Revisiting Quadrature, Infinity, and the Numbers

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TO CHRISTMAS AND EASTER

"Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about".

Old Testament, 1 Kings 7:23

A. WORKING HYPOTHESIS

"Two truths cannot contradict one another." ***Galileo Galilei*** **(1564-1642, [23, p.186])**

For us, here, the Aristotelian actual infinite is identified with number rationality and constructibility whereas the Aristotelian potential infinity is identified with number irrationality. It is a thesis which would be heretical if judged from the viewpoint of Euclid's theorem. Actual infinite consists of the rational numbers that may be formed on the basis of natural numbers whose number is equal to the product of all primes, symbolizing it via $\boldsymbol{\varpi}$. Potential infinite consists of the potential infinite of the decimal digits that might start being added at the end of a given rational endlessly, and by the potential infinite of the order/disorder with which decimal digits would keep piling up.

That is, our potential infinity is the outcome of the interplay of these two kinds of potential infinity with regard to each of the finitely infinite rational numbers, over the whole set of rational numbers. There are as many such "two-footed" potential infinities as finite rationals... This is one way to compromise the would-be constructibility of the Quadrature with its alleged impossibility: Constructible within the context of actual infinity but impossible from the viewpoint of potential infinity.

B. A BRIEF ACCOUNT OF THE PROBLEM

"The dear God has made the integer numbers, all the rest is man's work." ***Leopold Kronecker*** **(1823-1891[20, p.19])**

Constructing with a straightedge-ruler and a compass a square having area and perimeter equal to those of a given circle or vice versa, was deemed to be impossible by ancient Greeks: "…*Bryson (of Heraclea) declared the circle to be greater than all inscribed and less than all circumscribed polygons*" (*Themistius*, 317-c.390, [62]). That's the most that could be done with a ruler and a compass. Many attempts to refute the ancients have been made since then, but all have failed [32, 33]. In 1882, *Ferdinand von Lindemann* (1852-1939, [47]) proved that the squaring or quadrature of the circle is impossible, because π is a transcendental, rather than an algebraic number; that is, π is not a solution of any polynomial with rational coefficients. Hence, we cannot construct with a ruler and a compass a line segment x such that $x^2 = \pi R^2$, or setting the radius R of the circle equal to one, the number $x = \sqrt{\pi}$ is not constructible. It is quite clear that it is the inconstructibility of transcendence which is responsible for the impossibility of the Quadrature.

Is this thesis true or false? If inadequate computability is thought of corroborating some notion of potential infinity, this statement is indeed true. From the π=3.1605 of the Rhind papyrus in the 17th century B.C. [56] and the π=3.1415926535898732 of astronomer *Ghiyath al-Din Jamshid Mas'ud al-Kashi* (c.1380-1429) of Samarkand around 1430 [5, 18, 55] to the π with the 10000 decimal digits in 1958 and the π with the trillion decimal places being produced nowadays, there has always been a computation problem. Today, the problem is that real numbers are computed by finite, terminating algorithms. It is these computations that are taken to be the real numbers, not the real-real numbers *per se*. And, this presents problems like, for instance, that under the classical definition of a sequence, the set of computable numbers is not closed in so far as taking the *supremum* of a bounded sequence is concerned [7, 44]. Indeed, "*He who can properly define and divide is to be considered a god*" (Plato, 429-347 B.C., [68]).

Note for example that all numbers, rational and irrational, are representable through sums of infinite series. One such series is:

$$4\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right]=\pi,$$

which implies that we may write: $x^2 = (SR^2x^0)\left[1 + \frac{x2}{x}\right] + r \Rightarrow x^2 = SR^2 + r$, where S is a shorthand notation for the above series while r is a zero polynomial. Suppose that $r \neq 0$ and that $R \neq 1$ so that $r = 0$ and $x = SR'$ for some $R' \neq R$. Or, suppose that $r \neq 0$ and that $R = 1$ so that $r = 0$ and $x' = S$ for some $x' \neq x$. In either case, the fact remains that there is always some line segment x or x', call it uniformly y, such that $y = R\sqrt{S}$, $R > 0$. And, this is enough for us:

The transcendental number π comes up as the unique solution to the polynomial equation: $x^2 - SR^2x^0 = 0 \Rightarrow x = R\sqrt{S}$ and hence, $x = \sqrt{\pi}$ is constructible and the squaring of the circle is possible. The construction of the number $x = \sqrt{\pi}$ is possible as a line segment corresponding to an angle of tangent equal to $\sqrt{\pi}$ the way it is elaborated below. From still another point of view, let the numbers of series S (inside the brackets above) be polynomial roots so that S may be seen as the elementary symmetric polynomial $S \equiv e_1(x_1, x_2, \dots, x_v) = \sum_1^v x_i$, $i = 1,2, \dots, v$, coming out as a coefficient of the following linear factorization of a monic polynomial in μ:

$$\prod\nolimits_1^v (\mu - x_i) = \mu^v - e_1(x_1, \dots, x_v)\mu^{v-1} + e_2(x_1, \dots, x_v)\mu^{v-2} - \cdots + (-1)^v e_v(x_1, \dots, x_v)$$

Consequently, $v - 1, v - 2, \dots$ would be sensible if the infinite of S was the actual infinite, $\boldsymbol{\varpi}$; otherwise, $\infty - 1 = \infty - 2 = \cdots$

But, let us take the matter a little bit further. We do dismiss the transcendence of π, but do we retain its irrationality? We know from Euler that:

$$\frac{\pi}{4} = \frac{3}{4}\frac{5}{4}\frac{7}{8}\frac{11}{12}\frac{13}{12}\frac{17}{16}\frac{19}{20}\frac{23}{24}\frac{29}{28}\frac{31}{32}\cdots$$

The numerator is always a prime number while the denominator is always a multiple of four nearest to the numerator. Let us ignore our theorem about the finiteness of the primes, which was advanced in the Introduction, and let us abide by Euclid's Theorem that prime numbers are infinite. If it were not so, π would be a rational number. But, what kind of infinity is that of the prime numbers? One way to perceive it, is to let a computer adding terms to the right of this expression of $\pi/4$ ad infinitum, independently of man's presence on this earth. Another way is to view the product of fractions as product of polynomial roots in which case the product would be the elementary symmetric polynomial $e_v(x_1, \dots, x_v)$ with the same caveat about the infinite of v as with regard to e_1 in connection with series S. This in turn means that infinity is the actual rather than the potential one.

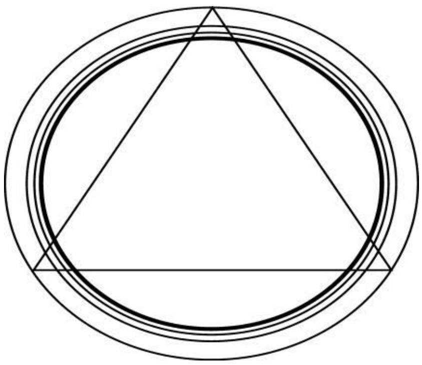


Fig. 1:Convergence to Finiteness

Any notion of actual infinite as signifying the presence of some extreme limit, would suffice to sustain the assertion that π is rational; rational though not computable until now. After all, what is π? It is the ratio of a circle's circumference to its diameter. That is, the ratio of the four sides of the square that squares the circle to the diameter of the circle. All of these magnitudes have endpoints; they are rational quantities and subsequently, π is the ratio of two rational quantities. π is proved to be an irrational number, because irrationality is taken to coincide with potentially infinite non-repeating decimal expansion. It is the potentiality of the example with the computer above, which is in disharmony with the physical world spatially-wise.

The key question is whether one accepts or not the truth of the statement that there is some square which has an area equal to the area of some circle. Once one does reckon this statement to be true, one puts in jeopardy any argument on the impossibility of the Quadrature. Because, a square is finite and so should a circle, or the same, 2π, be, being thereby equally constructible as a square. And, one does have to concede to the truth of this, because take, for example, the numbers $a_i = \sqrt{(3i-1)(3i+1)}/3i$ and the ratio of the perimeters of an equilateral triangle and of its circumcircle, $2\pi/3\sqrt{3}$. It may be shown that $\lim_{i\to\infty} 2\pi(a_1^2 a_2^2 \ldots a_i^2 \ldots) = 3\sqrt{3}$, where $i = 1,2,\ldots,\infty$ (Fig. 2.1, *Jean-Paul Delahaye*, [51]).

That is, the process of shrinking the circumcircle by multiplying its radius with the squares of the a's, *ends* by producing a circumference equal to the perimeter of the equilateral triangle. The sides of two such triangles form a hexagon from which an equal-perimeter square may be drawn, having perimeter equal to two such circumferences. In sum, there does exist some square perimeter corresponding to 2π. The end of the process of shrinking is a physical end, an end within the context of the two-dimensional space, not an end in the sphere of some abstract Platonic forms. The infinite in the lim above is the $\boldsymbol{\varpi}$ rather than the ∞.

Γ. CONSTRUCTION OF ANGLE WITH GRADIENT EQUAL TO $\sqrt{\pi}$

"God the Great Geometrizes continually." ***Plutrch*** **(46-119 AD attributed this belief to** ***Plato*** **429?–347 BC;** ***Convivialium disputationum*****, liber 8,2, [10]*)***

Problem:

Given line segment $\mathcal{E}$, construct with the use of a straightedge and a compass, a right triangle having $\mathcal{E}$ as one of its catheti and with the angle formed by $\mathcal{E}$ and the hypotenuse, having trigonometric tangent equal to $\sqrt{\pi}$ so that the other cathetus may be squaring the circle drawn with radius equal to $\mathcal{E}$, (or construct another line segment having length equal to the product $\mathcal{E}\sqrt{\pi}$ and being perpendicular at one of the endpoints of $\mathcal{E}$ so that the latter may be squaring the circle of radius $\mathcal{E}$).

Intuitive Observation:

Drawing a circle of circumference $L = 2\pi R$, (R=radius), both L and $R = L/2\pi$ are according to traditional mathematics irrational numbers, because π is such a number, and if in general y is a rational number and z is an irrational one, the numbers $z+y$, $z-y$, $y-z$, zy, z/y, and y/z, will be irrational as well. And, from our earlier discussion follows that the irrational numbers L and R should be as constructible as rational numbers are. Methodologically, I could take any number involving π for granted such as line segment $\sqrt{\pi}$, form the hypotenuse $\sqrt{2\pi}$ from the isosceles right triangle of side $\sqrt{\pi}$, separate $\sqrt{2}$ from π on the hypotenuse with a compass, and claim that the hypotenuse is the side x of the square squaring the circle with radius equal to $\sqrt{2}$: $x^2 = \pi(\sqrt{2})^2 \Rightarrow x^2 = 2\pi \Rightarrow x = \sqrt{2\pi}$ ... But, contrary to common sense [4], traditional mathematics do not allow me to consider π to be constructible, and so I have to find another, indirect, implicit, way through which π will be involved in my construction. And, this way is through trigonometry, because trigonometric numbers are based on radians of a rational multiple of π in bijection with rational number of degrees.

Analysis: Consider Fig. 2:

(i) Let $(\Sigma\Phi/\Omega\Sigma) = \tan\Phi\Omega\Sigma = \sqrt{3} \Rightarrow \Sigma\Phi = \Omega\Sigma\sqrt{3}$ and hence, according to Pythagorean Theorem, $\Omega\Phi = \sqrt{\Sigma\Phi^2 + \Omega\Sigma^2} = \sqrt{3\Omega\Sigma^2 + \Omega\Sigma^2} = \Omega\Sigma\sqrt{4} = 2\Omega\Sigma$. Or, if $\Omega\Sigma \equiv \mathcal{E}$, then $\Sigma\Phi = \mathcal{E}\sqrt{3}$, $\Omega\Phi = \Omega O = \Omega\Delta = \Omega Z = \Omega\Gamma = 2\mathcal{E}$ and consequently, $\Omega O' = OO' = \mathcal{E}\sqrt{2}$. And, since $E\Sigma = \Omega\Sigma$, it follows that $E\Sigma = \mathcal{E}$ and $\Omega E = \mathcal{E}\sqrt{2}$, concluding thus that line segments $\Omega O'$ and ΩE are radiuses of a circle with center at point Ω, $(\Omega, \mathcal{E}\sqrt{2})$, Ω being also the center of the circle $(\Omega, 2\mathcal{E})$. Moreover, $\Omega C = \sqrt{2(2\mathcal{E})^2} = 2\mathcal{E}\sqrt{2}$, $OC = \Omega C - \Omega O = 2\mathcal{E}\sqrt{2} - 2\mathcal{E} = 2\mathcal{E}(\sqrt{2}-1)$ and $EO = \Omega O - \Omega E = 2\mathcal{E} - \mathcal{E}\sqrt{2} = \mathcal{E}\sqrt{2}(\sqrt{2}-1) = OC/\sqrt{2}$ so that $EO + OC = \mathcal{E}\sqrt{2} = \Omega E$. Point E lies in the middles of ΩC and triangle $\triangle\ \Phi\Omega\Delta$ is an equilateral one.

(ii) Let next $(HZ/\Omega H) = \tan Z\Omega H = \sqrt{\pi} \Rightarrow HZ = \Omega H\sqrt{\pi}$ and hence, $\Omega Z = \sqrt{HZ^2 + \Omega H^2} = \sqrt{\Omega H^2 + \pi\Omega H^2} = \Omega\Sigma\sqrt{1+\pi}$. Or, if $\Omega H = H\Theta \equiv R \Rightarrow \Omega\Theta = R\sqrt{2}$ and $HZ = R\sqrt{\pi}$, the above magnitudes become $\Omega Z = \Omega\Phi = \Omega O = \Omega\Delta = \Omega\Gamma = R\sqrt{1+\pi} = 2\mathcal{E}$, $\Omega O' = OO' = \Omega E = EC = R\sqrt{1+\pi}/\sqrt{2}$, $\Omega C = R\sqrt{2}\sqrt{1+\pi}$, $OC = R(\sqrt{2}-1)\sqrt{1+\pi}$, and $EO = [R(\sqrt{2}-1)\sqrt{1+\pi}]/\sqrt{2}$. We also obtain the difference $H\Sigma = \mathcal{E} - R$ and $XP \parallel \Theta E = H\Sigma\sqrt{2} = XP$ and $\Lambda\Phi = \Theta E\sqrt{2}$, where X is the midpoint of $H\Lambda$ while P is the midpoint of $\Sigma\Phi$; $H\Lambda = R\sqrt{3}$, because of the similarity of triangles $\triangle\ \Phi\Omega\Sigma$ and $\triangle\ \Lambda\Omega H$, and given that $\Omega H \equiv R$.

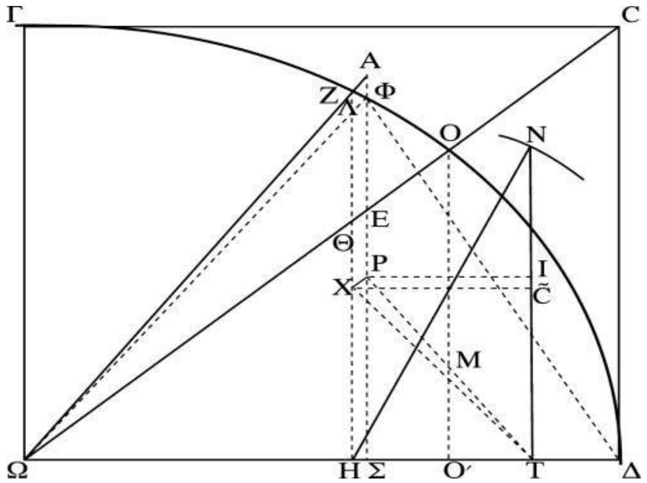


Fig. 2: Squaring the Circle

(iii) Furthermore, let $\mathcal{E}\sqrt{3} = HN \parallel \Omega A$ so that $(TN/HT) = \tan NHT = \sqrt{\pi} \Rightarrow TN = HT\sqrt{\pi}$ and hence, $HM = \sqrt{HT^2 + \pi HT^2} = HT\sqrt{1+\pi}$. Consequently, $\mathcal{E}\sqrt{3} = HT\sqrt{1+\pi} \Rightarrow HT = \mathcal{E}\sqrt{3}/\sqrt{1+\pi} = R\sqrt{3}/2$ and $TN = \left(R\sqrt{3}/2\right)\sqrt{\pi}$. The quadrilateral $HX\tilde{C}T$ is a square having side equal to $R\sqrt{3}/2$. Moreover, in Fig. 2, $I\tilde{C} = H\Sigma = \mathcal{E} - R$ while equalities $HX = HT = R\sqrt{3}/2$ and $H\Theta = H\Delta = R$ imply that $X\Theta = H\Theta - HX = R - \left(R\sqrt{3}/2\right) = H\Delta - HT = T\Delta = PE$.

(iv) Let finally, the upward extensions of ΩZ and $\Sigma\Phi$ meet at point A so that $(\Sigma A/\Omega\Sigma) = (\Sigma A/\mathcal{E}) = \sqrt{\pi} \Rightarrow \Sigma A = \mathcal{E}\sqrt{3}$ and subsequently, $\Omega A = \sqrt{\Omega\Sigma^2 + \Sigma A^2} = \sqrt{\mathcal{E}^2 + \pi\mathcal{E}^2} = \mathcal{E}\sqrt{1+\pi}$, obtaining also that $ZA = \Omega A - \Omega Z = \mathcal{E}\left(\sqrt{1+\pi} - 2\right)$ and $\Phi A = \mathcal{E}\left(\sqrt{\pi} - \sqrt{3}\right) = \Sigma A - \Sigma\Phi$.

Conclusion: The radius of circle $\left(\Omega, \Omega A = \mathcal{E}\sqrt{1+\pi}\right)$ gives through circle $\left(\Omega, \Omega Z = R\sqrt{1+\pi} = 2\mathcal{E}\right)$ rise to the cathetus $HZ = R\sqrt{\pi}$ that squares the circle $(\Omega, \Omega H = R)$, which has radius the other cathetus $\Omega H = R$ of the right triangle $\triangle\, \Omega ZH$; while the radius of the circle $\left(H, HN = \mathcal{E}\sqrt{3}\right)$ gives rise to the cathetus $TN = HT\sqrt{\pi}$ that squares the circle $\left(H, HT = R\sqrt{3}/2\right)$, which has radius the other cathetus HT of the right triangle $\triangle\, HNT$. It follows that if one starts with the equilateral triangle $\triangle\, \Phi\Omega\Delta$ in circle $\left(\Omega, R\sqrt{1+\pi}\right)$, obtain next $T\Delta = PE$ on $\Omega\Delta$, form afterwards square $HX\tilde{C}T$ from quadrilateral ΣPIT, and draw finally, from point H circle $\left(H, \mathcal{E}\sqrt{3}\right)$ to meet at point N the perpendicular at point T, the result will be $\tan NHT = \sqrt{\pi}$ and similar triangles $\triangle\, NHT$, $\triangle\, Z\Omega H$, and $\triangle\, A\Omega\Sigma$, having solved through the latter triangles the stated Problem.

Construction:

(a) Given line segment $\mathcal{E} = \Omega\Sigma$, draw with center endpoint Ω, circle $(\Omega, 2\mathcal{E})$, form equilateral triangle $\triangle\, \Phi\Omega\Delta$ in the northeast quadrant $\Gamma\Omega\Delta$, draw from Φ perpendicular $\Sigma\Phi$ to side $\Omega\Delta$, and receive the bisector ΩO of the right angle $\angle\Gamma\Omega\Delta$, where O is the intersection point of the bisector with the circumference of circle $(\Omega, 2\mathcal{E})$ while the bisector cuts also $\Sigma\Phi$ at point E. {Or, given line segment $\mathcal{E} = \Omega\Sigma$, draw with center endpoint Ω, circle $(\Omega, 2\mathcal{E})$, inscribe the northeast quadrant $\Gamma\Omega\Delta$ inside square $\Omega\Gamma C\Delta$, draw from the midpoint E of the diagonal ΩC line perpendicular to side $\Omega\Delta$ of angle $\angle\Gamma\Omega\Delta$, which perpendicular meets $\Omega\Delta$ at point Σ and cuts the circumference of circle $(\Omega, 2\mathcal{E})$ at point Φ, and form angle $\angle\Phi\Omega\Sigma$.}

(b) From the middle P of perpendicular $\Sigma\Phi$, receive distance equal to PE and transfer it on $\Omega\Delta$ as line segment $T\Delta$, drawing next at T perpendicular which cuts at point I the parallel to $\Omega\Delta$ drawn from P, forming this the parallelogram ΣPIT. With center M at the midpoint of diagonal TP of ΣPIT, draw on the left of TP, semi-circumference of radius $TP/2$, draw from P a half-line parallel to bisector ΩO (or to diagonal ΩC), which half-line meets the semi-circumference at point X and forms with ΣP angle $\angle\Sigma PX$, draw afterwards from X a parallel to ΣP, which cuts $\Omega\Delta$ at point H, and draw moreover a parallel to $\Omega\Delta$, which cuts TI at point $\tilde{C}$, receiving thus the quadrilateral $HX\tilde{C}T$.

(c) With center point H, draw circle $(H, \Sigma\Phi)$, which intersects the upward extension of $T\tilde{C}$ (or TI) at point N so that $HN = \Sigma\Phi$, draw from center Ω radius ΩZ parallel to HN, and finally, receive line segment HZ, forming the triangle $\triangle\, Z\Omega H$ and subsequently, $\triangle\, A\Omega\Sigma$, which is the sought triangle.

Proof (by Contradiction):

We have to prove three things: First, that the quadrilateral $HX\tilde{C}T$ is a square, next that $tan\,NHT = \sqrt{\pi}$ and finally, that HZ is perpendicular at H:

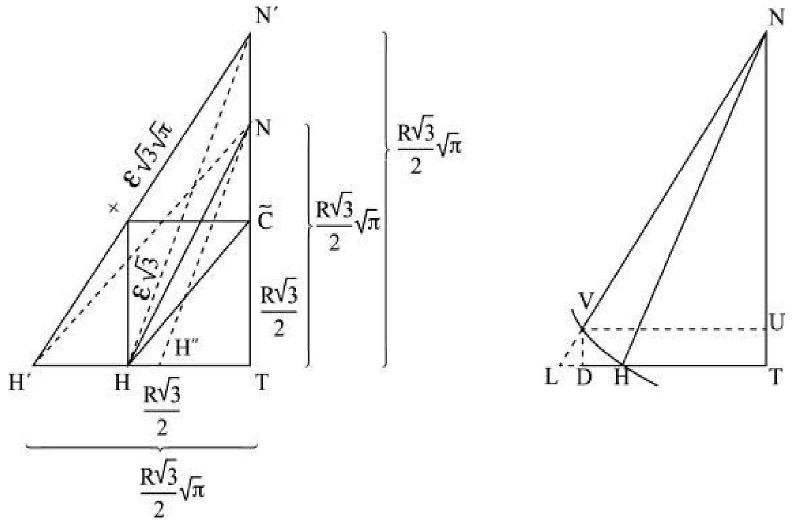
(α) Indeed, by construction, $\angle PX\tilde{C} = 45°$, because $XP \parallel \Omega O(\parallel \Omega\Sigma)$. And, since, drawing TX, triangle $\triangle TXP$ is inscribed in circle $(M, TP/2)$, $\angle TXP = 90°$ and hence, $\angle \tilde{C}XT = \angle TXP - \angle PX\tilde{C} = 45°$, which implies that TX is a diagonal of a square.

(β1) Let next $tan\,NHT = \psi \neq \sqrt{\pi}$. By construction, $\Sigma\Phi = \mathcal{E}\sqrt{3}$ and since, the upward extension of HX intersects $\Omega\Phi$ at point Λ and cuts $\Omega O(\Omega C)$ at Θ, then by the similarity of triangles $\triangle \Phi\Omega\Sigma$ and $\triangle \Lambda\Omega H$, $H\Lambda = \Omega H\sqrt{3}$ and $H\Theta = \Omega H\sqrt{2}$. Or, if $\Omega H \equiv R$, then $H\Lambda = R\sqrt{3}$, which implies that $HX = R\sqrt{3}/2$, since $XP \parallel \Omega O(\parallel \Omega\Sigma)$ and $\Sigma P = \mathcal{E}\sqrt{3}/2$ by construction. Consequently, $HT = R\sqrt{3}/2$, because $HX\tilde{C}T$ is a square. Therefore, if $\psi \neq \sqrt{\pi}$, TM should be equal to $\psi\left(R\sqrt{3}/2\right)$.

Consider now the left part of Fig. 3, which includes square $HX\tilde{C}T$, triangle $\triangle NHT$, and the similar triangle $\triangle N'H'T'$, which obtains through the multiplication of the sides of $\triangle NHT$ by $\sqrt{\pi}$. Let $\angle NH'T$ be the angle which is equal to $\sqrt{\pi}$, $tan\,NH'T = \sqrt{\pi}$. We have $TN = \psi\left(R\sqrt{3}/2\right)$, $TH' = \left(R\sqrt{3}/2\right)\sqrt{\pi}$, and $TN = TH'\sqrt{\pi}$; inserting the first two equalities in the last one yields that $\psi\left(R\sqrt{3}/2\right) = \left[\left(R\sqrt{3}/2\right)\sqrt{\pi}\right]\sqrt{\pi} \Rightarrow \psi = \pi$, which is not true, because π is a half-circle, and which moreover implies that $TN = \pi\left(R\sqrt{3}/2\right)$, giving rise to five contradictions:

The first is that $\triangle NH'T$ is a scaled-up by $\sqrt{\pi}$ version of $\triangle \tilde{C}HT$. How do we know that the hypotenuse of the smaller triangle coincides with diagonal $\tilde{C}H$? We know it, because, given that $(TH'/TH) = \left(R\sqrt{3}/2\right)\sqrt{\pi}/\left(R\sqrt{3}/2\right) = \sqrt{\pi}$, then by the similarity of the bigger with the smaller triangle, the same proportion $\sqrt{\pi}$ should hold for the other side TN of $\triangle NH'T$. And, given the length of TN, this proportion is provided by the ratio $TN/T\tilde{C}$. If $TN = \pi\left(R\sqrt{3}/2\right)$ as it seems to obtain when $\psi = \pi$, then $T\tilde{C}$ should be equal to $\left(R\sqrt{3}/2\right)\sqrt{\pi}$ to enable subsequently the derivation of $\left(T\tilde{C}/TH\right) = \sqrt{\pi}$.

This does not contradict only that $\left(T\tilde{C}/TH\right) = 1$ by construction; it also contradicts our assumption that $tan\,NH'T = \sqrt{\pi}$, because $\angle \tilde{C}HT = 45°$. The third contradiction is that if $\psi = \pi$ and $TN = \pi\left(R\sqrt{3}/2\right)$, then TN should coincide with TN'; but, it does not. And, there is a fourth contradiction, because if they did coincide, then $tan\,NH'T = tanN'H'T = \sqrt{\pi}$ and since, $NH \parallel N'H'$, we would have $\psi = \sqrt{\pi}$ rather than $\psi = \pi$. And, there is a fifth Figure 3: The



Contradiction

contradiction, because if $tanNHT = \psi = \pi$ and $tan\tilde{C}HT = \sqrt{\pi}$, the angle sum identity for $tan\left(\tilde{C}HT + NH\tilde{C}\right)$ would yield $tanNH\tilde{C} = \left(\pi - \sqrt{\pi}\right)/\left(1 + \pi\sqrt{\pi}\right)$. Given now that $\angle\tilde{C}HT + \angle NH\tilde{C} + \angle NHH' = \pi$ and that the sum $(tan\tilde{C}HT + tanNH\tilde{C} + tanNHH')$ is equal to the product $(tan\tilde{C}HTtanNH\tilde{C}tanNHH')$, one obtains that $NHH' = -\pi$, which is false.

Could it be at the other end that $tanN'HT = \sqrt{\pi}$? We understand through similar triangle $\triangle N'HT$ and $\triangle NH''T$ that the answer is negative. We should have $(TN'/TH) = \sqrt{\pi} = (TN/TH'') = \psi\left(R\sqrt{3}/2\right)/x \Rightarrow x = TH'' =$

$\psi(R\sqrt{3}/2)/\sqrt{\pi}$ and hence, $H''H = (R\sqrt{3}/2) - [\psi(R\sqrt{3}/2)/\sqrt{\pi}] = (R\sqrt{3}/2)[(\sqrt{\pi}-\psi)/\sqrt{\pi}]$, which given that $(TN/T\tilde{C}) = \sqrt{\pi}$, yields that $T\tilde{C} = H''H$, contradicting that $T\tilde{C} = R\sqrt{3}/2$, because $R\sqrt{3}/2 = (R\sqrt{3}/2)[(\sqrt{\pi}-\psi)/\sqrt{\pi}] \Rightarrow \sqrt{\pi} = \sqrt{\pi} - \psi \Rightarrow \psi = 0$. Note that the same result would obtain even if we accepted that $\sqrt{\pi} \neq (TN/T\tilde{C}) = \psi$ since, we should also have that $(TH''/H''H) = \psi$ as well.

The general conclusion is that square $HX\tilde{C}T$ along with the use of proportions do establish that $tanNHT = \sqrt{\pi}$ and consequently, that $\mathcal{E}\sqrt{3}$ is equal to the square root of the sum $\left[(R\sqrt{3}/2)^2 + [(R\sqrt{3}/2)\sqrt{\pi}]^2\right]$ from which it follows that $R\sqrt{1+\pi} = 2\mathcal{E}$.

(β2) But, do we really need $HX\tilde{C}T$ to prove that $tanNHT = \sqrt{\pi}$? Let us disregard it for a moment, and let us experiment not only with a different hypotenuse or different horizontal triangle side, but by altering both of them the way the right-hand part of Fig. 3 illustrates. Suppose that the triangle with the "real $\sqrt{\pi}$" is $\triangle NVU$ rather than $\triangle NHT$, with $tanNVU = tanVLD = \sqrt{\pi}$, $VU = R\sqrt{3}/2$ − because this is the length we should have according to the Analysis in order to have $\sqrt{\pi}$, too − and $NV = NH = \mathcal{E}\sqrt{3}$ on $NL = \mathcal{E}\sqrt{3}\sqrt{\pi}$ so that LT is some multiple λ of $R\sqrt{3}/2 = DT = VU$.

From the differences $LD = LT - HT = LT - VU = (R\sqrt{3}/2)(\lambda - 1)$ and $LV = LN - VN = \mathcal{E}\sqrt{3}(\sqrt{\pi}-1)$, and from the similarity of triangles $\triangle VLD$ and $\triangle NLT$, we obtain the proportions:

$$\frac{\mathcal{E}\sqrt{3}\sqrt{\pi}}{\lambda(R\sqrt{3}/2)} = \frac{\mathcal{E}\sqrt{3}(\sqrt{\pi}-1)}{(R\sqrt{3}/2)(\lambda-1)}$$

from which it follows that:

$$\frac{\sqrt{3}\sqrt{\pi}}{\lambda} = \frac{\sqrt{\pi}-1}{\lambda-1} \Rightarrow \lambda^2(2\sqrt{\pi}-1) - \lambda 2\pi + \pi = 0$$

which equation in λ yields the solutions $\lambda = \sqrt{\pi}$ and $\lambda = \sqrt{\pi}/(2\sqrt{\pi}-1)$. The latter solution is rejected because it implies that $LT = (R\sqrt{3}/2)[\sqrt{\pi}/(2\sqrt{\pi}-1)]$ and hence, that $LD = LT - DT = (R\sqrt{3}/2)\{[\sqrt{\pi}/(2\sqrt{\pi}-1)] - 1\}$, with the right-hand side becoming $-(R\sqrt{3}/2)[(1+\sqrt{\pi})/(2\sqrt{\pi}-1)] < 0$. Consequently, multiple $\lambda = \sqrt{\pi}$ reflects the angle $\angle NVU = \angle VLD$, the tangent of which has been assumed to be $\sqrt{\pi}$. It follows that $UN = (R\sqrt{3}/2)\sqrt{\pi}$ given that $VU = R\sqrt{3}/2$, and therefore, $TN = (R\sqrt{3}/2)\pi$, which if rewritten as $TN = [(R\sqrt{3}/2)\sqrt{\pi}]\sqrt{\pi}$, is consistent with $tanNHT = \sqrt{\pi}$ and $HT = R\sqrt{3}/2$, contrary to what we have assumed.

But, more important is the observation that if $TN = [(R\sqrt{3}/2)\sqrt{\pi}]\sqrt{\pi}$, triangle $\triangle NLT$ should be the multiple $\sqrt{\pi}$ of the sides of another triangle, similar to $\triangle NLT$, with sides equal to $R\sqrt{3}/2$ and $(R\sqrt{3}/2)\sqrt{\pi}$, and a hypotenuse $\mathcal{E}\sqrt{3}$, having the angle facing the side equal to $(R\sqrt{3}/2)\sqrt{\pi}$, tangent equal to $\sqrt{\pi}$. This is a quite interesting result, because it suggests that even if the Construction was wrong, it would lead to the correction of the error by simply drawing a parallel to NL so that $\sqrt{\pi}$ may be obtained.

(γ) We must finally show that HZ is perpendicular at H on $\Omega\Delta$. If it was at $\underline{H} \neq H$, then $tanN\underline{H}T = \sqrt{\pi}$, which contradicts that $tanNHT = \sqrt{\pi}$ unless $\underline{H}$ and H coincide. Also, if the upward extension of HX did not intersect the circumference of circle $(\Omega, 2\mathcal{E})$ at Z but at $\underline{Z}$, we should have $tan\underline{Z}\Omega H = \sqrt{\pi}$ and hence, $\underline{Z}$ and Z should coincide, given moreover that by construction, $\Omega Z \parallel HN$: The parallels ensure the verticality. If not anything else, $HZ = \sqrt{\Omega Z^2 - \Omega H^2}$, which is equal to the square root of $\left[(R\sqrt{1+\pi})^2 - R^2\right]$, implying that $HZ = R\sqrt{\pi}$, which is true and therefore, $HZ \perp \Omega\Delta$. It follows that the sought triangle is $\triangle Z\Omega H$, with its hypotenuse ΩZ being the side of the square squaring the circle with radius equal to side ΩH…*Quod Erat Demonstrandum*…

Δ. ON CONSTRUCTIBILITY

"There is more danger of numerical sequences continued indefinitely than of trees growing up to heaven. Each will some time reach its greatest height." ***Friedrich Ludwig Gottlob Frege*** **(1848-1925, [22, p. 204])**

Is there any irrational number that cannot be constructed? In so far as space is concerned, the answer is negative, because tangent runs from zero to infinity while secant runs from one to infinity: All irrational numbers are there; even infinity by itself is there. Infinity, the cosmos, is constructible, and this is why it has to be the actual, the proper infinity.

We have one more proof that spatial infinity has to be the actual one. Spatially-wise, there is no such thing as irrationality, because simply a never ending non-repeating decimal part of a decimal number could not be constructible: When and where our line segment would end? Irrationality should be attributed to computation inadequacies and/or non-spatial considerations like time as a physical phenomenon. The difficulty of constructing irrationals lies in the difficulty of determining which exactly rationals give rise to them. This is the reason in the first place the Quadrature above has been so cumbersome.

One might object to the constructibility of numbers like $\sqrt{2}$ or π [20, 21]. Consider, for instance, $\sqrt{2}$, which had prompted much skepticism on the part of Pythagoreans. If its construction was not possible as a hypotenuse of an isosceles right triangle of unit legs, triangle inscribable into semi-circumference, the proposition that an angle inscribed in a semicircle is a right angle, would not hold. This proposition and hence, the axiom of parallel lines would be violated. It would be impossible to construct the unit *per se* as the hypotenuse of another isosceles right triangle of legs equal to $1/\sqrt{2}$. And, the construction of this leg-side in turn, as the hypotenuse of still another isosceles right triangle of legs equal to $1/\sqrt{4}$, and so on, since none of these sides-hypotenuses could constitute circle diameter.

What would ensure that such triangles are right triangles once the axiom of parallel lines is rejected? One might replace this axiom by setting some magnitude equal to the unit and prompting subsequently the emergence of number $\sqrt{2}$, too. But, how, construction-wise, if one did not also postulate some axiom analogous to that of parallel lines? The fact, yes fact, that $\sqrt{2}$ is constructible, that it has a beginning and an end, stems if not anything else from the fact also that constructible are numbers greater that $\sqrt{2} = 1.41421 \ldots$, numbers like 1.5. As soon as $\sqrt{2} < 1.5$, if their construction started from a single point, the construction representing $\sqrt{2}$ should have an end before the end of the construction representing 1.5. And, hence, the number of the decimal digits capturing $\sqrt{2}$ should have an end as well, even if the axiom of parallel lines was disregarded, and we defined instead some magnitude to be our unit. After all, the notion of Dedekind cut *per se* relies on general number constructibility: Cut of the real line in two distinct half-lines. If it were not so, where would the cut capturing an irrational number be placed? Unless irrationality captures the cut per se, the abrupt disruption of continuity when time is introduced in the discussion.

But, in so far as space alone is concerned, we have to distinguish between infinite but countable decimal digits and infinite uncountable digits accompanying the integer of a decimal number. Toward this end, consider the sequence of sides-hypotenuses, $1/\sqrt{2}$, $1/\sqrt{2}\sqrt{2}$,..., $1/\sqrt{2}^{\kappa}$, where κ is an integer. This sequence tends to zero. How could one start constructing the unit out of zero? The key to the answer is the word "tends"; zero should be out of reach, never reached, because only then, out of something, not out of nothing, one might start constructing the unit. Decimal digits keep coming one after the other, impossible practically to calculate their number, but they have to stop at the gate of zero. Otherwise, the unit would not be constructible.

Or, consider the example of the number π. The infinite division of polygon sides must have an end if the points comprising a circle circumference and not thin air, a complete vacuum, is to be produced. Consequently, the decimal digits of π must have an end. In general, given constructability *per se*, and the constructibility of a number greater than another number with infinite decimal digits, it follows logically that the latter number should be constructible as well. Decimal digits must have an end; they are infinite but countably so. The *ad infinitum* counting must come to a halt to allow the construction of a number which is smaller than a greater known to be constructible number. All numbers with infinite decimals are countably infinite, because there is always a greater number known to be constructible. There are no uncountably infinite decimals.

What we have, in other words, is infinity in the Aristotelian sense of actual as opposed to potential infinity. Any in general irrational number is one with an actualization in nature and hence, with a number of decimal digits in line with the Aristotelian notion of actual infinity; with decimals that sooner or later become repeating. It all comes down to the fifth axiom of Euclid: "*That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles*." The first four are: "*Let the following be postulated'": 1. "To draw a straight line from any point to any point." 2."To produce [extend] a finite straight line continuously in a straight line." 3."To describe a circle with any centre and distance [radius]." 4."That all right angles are equal to one another.*" [13, 30, 31]

Consider Fig. 4. If the two lines ϵ and η could not meet at point A, no circle $(O, OA = OB)$ could be drawn, because no line $A\Gamma$ could be drawn too, in violation of axiom 3, which refers to *any* circle of *any* radius. As a matter of fact, no circle at all could be drawn, because one must always be able to draw from a point like A a line like ϵ, intersecting a radius like OB. But, also, axiom 4 would be violated, because angles a and b would have to be right angles, and $a \neq b$. Axiom 5 follows from and completes axioms 3 and 4 in fully describing the plane, the two-dimensional space, following axioms 1 and 2, which fully describe the one-dimensional space. More precisely, axiom 5 ensures continuity in the two-dimensional space the same way axiom 2 ensures continuity in the one-dimensional space. If lines like ϵ could not meet the horizontal axis, no circle at all could be drawn; there would be no two-dimensional space, contrary to what axioms 3 and 4 postulate.

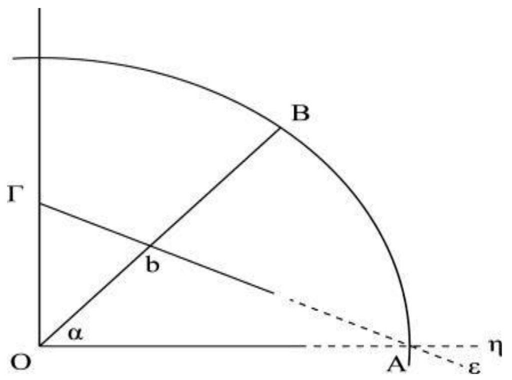


Fig. 4: The Euclidean Axiom

This is the reason Euclid formulated axiom 5 the way he did and not as a Playfair or other similar axiom; the wording was chosen carefully in serving the purpose of this axiom. But, axiom 5 does much more than completely defining along with axioms 3 and 4 the plane. It puts an end to infinity: this "*produced indefinitely*" has an end, be it one next to the origin of axes, O, or to zillions miles away from it, the end being point A, the intersection point, because intersection takes place in the infinity, after indefinite extension of ϵ and η. If space ended before the intersection, neither axiom 4 nor axiom 5 would hold; and if space ended after A, the extension of ϵ and η would have not been indefinite, because intersection at A occurs after such an extension.

The finiteness of the infinity is in the core of geometry, and this is the reason it underlines non-Euclidean geometries as well. These geometries replace the *John Playfair* (1748-1819)-axiom side of axiom 5, but retain the actual infinity side, and this is the reason they continue being geometries, i.e. studies of space, each viewing it analytically from its own standpoint given that space in reality is only one. Any other axiomatic theoretical construction dismissing axiom 5 altogether, simply is not geometry. When *Aristotle* (384-322 B.C. [3, ch. 6]) said: "*For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different*", he said it literally: He did not say that the cosmos is limitless but that the limits are in continuous change, and trying to catch up with them is futile. This position is very important analytically, because it implies that statically viewed, the cosmos is susceptible to scientific inquiry including the dynamics inside its borders...

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