum estimate M converges, only knowing that this number must be unique.

Now we can explicitly define the limit of the sequence $\{c^k(n'_0)\}$, which is the main goal of this section:

$$\lim_{k\to\infty} c^k(n'_0) = \lim_{k\to\infty} 2^{M-\Delta_{(M)k}} \left(\frac{3}{4}\right)^{k-1} s'_k = \lim_{k\to\infty} 2^{M-\Delta_{(M)k}} \left(\frac{1}{2}\right)^{M} 2^{\Delta_{(M)1}} = 1, \tag{2.19}$$

where $\Delta_{(M)k} = \Delta_{(M)1}$, since we have established that for a value of k (the number of iterations of the Collatz algorithm) tending to infinity, $\Delta_{(M)i-1}$ can take only single value.

We have obtained a contradiction to the statement (1) of Proposition 1.1. There is no such number as n'_0 with the specified properties because, by definition, it should refute the Collatz hypothesis, but at the same time, it produces a sequence converging to the unit.

3. Investigation of the assumption of the existence of a divergent to infinity sequence

Write down the sequence formed by the number n''_0 with properties defined according to Proposition 1.1:

$$\{c^k(n''_0)\} = \{2^{m''_k}\}\left[\left\{\left(\frac{3}{4}\right)^k a''_0\right\} + \left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}\right], \tag{3.1}$$

where $n''_0 = a''_0 2^{m''_0}$ by formula (1.2); s''_k is the partial sum of the series corresponding to the number n''_0.

To ensure the fulfillment of the conditions stated in (2), namely $C(n''_0) = \infty$, it is necessary to accept the following relation:

$$\lim_{k\to\infty} 2^{m''_k} = \infty, \tag{3.2}$$

since the expression in square brackets cannot exceed the unit by definition, i.e., it is bounded.

Let us provide an upper estimate for the sequence $\{2^{m''_k}\}$. Since, by definition, m_k implies the property $0 \le m_{k+1} \le m_k + 1$, it can be said that $\{2^{m_k}\}$ is bounded by the sequence $\{2^{m_0+k}\}$, which is also applicable to the hypothetical sequence formed by the number m''_0. Thus, $\{2^{m''_k}\} \le \{2^{m''_0+k}\}$.

Define the lower estimate of the partial sum of the series s''_k by replacing the parameter m''_{i-1} with its upper estimation $(m''_0 + i - 1)$:

$$\underline{s}''_k = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{(m''_0+i-1)+2}} = \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{i-1} \frac{1}{2^{(i-1)}} \frac{1}{2^{m''_0+2}} = \frac{1}{2^{m''_0+2}} \sum_{i=1}^{k} \left(\frac{2}{3}\right)^{i-1} =$$
$$\frac{1}{2^{m''_0+2}} \left(\frac{\left({}^2/_3\right)^k - 1}{\left({}^2/_3\right) - 1}\right) = \frac{1}{2^{m''_0+2}} 3\left(1 - \left({}^2/_3\right)^k\right); \tag{3.3}$$

the expression of the series has been transformed into its sum using the formula for the sum of a geometric progression. This estimate is a lower bound because the maximum estimate of the parameter m''_{i-1} appears in the denominator.

For the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$, we can obtain the corresponding estimate through the lower bound of the series:

$$\left(\frac{3}{4}\right)^{k-1} \underline{s}''_k = \frac{4}{3}\left(\frac{3}{4}\right)^k \frac{1}{2^{m''_0+2}} 3\left(1-\left({}^2/_3\right)^k\right) = \frac{1}{2^{m''_0}}\left(\frac{3}{4}\right)^k \left(1-\left({}^2/_3\right)^k\right) = \frac{1}{2^{m''_0}}\left(\left(\frac{3}{4}\right)^k - \left(\frac{1}{2}\right)^k\right). \quad (3.4)$$

According to the condition of the existence of n''_0, the parameter m''_{i-1} should tend to infinity; while we have limited this process with an upper estimate, $\overline{m}''_{i-1} = m''_0 + i - 1$. However, we can consider another option; for this purpose, we introduce the parameter $p_{(m)} \in \mathbb{N}$: $m''\left(p_{(m)}\right)_{i-1} = m''_0 + \left\lfloor\frac{i-1}{p_{(m)}}\right\rfloor$, where $\left\lfloor\frac{i-1}{p_{(m)}}\right\rfloor$ is the floor function of $\frac{i-1}{p_{(m)}}$. We can interpret the upper estimate as: $\overline{m}''_{i-1} = m''(1)_{i-1}$. However, considering the parameter m''_{i-1} as a function allows obtaining the function of estimates for the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$ by replacing the series with $s''_k\left(p_{(m)}\right)$:

$$\left(\frac{3}{4}\right)^{k-1} s''_k\left(p_{(m)}\right) = \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m''\left(p_{(m)}\right)_{i-1}+2}} = \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m''_0+2}} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{i-1} \frac{1}{2^{\left\lfloor\frac{i-1}{p_{(m)}}\right\rfloor}}. \quad (3.5)$$

Let us consider it obvious that:

$$\lim_{i\to\infty} \frac{\left\lfloor\frac{i-1}{p_{(m)}}\right\rfloor}{\frac{i-1}{p_{(m)}}} = 1, \quad (3.6)$$

so that in the limit, the floor function can be replaced by the expression being floored. Moreover, the problem formulation also allows us to replace the series with any of its remainders, so we obtain:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} s''_k\left(p_{(m)}\right) = \lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m''_0+2}} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{i-1} \frac{1}{2^{\frac{i-1}{p_{(m)}}}} =$$

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m''_0+2}} \sum_{i=1}^{k} \left(\frac{4}{3}\right)^{i-1} \frac{1}{\left(2^{\frac{1}{p_{(m)}}}\right)^{i-1}} = \lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m''_0+2}} \sum_{i=1}^{k} \left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right)^{i-1} =$$

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m''_0+2}} \left(\frac{\left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right)^{k} - 1}{\left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right) - 1}\right) = \lim_{k\to\infty} \frac{1}{2^{m''_0}} \frac{1}{\left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right) - 1} \frac{1}{3}\left(\left(\frac{1}{2^{\frac{1}{p_{(m)}}}}\right)^k - \left(\frac{3}{4}\right)^k\right) = 0. \quad (3.7)$$

This result also implies that any convergence estimates for the number m''_k have a unique limit, which is zero. In other words, we can choose a number $p_{(m)}$ such that the sequence $\left\{m''\left(p_{(m)}\right)_{i-1}\right\} \leq \{m''_{i-1}\}$, meaning it serves as a lower bound for the sequence $\{m''_{i-1}\}$.

If we assume that we cannot find $p_{(m)}$ such that $\left\{m''\left(p_{(m)}\right)_{i-1}\right\} \leq \{m''_{i-1}\}$, this would lead to a contradiction with the definition of the properties of the number n''_0, because then the sequence formed by the number n''_0 would not diverge to infinity.

Therefore, we can conclude the convergence to zero of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$ for any n''_0, based on the theorem that the considered sequence is bounded between two sequences converging to the same limit.

Note 3.1

A completely analogous reasoning applies to the following consideration: for the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$, we can extract a subsequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_{k'}\right\}$, where k' is any natural number taken by k, which will be monotonically decreasing and have a restriction that endows this subsequence with the property of convergence by theorem. However, the convergence of a subsequence does not necessarily imply the convergence of the sequence under consideration. But given that in the considered sequence, we cannot in any way extract a non-decreasing subsequence, as it would lead to a contradiction with the definition of n''_0, we should consider the original sequence as consisting only of monotonically decreasing bounded subsequences. The contradiction arises from the requirement (3.2) or $\lim_{k\to\infty} m''_k = \infty$. Therefore, the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$ must unequivocally be considered as convergent.

It is evident that the sequence $\left\{\left(\frac{3}{4}\right)^{k} a''_0\right\}$ converges unequivocally for any n''_0, which can be formally expressed as:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k} a''_0 = 0. \tag{3.8}$$

Based on the theorem about the sum of limits of convergent sequences, the limit of the sequence inside the square brackets for any n''_0converges to zero:

$$\lim_{k\to\infty}\left[\left(\frac{3}{4}\right)^{k} a''_0 + \left(\frac{3}{4}\right)^{k-1} s''_k\right] = 0. \qquad (3.9)$$

As a result, considering the sequence $\{c^k(n''_0)\}$, we encounter an indeterminate form $\infty[0]$ in the context of the expansion formula (1.13). The task of the preceding reasoning can be considered as clarifying this particular fact.

Definition 3.1

Rewrite the limit formula (1.13) incorporating the parameter z_k:

$$C(n_0) = \lim_{k\to\infty} 2^{m_k - z_k} \left[\!\left[2^{z_k}\left[\left(\frac{3}{4}\right)^{k} a_0 + \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right]\right]\!\right]. \qquad (3.10)$$

The expression in double square brackets represents the parameter a for some k-th odd number in the sequence n_k, formed from n_0. For convenience in reasoning, let us provide explanatory notations:

$$c(2^{m_0}[\![a_0]\!]) = 2^{m_1}2^{-z_1}\left[\!\left[2^{z_1}\left(\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right)\right]\!\right] = 2^{m_1 - z_1}[\![a_1]\!]; \qquad (3.11)$$

$$C(n_0) = \lim_{k\to\infty} 2^{m_k - z_k}[\![a_k]\!]. \qquad (3.12)$$

This parameter is constructively defined for any n_0 passed to the algorithm; at each iteration of the algorithm, it is determined in a straightforward manner. Note that this representation is valid for any n_0, which is also evident in the interpretation of the normal algorithm's operation $\mathbb{C}: a_0 2^{m_0} \vdash a_1 2^{m_1 - z_1} \models a_k 2^{m_k - z_k}$.

The purpose of this representation is to define the constraint of the expression in double square brackets as identical to the constraint of the parameter a_0, i.e., $[\![a_k]\!] \in \mathbb{Q}\left(\frac{1}{2}; 1\right]$.

We will consider that at each iteration, a correction $\Delta_{(z)k}$ is introduced in the sum constituting the parameter z_k. Let us clarify the derivation of the partial sum of the series similar to the derivation in (1.12), but considering the newly introduced parameter. Note that we will conduct the reasoning in the context of an arbitrarily chosen number, not associated with the hypothesis. We obtain:

$$c^1(n_0) = 2^{m_1 - z_1}\left[\!\left[2^{z_1}\left[\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right]\right]\!\right] = 2^{m_1 - \Delta_{(z)1}}\left[\!\left[2^{\Delta_{(z)1}}\left[\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right]\right]\!\right],$$

$$c^2(n_0) = 2^{m_2 - \Delta_{(z)1} - \Delta_{(z)2}}\left[\!\left[2^{-\Delta_{(z)1} - \Delta_{(z)2}}\left[\frac{3}{4}\left[\!\left[2^{\Delta_{(z)1}}\left[\frac{3}{4}a_0 + \frac{1}{2^{m_0+2}}\right]\right]\!\right] + 2^{\Delta_{(z)1}}\frac{1}{2^{m_1+2}}\right]\right]\!\right] =$$
$$2^{m_2 - \Delta_{(z)1} - \Delta_{(z)2}}\left[\!\left[2^{-\Delta_{(z)1} - \Delta_{(z)2}}\left[\left(\frac{3}{4}\right)^{2} a_0 + \left(\frac{3}{4}\right)^{1}\frac{1}{2^{m_0+2}} + \frac{1}{2^{m_1+2}}\right]\right]\!\right];$$

and for an arbitrary term:

$c^k(n_0) = 2^{m_k} \prod_{i=1}^{k} 2^{-\Delta_{(z)i}} \left[\!\left[\prod_{i=1}^{k} 2^{\Delta_{(z)i}} \left[\left(\frac{3}{4}\right)^k a_0 + \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}} \right] \right]\!\right] =$
$2^{m_k - z_k} \left[\!\left[2^{z_k} \left[\left(\frac{3}{4}\right)^k a_0 + \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}} \right] \right]\!\right].$

Formally, we write:

$$z_k = \sum_{i=1}^{k} \Delta_{(z)i}. \tag{3.13}$$

It is easy to see the identity of formulations in this expression, representing an algebraic tautology; at the same time, a new variable with some semantic component is introduced. Naturally, constraints on the newly introduced parameter follow: $\Delta_{(z)i} \geq 0$. Thus, $\forall k (z_k \leq z_{k+1})$; we conclude that the sequence $\{z_k\}$ is monotonic, namely, non-decreasing.

Example 3.1

Consider an example of algorithmic transformations for the number $n_0 = 19$, elucidating the proposed definition. This example is conveniently examined in the context of Examples 1.1 and 1.2. We present the sequence formed by the Collatz algorithm in the corresponding representation in Definition 3.1:

$\mathbb{C}\colon\ 19 = 2^5 \left[\!\left[\frac{19}{32}\right]\!\right] \vdash c^1(19) = 2^{6-1} \left[\!\left[2 \left[\left(\frac{3}{4}\right)^1 \frac{19}{32} + \left(\frac{3}{4}\right)^{1-1} \left(\left(\frac{3}{4}\right)^{1-1} \frac{1}{2^{5+2}} \right) \right] \right]\!\right] \vdash c^2(19) =$
$2^{5-1} \left[\!\left[2 \left[\left(\frac{3}{4}\right)^2 \frac{19}{32} + \left(\frac{3}{4}\right)^{2-1} \left(\left(\frac{3}{4}\right)^{1-1} \frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{1-2} \frac{1}{2^{6+2}} \right) \right] \right]\!\right] \vdash c^3(19) = 2^{6-1} \left[\!\left[2 \left[\frac{17}{64}\right] \right]\!\right] \vdash c^4(19) =$
$2^{6-2} \left[\!\left[2^2 \left[\frac{13}{64}\right] \right]\!\right] \vdash c^5(19) = 2^{5-2} \left[\!\left[2^2 \left[\frac{5}{32}\right] \right]\!\right] \vdash c^6(19) = 2^{3-3} \left[\!\left[2^3 \left[\frac{1}{8}\right] \right]\!\right].$

Then, in the context of Definition 3.1, we can consider the limit of the sequence $\{c^k(n''_0)\}$, related to Proposition 1.1, as an unfolding of the indeterminacy of the form: $\infty[\![\infty[0]]\!]$. It is evident that $\lim_{k\to\infty} 2^{m''_k - z''_k} = \infty$, based on the conditions of the definition of n''_0; at the same time, $\lim_{k\to\infty} 2^{z''_k} = \infty$, since, as shown, the $\left\{ \left(\frac{3}{4}\right)^k a''_0 + \left(\frac{3}{4}\right)^{k-1} s''_k \right\}$ must converge to zero, and the expression in double square brackets must take a non-zero numerical value, limited by the condition imposed on the parameter a.

We proceed to examine the limit of the sequence $\left\{ 2^{m''_k - z''_k} \left[\!\left[2^{z''_k} \frac{n''_k}{2^{m''_k}} \right]\!\right] \right\}$. Based on the definition of n''_0, the following relationships, or statements about the limits of the sequences constituting the expression in double brackets, are evident: $\lim_{k\to\infty} \frac{2^{z''_k}}{2^{m''_k}} = 0$, $\lim_{k\to\infty} \frac{n''_k}{2^{m''_k}} = 0$, $\lim_{k\to\infty} 2^{z''_k} = \infty$, $\lim_{k\to\infty} n''_k = \infty$.

Recall that according to (3.7), when considering the limit, we can replace $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$ with an estimate in the form of a functional sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\left(p_{(m)}\right)\right\}$. Depending on the parameter $p_{(m)}$, it serves as both an upper bound $p_{(m)} = p_{(m)max}$: $\forall k\left(\left(\frac{3}{4}\right)^{k-1} s''_k\left(p_{(m)max}\right) \geq \left(\frac{3}{4}\right)^{k-1} s''_k\right)$ and a lower bound $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$, or $p_{(m)} = 1$: $\forall k\left(\left(\frac{3}{4}\right)^{k-1} s''_k(1) \leq \left(\frac{3}{4}\right)^{k-1} s''_k\right)$. Since, for any $p_{(m)}$, convergence to a common limit occurs, the estimated sequence $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\right\}$ also converges to the same limit. It is important to emphasize that the replacement of $\left(\frac{3}{4}\right)^{k-1} s''_k$ with $\left(\frac{3}{4}\right)^{k-1} s''_k\left(p_{(m)}\right)$ is done in the context of finding the limit, making it permissible according to the theorems on limits.

The determination of the estimate $\left\{\left(\frac{3}{4}\right)^{k-1} s''_k\left(p_{(m)}\right)\right\}$ was carried out based on the sequence of the parameter $\{m''_k\}$, which, according to the definition of finding this parameter, was replaced by its estimation $\left\{m''\left(p_{(m)}\right)_k\right\}$. This estimate, depending on the value of the finite parameter $p_{(m)}$, can also serve as an upper or lower bound for $\{m''_k\}$. We can express this as:

$$m''\left(p_{(m)}\right)_k = m''_0 + \left\lfloor \frac{k}{p_{(m)}} \right\rfloor, \tag{3.14}$$

which also means $\exists p_{(m)} \forall k\left(\left(m''_0 + \left\lfloor \frac{k}{p_{(m)}} \right\rfloor\right) \leq m''_k\right)$ and also $\forall k\left(\left(m''_0 + \left\lfloor \frac{k}{1} \right\rfloor\right) \geq m''_k\right)$; according to (3.6), in the context of finding the limit, we can use the notation without rounding, i.e., $\lim_{k\to\infty} m''\left(p_{(m)}\right)_k = \lim_{k\to\infty}\left(m''_0 + \frac{k}{p_{(m)}}\right)$.

In a similar manner, let us provide a functional estimate for the parameter z''_k. We conclude that the sequence $\{z''_k\}$, being a particular case of the sequence $\{z_k\}$, must also be monotonically non-decreasing. Based on this assumption, let us give a functional estimate based on some natural integer parameter $p_{(z)}$:

$$z''_k\left(p_{(z)}\right) = \left\lfloor \frac{k}{p_{(z)}} \right\rfloor. \tag{3.15}$$

We have taken into account that according to Definitions 1.1 and 3.1, $z''_0 = 0$. In the limit transition, we can replace the rounded value with the rounded one: $\lim_{k\to\infty} 2^{z''_k\left(p_{(z)}\right)} = \lim_{k\to\infty} 2^{\frac{k}{p_{(z)}}} = \infty$.

To estimate the values of the parameter $p_{(z)}$, consider the requirement $\lim_{k\to\infty} 2^{m''_k - z''_k} = \infty$, which should hold for the estimates as well. This can be expressed as:

$$\lim_{k\to\infty}\left(\left\lfloor\frac{k}{p_{(m)}}\right\rfloor-\left\lfloor\frac{k}{p_{(z)}}\right\rfloor\right)=\lim_{k\to\infty}\frac{k\left(p_{(z)}-p_{(m)}\right)}{p_{(m)}p_{(z)}}=\infty, \tag{3.16}$$

that obviously achieved by ensuring the condition $p_{(z)}>p_{(m)}$, so that $p_{(z)}$ can take values $p_{(z)}\geq 2$.

Rewrite the limit formula (3.10) with the substitution of parameters by their estimates:

$$C(n''_0)=\lim_{k\to\infty}2^{m''_0+\frac{k}{p_{(m)}}-\frac{k}{p_{(z)}}}\left[\!\left[2^{\frac{k}{p_{(z)}}}\left[\left(\frac{3}{4}\right)^k a''_0+\left(\frac{3}{4}\right)^{k-1}s''_k\left(p_{(m)}\right)\right]\right]\!\right]. \tag{3.17}$$

Now, examine the sequence of the expression in double square brackets for convergence:

$$\lim_{k\to\infty}[\![a''_k]\!]=\lim_{k\to\infty}2^{\frac{k}{p_{(z)}}}\left[\left(\frac{3}{4}\right)^k a''_0+\left(\frac{3}{4}\right)^{k-1}s''_k\left(p_{(m)}\right)\right]=\lim_{k\to\infty}\left(2^{\frac{k}{p_{(z)}}}\left(\frac{3}{4}\right)^k a''_0+\right.$$

$$\left.\left(2^{\frac{1}{p_{(z)}}}\right)^k\frac{1}{2^{m''_0}}\frac{1}{\left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right)-1}\frac{1}{3}\left(\left(\frac{1}{2^{\frac{1}{p_{(m)}}}}\right)^k-\left(\frac{3}{4}\right)^k\right)\right)=\lim_{k\to\infty}\left(\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k a''_0+\right.$$

$$\left.\frac{1}{2^{m''_0}}\frac{1}{\left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right)-1}\frac{1}{3}\left(\left(\frac{2^{\frac{1}{p_{(z)}}}}{2^{\frac{1}{p_{(m)}}}}\right)^k-\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k\right)\right). \tag{3.18}$$

The expression $\left(\frac{1}{2^{m''_0}}\frac{1}{\left(\frac{4}{3\cdot 2^{\frac{1}{p_{(m)}}}}\right)-1}\frac{1}{3}\right)$ will be considered simply as some constant expressed in terms of the parameters m''_0 and $p_{(m)}$, and we will denote it as $b\left(m''_0,p_{(m)}\right)$. Depending on the value of $p_{(m)}$, the expression can be either positive or negative; however, we will write $\pm b\left(m''_0,p_{(m)}\right)$, emphasizing these possibilities.

The sequence $\left\{\left(\frac{2^{\frac{1}{p_{(z)}}}}{2^{\frac{1}{p_{(m)}}}}\right)^k\right\}$ converges to zero because the condition (3.16) must be satisfied:

$$\lim_{k\to\infty}\left(\frac{2^{\frac{1}{p_{(z)}}}}{2^{\frac{1}{p_{(m)}}}}\right)^k=\lim_{k\to\infty}\left(\frac{1}{2^{\frac{1}{p_{(m)}}-\frac{1}{p_{(z)}}}}\right)^k=\lim_{k\to\infty}\left(\frac{1}{2^{\frac{p_{(z)}-p_{(m)}}{p_{(m)}p_{(z)}}}}\right)^k=0. \tag{3.19}$$

The analysis of the limit (3.18) can be simplified:

$$\lim_{k\to\infty}[\![a''_k]\!] = \lim_{k\to\infty}\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k a''_0 \pm \left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k b(m''_0, p_{(m)}) = \lim_{k\to\infty}\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k \left(a''_0 \pm b(m''_0, p_{(m)})\right). \quad (3.20)$$

Here, we finally arrive at a contradiction. Regardless of the admissible value taken by the parameter $p_{(z)}$, the sequence $\{a''_k\}$ has a monotonic nature, i.e., it is either non-increasing or non-decreasing. In the limiting approximation, this sequence is decreasing or increasing, respectively. According to the theorem on sequences, every bounded monotonic sequence has a limit; therefore, it converges to a specific number, let us denote it as $\lim_{k\to\infty}[\![a''_k]\!] = a''_{\text{lim}}$. The existence of the limit of the sequence in the context of the problem is equivalent to the statement about the existence of a segment of the sequence with identical elements, i.e., $\{a''_{\text{lim}}\}$ is a segment of the sequence $\{a''_k\}$. This also implies, based on the requirement for the existence of the number n''_0, that a segment of the sequence $\{2^{m''_k - z''_k} a''_{\text{lim}}\}$ must be formed. However, in the context of the Collatz algorithm, this is the same number. Therefore, the sequence formed by the number n''_0 does not diverge to infinity. This contradicts the definition of n''_0, which means that numbers with the property, according to statement (2) of Proposition 1.1, do not exist.

Note 3.2

The scenario assuming infinite approximation by the sequence $\{a''_k\}$ towards the value a''_{lim}, without actually reaching it, is invalid.

Let us examine the limit of the sequence $\left\{\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k\right\}$ outside the context of the Collatz algorithm:

$$\lim_{k\to\infty}\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k\Bigg|_{p_{(z)}=2} = \infty,$$

$$\lim_{k\to\infty}\left(\frac{3\cdot 2^{\frac{1}{p_{(z)}}}}{4}\right)^k\Bigg|_{p_{(z)}>2} = 0. \quad (3.21)$$

From this, it is evident that the limit (3.20) cannot be arbitrarily close to the boundaries of the parameter a_k, therefore, either accepting a contradiction to the definition of a_k or acknowledging the convergence of the limit to the limit by the definition of a_k is necessary. If we assume that the sequence $\{a''_k\}$ is decreasing, the lower bound suggests a value of $\frac{1}{2}$, but according to the definition, the parameter cannot take such a value. It remains to assume that the sequence $\{a''_k\}$ is increasing; in this case, it is bounded by the number 1, which is admissible for the parameter, so $a''_{\lim} = 1$. However, this implies the convergence of the sequence and confirms the Collatz conjecture, which contradicts the definition of n''_0.

Since all statements of Proposition 1.1 lead to a contradiction, we suggest considering the Collatz conjecture proven by methods of sequence theory.