An Operator Theory Problem Book: Chapter 5

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CHAPTER 5

Positive operators. Square root

5.1. Exercises with Solutions

Exercise 5.1.1. Are the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

positive?

Exercise 5.1.2. Let S be the shift operator on $\ell^2(\mathbb{N})$. Is $I - SS^*$ positive?

Exercise 5.1.3. Let $A \in B(\ell^2)$ be the multiplication operator defined by:

$$A(x_1, x_2, \cdots, x_n, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots)$$

where $(\alpha_n)_n \in \ell^\infty$. Show that

$$A \geq 0 \Longleftrightarrow \alpha_n \geq 0,\ \forall n \in \mathbb{N}.$$

Exercise 5.1.4. Let $A \in B(H)$ be self-adjoint. Show that e^A is positive.

Exercise 5.1.5. Let $A, B \in B(H)$ be both positive. Does it follow that $AB + BA \geq 0$?

Exercise 5.1.6. Let A and B be two bounded and positive operators on a *complex* Hilbert space H. Show that if $A + B = 0$, then $A = B = 0$.

Exercise 5.1.7. Let A be a matrix on a finite dimensional space such that $A \geq 0$ and tr$A$=0. Show that $A = 0$.

Exercise 5.1.8. Let $A, B, T \in B(H)$ where A and B are self-adjoint.

(1) Show that:

$$A \geq 0 \Longrightarrow T^*AT \geq 0 \text{ and } TAT^* \geq 0.$$

(2) Show that:

$$A \geq B \Longrightarrow T^*AT \geq T^*BT \text{ and } TAT^* \geq TBT^*.$$

Exercise 5.1.9. Let $P, Q \in B(H)$ be two orthogonal projections. Show that $P - Q$ is an orthogonal projection iff $P \geq Q$.

Exercise 5.1.10. Let $A \in B(H)$ be positive.

(1) Show that
$$| < Ax, y > |^2 \leq < Ax, x >< Ay, y >$$
for all $x, y \in H$.

(2) Infer that for every $x \in H$,
$$\|Ax\|^2 \leq \|A\| < Ax, x > .$$

Exercise 5.1.11. Let $A \in B(H)$ be self-adjoint.

(1) Show that
$$-I \leq A \leq I \Longleftrightarrow \|A\| \leq 1.$$

(2) Let $\alpha \geq 0$. Show that
$$-\alpha I \leq A \leq \alpha I \Longleftrightarrow \|A\| \leq \alpha.$$

Exercise 5.1.12. Let $A, B \in B(H)$ be self-adjoint where $A \geq 0$. Show that
$$-A \leq B \leq A \Longrightarrow \|B\| \leq \|A\|.$$

Exercise 5.1.13. Let $A, B \in B(H)$ be both positive. Show that
$$\|A - B\| \leq \max(\|A\|, \|B\|).$$

Exercise 5.1.14. Let $A, K \in B(H)$ be such that A is positive and AK is self-adjoint. Prove that
$$| < AKx, x > | \leq ||K|| < Ax, x >$$
for all $x \in H$.

Exercise 5.1.15. (cf. Exercise 5.1.29) Let $A \in B(H)$ be positive and let $K \in B(H)$ be a contraction. Show that if $AK^* = KA$, then
$$K^2 A = A(K^*)^2 = KAK^* \leq A.$$

Exercise 5.1.16. Let $A, B \in B(H)$ be commuting and positive. Using the Reid Inequality, show that $AB \geq 0$.

Exercise 5.1.17. Let $A \in B(H)$ be positive. Show that A^n is also positive for each $n \in \mathbb{N}$.

Exercise 5.1.18. (cf. Exercise 5.1.19) Let $A, B \in B(H)$ be such that $A \geq B \geq 0$.

(1) Does it follow that $A^2 \geq B^2$?
(2) Show that $A^2 \geq B^2$ whenever $AB = BA$.

Exercise 5.1.19. Let $A, B \in B(H)$ be such that $0 \leq A \leq B$ and $AB = BA$. Show that $0 \leq A^n \leq B^n$ for all $n \in \mathbb{N}$.

Exercise 5.1.20. Let A be a bounded self-adjoint operator on an $\mathbb{R}$-Hilbert space H such that

$$\exists c > 0, \forall x \in H : < Ax, x > \geq c\|x\|^2.$$

(1) Show that A is invertible.
(2) Let $p(t) = t^2 + at + b$ be a real polynomial having a *strictly negative* discriminant. Show that $p(A)$ is invertible.
(3) *Application*: Check that $A^2 + A + I$ is invertible whenever A is self-adjoint.
(4) Show that the hypothesis A being self-adjoint cannot be simply dropped.

Exercise 5.1.21. Let $A \in B(H)$ be self-adjoint. Let

$$U = (A - iI)(A + iI)^{-1}$$

(U is called the **Cayley Transform** of A).

(1) Explain why $A + iI$ is invertible (so that $(A + iI)^{-1}$ makes sense!).
(2) Show that U is unitary.

Exercise 5.1.22. ([**29**]) Let $U, V \in B(H)$ be both unitary. Show that the following assertions are equivalent:

(1) $\|U - V\| < 2$;
(2) $U + V$ is invertible.

Exercise 5.1.23. Let $A \in B(H)$. Show that

$$\text{Re}A \geq 0 \Longleftrightarrow (A - \alpha I)^*(A - \alpha I) \geq \alpha^2 I, \ \forall \alpha < 0.$$

Exercise 5.1.24. Find the square root (if it exists) of the following operators:

(1) $A : \ell^2 \to \ell^2$ defined by

$$A(x_1, x_2, \cdots) = (0, 0, x_3, x_4, \cdots).$$

(2) S is the shift operator on ℓ^2. What about S^*?

Exercise 5.1.25. Let (A_n) be a sequence of self-adjoint operators in $B(H)$. Prove that if (A_n) is bounded monotone increasing, then it is strongly convergent to a self-adjoint operator in $B(H)$.

Exercise 5.1.26. Let $A \in B(H)$ be positive.

(1) Suppose that $\|A\| \leq 1$. Define a sequence (B_n) of operators in $B(H)$ by
$$\begin{cases} B_0 = 0, \\ B_{n+1} = B_n + \frac{1}{2}(A - B_n^2). \end{cases}$$
Show that (B_n) is a sequence of positive self-adjoint operators which is also bounded monotone increasing.
(2) Deduce that (B_n) strongly converges to a positive $B \in B(H)$ such that $B^2 = A$. Infer also that any operator which commutes with A commutes with B.
(3) Obtain the same conclusion by making no assumption this time on the norm $\|A\|$.
(4) Show that if B and C are positive and such that $B^2 = A$ and $C^2 = A$, then $B = C$.

Exercise 5.1.27. Give another proof of the uniqueness of the positive square root of positive operators (**hint**: if $T \in B(H)$ is self-adjoint, what is $\|T^4\|$?).

Exercise 5.1.28. Let A and B be two *positive* operators on a complex Hilbert space H.
(1) Show that if A and B commute, then AB (and hence BA) is positive. Infer that
$$(AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}.$$
(2) Give an example showing the importance of the commutativity of A and B for the result to hold.
(3) Prove the converse of the result in Question 1, that is, prove that if A, B and AB are all positive operators, then A and B must commute.

Exercise 5.1.29. (cf. Exercise 5.1.15) Let $A, K \in B(H)$ where A is positive and K is a contraction. Show that if $AK = KA$, then $K^*AK \leq A$.

Exercise 5.1.30. Let $A, B \in B(H)$ be such that $0 \leq A \leq B$.
(1) Show that $\sqrt{A} \leq \sqrt{B}$.
(2) If further A is taken to be invertible, then show that B too is invertible and that $B^{-1} \leq A^{-1}$.

Exercise 5.1.31. Let $A, B \in B(H)$ be such that $AB = BA$ and $A, B \geq 0$. Show that
$$\sqrt{A+B} \leq \sqrt{A} + \sqrt{B} \leq \sqrt{2(A+B)}.$$

Exercise 5.1.32. Let A be a self-adjoint operator on a complex Hilbert space H such that $\|A\| \leq 1$. Let I be the identity operator on H.

(1) Justify the existence of $(I - A^2)^{\frac{1}{2}}$.
(2) Set $U_\pm = A \pm i(I - A^2)^{\frac{1}{2}}$. Show that $U_\pm$ are unitary operators on H.

Exercise 5.1.33. Show that any $A \in B(H)$ may be written as a linear combination of four *unitary* operators.

Exercise 5.1.34. Let H be a complex Hilbert space. If $A, B \in B(H)$ are self-adjoint and $BA \geq 0$, then show that

$$\forall x \in H: \ \|Ax\| \leq \|Bx\| \Longleftrightarrow \exists K \in B(H) \text{ **positive** contraction}: \ A = KB.$$

Exercise 5.1.35. Let $A, B \in B(H)$ be positive and commuting. Show that

$$0 \leq A \leq B \Longrightarrow A^2 \leq B^2.$$

Exercise 5.1.36. ([**7**]) Let $A, B, C \in B(H)$ be such that $A, B \geq 0$. Define an operator T on $B(H \oplus H)$ by

$$T = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}.$$

Show that

$$T \geq 0 \Longleftrightarrow |< Cx, y >|^2 \leq < Ax, x >< By, y >, \ \forall x, y \in H.$$

Exercise 5.1.37. Let $A, B, C \in B(H)$ be such that B and C are positive. Show that if $BA = AC$, then

$$\sqrt{B}A = A\sqrt{C}.$$

Exercise 5.1.38. ([**121**]). Let $A, B \in B(H)$ be such that either A *or* B is positive. We want to show that

$$\|[A, B]\| \leq \|A\|\|B\| ...(1)$$

WLOG, we choose $A \geq 0$.

(1) If B is a self-adjoint contraction, show that

$$\|[A, B]\| \leq \|A\|.$$

(2) Deduce that if B is self-adjoint but not necessarily a contraction this time, then Inequality (1) still holds.
(3) Show, via an operator matrix trick, that Inequality (1) holds for any $B \in B(H)$.

Exercise 5.1.39. ([**158**], cf. Exercise 5.1.40) Let $T \in B(H)$ be such that $T^2 = 0$ and $\operatorname{Re} T \geq 0$ (or $\operatorname{Im} T \geq 0$). Show that T is normal and so $T = 0$.

Exercise 5.1.40. ([**77**]) Let $T = A + iB \in B(H)$ and let $n \geq 2$. Show that if $T^n = 0$ and $A \geq 0$ (or $B \geq 0$), then $T = 0$.

Exercise 5.1.41. Let p and q be two relatively prime numbers, and let $A, B \in B(H)$ be such that $A^p = B^p$ and $A^q = B^q$. Show that $A = B$ whenever A is invertible.

5.2. Solutions

Solution 5.2.1. Both A and B are positive. Let $x, y \in \mathbb{R}$. Then

$$< \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} >= < \begin{pmatrix} x+y \\ x+y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} >$$

$$\begin{aligned} &= x^2 + yx + xy + y^2 \\ &= (x+y)^2 \geq 0. \end{aligned}$$

As for B, despite the fact that

$$\begin{aligned} < \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > &=< \begin{pmatrix} x+y \\ 2y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > \\ &= x^2 + yx + 2y^2 \\ &= \left(x + \frac{y}{2}\right)^2 + \frac{7}{4}y^2 > 0, \end{aligned}$$

we cannot consider it as a positive matrix as B is not symmetric!

In fine, C is not positive because

$$< \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} >= x^2 + 4xy + 2y^2$$

can be negative (e.g. if $x = 1$ and $y = -1$).

Solution 5.2.2. The answer is yes. Let $x = (x_1, x_2, \cdots) \in \ell^2$. Then we already know that

$$S(S^*x) = S(x_2, x_3, \cdots) = (0, x_2, x_3, \cdots).$$

Hence

$$(I - SS^*)(x_1, x_2, \cdots) = (x_1, x_2, \cdots) - (0, x_2, \cdots) = (x_1, 0, 0, \cdots).$$

Thence

$$< (I - SS^*)x, x >=< (x_1, 0, 0, \cdots), (x_1, x_2, \cdots) >= x_1\overline{x_1} + 0 + \cdots = |x_1|^2.$$

Therefore, $I - SS^*$ is positive.

Remark. We know that $S^*S = I$. This means that we have just shown that $SS^* \leq S^*S$. In fact, any isometry A verifies $AA^* \leq A^*A$. This seems to be an unnecessary observation but this shows that the shift operator belongs to an important class of operators (see Hyponormal Operators).

Solution 5.2.3. We know that A is self-adjoint iff α_n is real-valued for each n. If $\alpha_n \geq 0$ for all n, then clearly for any $x = (x_1, x_2, \cdots, x_n, \cdots) \in \ell^2$

$$< Ax, x >= \sum_{n=1}^{\infty} \alpha_n |x_n|^2 \geq 0,$$

i.e. $A \geq 0$.

Conversely, if $A \geq 0$, then for *any* $x = (x_1, x_2, \cdots, x_n, \cdots) \in \ell^2$

$$< Ax, x >= \sum_{n=1}^{\infty} \alpha_n |x_n|^2 \geq 0.$$

In particular, for $x = e_n$ (from the usual orthonormal basis), we have that $\alpha_n \geq 0$ for all n, as needed.

Solution 5.2.4. Let $x \in H$. Since A is self-adjoint, $A/2$ too is self-adjoint so that $e^{\frac{A}{2}}$ is self-adjoint. We may then write for all $x \in H$

$$< e^A x, x >=< e^{\frac{A}{2}} e^{\frac{A}{2}} x, x >=< e^{\frac{A}{2}} x, e^{\frac{A}{2}} x >= \|e^{\frac{A}{2}} x\|^2 \geq 0.$$

Solution 5.2.5. No! Consider the positive matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } BA = (AB)^* = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

But

$$AB + BA = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

is not positive (why?).

Solution 5.2.6. Let $x \in H$. We may write for all $x \in H$

$$0 =< (A + B)x, x >=< Ax, x > + < Bx, x > .$$

But $< Ax, x >$ and $< Bx, x >$ are two positive *real* numbers because A and B are positive operators. Therefore,

$$< Ax, x >= 0 \text{ and } < Bx, x >= 0 \text{ for all } x \in H,$$

i.e. $A = B = 0$.

Solution 5.2.7. Since $A \geq 0$, A is self-adjoint. Hence it is diagonalizable (a well known fact or see e.g. [**10**]). Thus, for some invertible P,

$$P^{-1}AP = D,$$

where D is a diagonal matrix whose diagonal contains the eigenvalues of A which are all positive (why?). But, clearly

$$\mathrm{tr}D = \mathrm{tr}(P^{-1}AP) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}A.$$

Since $\mathrm{tr}A$=0, $\mathrm{tr}D$=0, that is, the sum of the *positive* eigenvalues vanishes. This forces $D = 0$ or $A = 0$.

Solution 5.2.8.

(1) Let $x \in H$. Then

$$< T^*ATx, x >=< ATx, T^{**}x >=< ATx, Tx >\geq 0$$

since A is positive. A similar argument applies to prove the other inequality.

(2) Since $A - B \geq 0$, we may just apply the previous results to have

$$T^*(A-B)T \geq 0 \text{ or } T^*AT \geq T^*BT$$

(since also T^*AT and T^*BT are self-adjoint) and

$$T(A-B)T^* \geq 0 \text{ or } TAT^* \geq TBT^*.$$

Solution 5.2.9. Assume that $P - Q$ is an orthogonal projection. Then $(P-Q)^2 = P - Q$ so that for all $x \in H$, we have

$$< (P-Q)x, x >=< (P-Q)^2x, x >=< (P-Q)x, (P-Q)x >= \|(P-Q)x\|^2 \geq 0,$$

meaning that $P \geq Q$.

Conversely, assume that $P \geq Q$. Then we leave it to the reader to show that this is equivalent to saying that $PQ = Q$, and also equivalent to $QP = Q$. Hence

$$(P-Q)^2 = P^2 - PQ - QP + Q^2 = P - Q - Q + Q = P - Q.$$

Accordingly, $P - Q$ is an orthogonal projection (because $P - Q$ is also self-adjoint).

Solution 5.2.10.

(1) Let $x, y \in H$. Define

$$[x, y] =< Ax, y > .$$

Then $[\cdot, \cdot]$ verifies all the properties of an inner product except perhaps that we may have $[x, x] = 0$ for some $x \neq 0$. So, to establish the required inequality, just proceed as in the first question of Exercise 3.3.7.

Remark. ([**132**]) Another way of establishing the previous inequality is to set $< x,y >_r = < Ax,y > + r < x,y >$ where $r > 0$. Then show that $< \cdot,\cdot >_r$ is an inner product, apply the standard Cauchy-Schwarz Inequality to it, send $r \to 0$ and finally get the desired generalization!

(2) Setting $y = Ax$ in the previous result, we get

$$\|Ax\|^4 = |< Ax, Ax >|^2 \leq < Ax,x >< A^2x, Ax > \leq < Ax,x > \|A^2x\|\|Ax\|.$$

Whence

$$\|Ax\|^4 \leq < Ax,x > \|A\|\|Ax\|\|Ax\| \Longrightarrow \|Ax\|^4 \leq < Ax,x > \|A\|\|Ax\|^2.$$

Thus

$$\|Ax\|^2 \leq \|A\| < Ax,x > .$$

Remark. Another way of proving the previous inequality is via the Reid Inequality (as observed in [**183**]). Indeed, setting $A = K$ in the Reid Inequality gives a shorter proof of this result.

Solution 5.2.11.

(1) Since A is self-adjoint, $< Ax,x >$ is real (for all $x \in H$). We may then write

$$\begin{aligned} < (\pm A - I)x,x > &= \pm < Ax,x > - \|x\|^2 \\ &= |< Ax,x >| - \|x\|^2 \\ &\leq \|Ax\|\|x\| - \|x\|^2 \text{ (by the Cauchy-Schwarz Inequality)} \\ &= (\|Ax\| - \|x\|)\|x\|. \end{aligned}$$

If $\|A\| \leq 1$, then clearly $\|Ax\| \leq \|A\|\|x\| \leq \|x\|$ for each $x \in H$. Hence

$$< (\pm A - I)x,x > \leq 0 \text{ or merely } \pm A \leq I,$$

i.e. $-I \leq A \leq I$.

To prove the other implication, notice that if $-I \leq A \leq I$, then

$$\forall x \in H: \ \pm < Ax,x > \leq \|x\|^2 \text{ or } |< Ax,x >| \leq \|x\|^2$$

for all $x \in H$. Passing to the supremum over $\|x\| = 1$ yields (by taking into account the self-adjointness of A)

$$\|A\| = \sup_{\|x\|=1} |< Ax,x >| \leq 1$$

and this marks the end of the proof.

(2) If $\alpha = 0$, then the results is obvious. If $\alpha > 0$, then apply the previous question with $\frac{1}{\alpha}A$ instead of A.

Solution 5.2.12. By assumption, for all $x \in H$

$$- < Ax, x > \leq < Bx, x > \leq < Ax, x > \text{ or merely } | < Bx, x > | \leq < Ax, x > .$$

Therefore,

$$\|B\| = \sup_{\|x\|=1} | < Bx, x > | \leq \sup_{\|x\|=1} < Ax, x > = \|A\|,$$

as desired.

Solution 5.2.13. WLOG, we may assume that $\|A\| \geq \|B\|$. So we must show that

$$\|A - B\| \leq \|A\|.$$

Since $A, B \geq 0$, they are self-adjoint, and so is then $A - B$. Again, since $A, B \geq 0$, we have

$$-B \leq A - B \leq A.$$

Also for all $x \in H$, we have (by the Cauchy-Schwarz Inequality)

$$< Ax, x > \leq \|Ax\|\|x\| \leq \|A\| < Ix, x > = < \|A\|Ix, x >,$$

i.e. $A \leq \|A\|I$. Similarly, we find that $-B \geq -\|B\|I$. Thus,

$$-\|B\|I \leq A - B \leq \|A\|I.$$

Taking into account the choice $\|A\| \geq \|B\|$ yields

$$-\|A\|I \leq A - B \leq \|A\|I.$$

Finally, by Exercise 5.1.11, we then obtain

$$\|A - B\| \leq \|A\| = \max(\|A\|, \|B\|).$$

Solution 5.2.14. The proof presented here is mostly due to Reid in [**183**]. WLOG, we may assume that $\|K\| \leq 1$ (why?). Therefore, we need only show

$$| < AKx, x > | \leq < Ax, x >$$

for all $x \in H$.

Since AK is self-adjoint, it follows that $AK = K^*A$. Hence

$$AK^2 = K^*AK = (K^*)^2A = (AK^2)^*, \ AK^3 = (K^*)^2AK = (K^*)^3A = (AK^3)^*, \cdots ,$$

so by induction, for each n, AK^n is self-adjoint.

Since $A \geq 0$, Corollary **??** yields for all $x \in H$:

$$\begin{aligned} | < AKx, x > | \leq & \frac{1}{2}[< Ax, x > + < AKx, Kx >] \\ = & \frac{1}{2}[< Ax, x > + < K^*AKx, x >] \\ = & \frac{1}{2}[< Ax, x > + < AK^2x, x >]. \end{aligned}$$

Thanks to the previous inequality and by doing a little induction, we get for all n (and all x)

$$| < AKx, x > | \leq (2^{-1} + \cdots + 2^{-n}) < Ax, x > + 2^{-n} < AK^{2^n}x, x > ...(1)$$

Since $\|K\| \leq 1$, we have by the Cauchy-Schwarz Inequality

$$| < AK^{2^n}x, x > | \leq \|AK^{2^n}x\|\|x\| \leq \|A\|\|K^{2^n}\|\|x\|^2 \leq \|A\|\|K\|^{2^n}\|x\|^2 \leq \|A\|\|x\|^2$$

and so passing to the limit as $n \to \infty$ in (1) gives clearly

$$| < AKx, x > | \leq < Ax, x >,$$

as suggested.

Solution 5.2.15. First, observe that

$$AK^* = KA \Longrightarrow A(K^*)^2 = KAK^* = K^2A.$$

Since A is positive, so is KAK^* or $A(K^*)^2$. Thereupon, using Reid Inequality, we know that

$$< KAK^*x, x > = < A(K^*)^2x, x > = | < A(K^*)^2x, x > | \leq < Ax, x > .$$

So much for the proof.

Solution 5.2.16. WLOG, we may suppose that $0 \leq B \leq I$ (otherwise work with $\frac{B}{\|B\|}$). Hence $\|I - B\| \leq 1$. Since $A(I - B)$ is clearly self-adjoint and $A \geq 0$, it follows from Reid Inequality that

$$AB = A - A(I - B) \geq 0.$$

Solution 5.2.17. The proof follows by induction (using the fact that the product of two positive commuting operators remains positive). Alternatively, we can treat two cases: n being even and n being odd (remembering that a positive operator is self-adjoint). Details are left to the reader.

Solution 5.2.18.

(1) The answer is no! Anticipating a little bit, we know from Question 2 that we need to choose two non-commuting A and B. Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Observe that both A and B are positive. So it only remains to check that $A \geq B$ whereas $A^2 \not\geq B^2$, that is, we need to verify that $A - B \geq 0$ and that $A^2 - B^2 \not\geq 0$. We see that

$$A - B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0$$

whereas

$$A^2 - B^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \not\geq 0$$

(check it).

(2) Since $AB = BA$, we clearly have

$$A^2 - B^2 = (A + B)(A - B).$$

But, $A \geq B$ means that $A - B \geq 0$. Also, it is plain that $A + B \geq 0$.

The fact that $A - B$ commutes with $A + B$ (as $AB = BA$) imply that

$$(A + B)(A - B) = A^2 - B^2 \geq 0,$$

and hence $A^2 \geq B^2$ (***remember that A^2 and B^2 are self-adjoint, a simple but a crucial point!***). This marks the end of the proof.

Solution 5.2.19. Since $AB = BA$, we have

$$0 \leq A \leq B \Longrightarrow 0 \leq A^2 \leq AB$$

and

$$0 \leq A \leq B \Longrightarrow 0 \leq AB \leq B^2.$$

Hence

$$A^2 \leq B^2$$

(which is another proof of the result of Exercise 5.1.18). Using a similar argument, and a proof by induction, we can easily prove the required inequality for $n \in \mathbb{N}$...

Solution 5.2.20.

(1) Let $x \in H$. By the Cauchy-Schwarz Inequality

$$c\|x\|^2 \leq < Ax, x > \leq \|Ax\| \|x\|.$$

Therefore $\|Ax\| \geq c\|x\|$. Since A is self-adjoint, the result follows.

(2) By hypothesis $\triangle = a^2 - 4b < 0$. Then

$$p(A) = A^2 + aA + bI$$

is self-adjoint. We may write

$$A^2 + aA + bI = \left(A + \frac{a}{2}I\right)^2 + b - \frac{a^2}{4} = \left(A + \frac{a}{2}I\right)^2 - 4\triangle.$$

Since $A + a/2I$ is self-adjoint, $(A + \frac{a}{2}I)^2$ is positive. Hence for all $x \in H$

$$< p(A)x, x > \geq \underbrace{-4\triangle}_{>0} < x, x > .$$

Thus $p(A)$ is invertible by the foregoing question.

(3) Straightforward!

(4) Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then A is not self-adjoint. It is also easy to see that

$$A^2 = -I \text{ or } A^2 + I = 0.$$

With the above notation, $a = 0$ and $b = 1$ and so $a^2 - 4b < 0$. In the end, it is clear that $A^2 + I$ is not invertible.

Solution 5.2.21.

(1) Let $x \in H$. By considering

$$\|(A + iI)x\|^2 = < (A + iI)x, (A + iI)x >,$$

one can easily see that

$$\forall x \in H : \ \|(A + iI)x\| \geq \|x\|.$$

Hence $A + iI$ is bounded below. Since A is self-adjoint, $A + iI$ is normal. Therefore, $A + iI$ is invertible.

(2) First we compute U^*. We have

$$\begin{aligned} U^* =&[(A-iI)(A+iI)^{-1}]^* \\ =&[(A+iI)^{-1}]^*(A-iI)^* \\ =&[(A+iI)^*]^{-1}(A^*+iI^*) \\ =&[(A^*-iI^*)]^{-1}(A^*+iI) \\ =&(A-iI)^{-1}(A+iI) \text{ (because } A \text{ is self-adjoint).} \end{aligned}$$

Since A commutes with multiples of the identity, we easily see that

$$\begin{aligned} U^*U =&[(A-iI)]^{-1}(A+iI)(A-iI)(A+iI)^{-1} \\ =&\underbrace{[(A-iI)]^{-1}(A-iI)}_{I}\underbrace{(A+iI)(A+iI)^{-1}}_{I} \\ =&I. \end{aligned}$$

In a similar vein, we find that $UU^* = I$, that is, U is unitary.

Solution 5.2.22.

(1) "(1) $\Rightarrow$ (2)": First, we set

$$A = \frac{1}{2}(U+I) \text{ and } B = \frac{1}{2}(V+I).$$

Then it is clear that

$$\|A-B\| = \frac{1}{2}\|U-V\| < 1.$$

Hence $\|A-B\|^2 < 1$ so that there exists some $\alpha > 0$ such that

$$\|(A-B)^*(A-B)\| = \|A-B\|^2 \leq 1-\alpha.$$

Whence

$$(A-B)^*(A-B) \leq (1-\alpha)I$$

or after simplification,

$$I - A^*A - B^*B + A^*B + B^*A \geq \alpha I.$$

It is clear that

$$A^*A = \frac{1}{2}(A+A^*) \text{ and } B^*B = \frac{1}{2}(B+B^*).$$

Since $U = 2A - I$ and $V = 2B - I$, we have

$$\begin{aligned} (U+V)^*(U+V) =&4(A^*+B^*-I)(A+B-I) \\ =&4(I-A^*A-B^*B+A^*B+B^*A) \\ \geq&4\alpha I. \end{aligned}$$

Similarly, by considering

$$A^* = \frac{1}{2}(U^* + I) \text{ and } B^* = \frac{1}{2}(V^* + I),$$

we may show that

$$(U + V)(U + V)^* \geq 4\alpha I.$$

Thus $U + V$ is invertible.

(2) The other implication may be proved by going backwards in the previous proof (do the details!).

Solution 5.2.23. It is clear that if $\alpha \in \mathbb{R}$, then

$$(A-\alpha I)^*(A-\alpha I)-\alpha^2 I = (A^*-\alpha I)(A-\alpha I)-\alpha^2 I = A^*A-\alpha(A^*+A)...(1)$$

If the previous quantity is positive for all $\alpha < 0$, then we have

$$\alpha(A^* + A) \leq A^*A \text{ or } A^* + A \geq \frac{1}{\alpha}A^*A.$$

Taking the limit as $\alpha \to -\infty$ gives

$$A + A^* \geq 0, \text{ i.e. } \mathrm{Re}A \geq 0$$

and this proves "$\Leftarrow$".

Now assume that $\mathrm{Re}A \geq 0$ and let $\alpha < 0$. Since A^*A is positive, it is evident that

$$A + A^* \geq 0 \geq \frac{A^*A}{\alpha}.$$

This means that the quantities on each side of the equalities involved in Equation (1) are greater than or equal to zero, so that for any $\alpha < 0$,

$$(A - \alpha I)^*(A - \alpha I) \geq \alpha^2 I,$$

establishing "$\Rightarrow$".

Solution 5.2.24.

(1) It is easy to see that A is positive (do the details!). It then follows that A has one and only one positive square root. As clearly $A^2 = A$, then $\sqrt{A} = A$ is the (unique) positive square root of A.

(2) The shift operator and its adjoint do not possess any square root whatsoever. Assume for the sake of contradiction that e.g. S^* does, i.e. $A^2 = S^*$, where $A \in B(H)$. Then, $A^2S = S^*S = I$ and by the general theory A is right invertible and so it is surjective. Notice also that A cannot be injective (indeed, this would imply that $A^2 = S^*$ is injective and this is untrue).

Now, we show that $\ker A = \ker S^* = \mathbb{R}e_1$, where $e_1 = (1, 0, 0, \cdots)$. The equality $\ker S^* = \mathbb{R}e_1$ is known and clear.

It also implies that $\dim\ker S^* = 1$. Now, we obviously have $\ker A \subset \ker S^*$ because $A^2 = S^*$. Since A is not injective, we are forced to have $\ker A = \ker S^*$ as $\ker A$ and $\ker S^*$ are vector spaces.

Since A is onto, for all $y \in \ell^2$, in particular for $e_1 \in \ell^2$, there is an $x \in \ell^2$ such that $Ax = e_1$ (and so $x \notin \ker A = \ker S^*$). Thus (as $e_1 \in \ker A$)

$$A^2x = Ae_1 = 0 \neq S^*x.$$

This shows that S^* does not have any square root.

If S had a square root, then we would have $S = B^2$, where $B \in B(\ell^2)$. Therefore, $S^* = (B^2)^* = (B^*)^2$, i.e. S^* would possess a square root! This is a contradiction with what we have just seen. Accordingly, S cannot have a square root either!

Solution 5.2.25. By assumption, we know that $A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq A$ for some self-adjoint $A \in B(H)$. WLOG we may assume that $A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq I$ (just divide each A_i by $\|A\|$ and relabel $\frac{A_i}{\|A\|}$ as A_i). There is also no loss of generality in assume that all $A_n \geq 0$ (e.g. we could use the sequence $(A_n - A_1)_n$, say). Therefore, we may work with $0 \leq A_1 \leq A_2 \leq \cdots \leq A_n \leq \cdots \leq I$.

The primary aim is to show that (A_nx) converges for each x in H. By the completeness of H, this means that it suffices then to show that (A_nx) is Cauchy. Let $n > m$ and let $x \in H$. Then $A_n - A_m \geq 0$ and $A_n - A_m \leq I$. Hence $\|A_n - A_m\| \leq 1$. Now, we may write

$$\begin{aligned}
\|A_nx - A_mx\|^4 &= < (A_n - A_m)x, (A_n - A_m)x >^2 \\
&\leq < (A_n - A_m)x, x >< (A_n - A_m)^2x, (A_n - A_m)x > \\
&\leq < (A_n - A_m)x, x > \|(A_n - A_m)^2x\|\|(A_n - A_m)x\| \\
&\leq < (A_n - A_m)x, x > \|A_n - A_m\|\|(A_n - A_m)x\|^2 \\
&\leq < (A_n - A_m)x, x > \|A_nx - A_mx\|^2
\end{aligned}$$

where we have used Theorem **??** in the first inequality. Therefore,

$$\|A_nx - A_mx\|^2 \leq < (A_n - A_m)x, x > = < A_nx, x > - < A_mx, x > .$$

But $(< A_nx, x >)_n$ is an increasing *real* sequence which is bounded above by $(\|x\|^2)$. Whence, it converges and so it is Cauchy. Thereupon,

$$\lim_{n,m\to\infty} \|A_nx - A_mx\| = 0.$$

This means, as already observed above, that $\lim_{n\to\infty} A_nx$ exists for each $x \in H$.

Define now for each x

$$Ax = \lim_{n\to\infty} A_n x$$

(in the sense that $\|A_n x - Ax\| \to 0$ for all x). Then A is clearly linear. It only remains to see why A is bounded and self-adjoint. We prove these two requirements together: By the continuity of the inner product, we have for all $x, y \in H$

$$< Ax, y >= \lim_{n\to\infty} < A_n x, y >= \lim_{n\to\infty} < x, A_n y >=< x, Ay > .$$

Calling on the Hellinger-Toeplitz Theorem, we obtain that $A \in B(H)$, and clearly A is self-adjoint.

To summarize, the bounded monotone increasing sequence (A_n) converges strongly to the self-adjoint bounded operator A.

Solution 5.2.26.

(1) Observe first that since A is positive and $\|A\| \leq 1$, we have $0 \leq A \leq I$. Another equally important observation is that the sequence (B_n) is a "polynomial" of A. This implies that all of B_n are pairwise commuting.

Next, $B_0 = 0$ is evidently self-adjoint. So, assuming that B_n is self-adjoint (and recalling that A is self-adjoint), we can easily check that B_{n+1} too is self-adjoint. Therefore, all B_n are self-adjoint.

Now, we claim that $B_n \leq I$ for all n. This is obviously true for $n = 0$. Assume that $B_n \leq I$. Observing that $(I - B_n)^2 \geq 0$ (why?), we then have

$$I - B_{n+1} = I - B_n - \frac{1}{2}(A - B_n^2) = \frac{1}{2}(I - B_n)^2 + \frac{1}{2}(I - A) \geq 0.$$

To prove that (B_n) is increasing, observe first that $B_0 \leq \frac{1}{2}A = B_1$. Assuming that $B_n \geq B_{n-1}$, we may write

$$B_{n+1} - B_n = \frac{1}{2}[(I - B_{n-1}) + (I - B_n)](B_n - B_{n-1})$$

which, being a product of commuting positive operators, itself is positive.

Consequently, we have shown that

$$0 = B_0 \leq B_1 \leq \cdots \leq B_n \leq \cdots \leq I,$$

as needed.

(2) Since (B_n) is bounded monotone increasing, by Theorem **??** we know that (B_n) converges strongly to some self-adjoint $B \in B(H)$. Since each B_n is positive, we have

$$< Bx, x >= \lim_{n\to\infty} < B_n x, x > \geq 0$$

as strong convergence implies weak one. Thus, $B \geq 0$.

It remains to show that $B^2 = A$. Let $x \in H$. We have by hypothesis

$$B_{n+1}x = B_n x + \frac{1}{2}(Ax - B_n^2 x).$$

Passing to the strong limit and using $\|B_n^2 x - B^2 x\| \to 0$ (why?), we finally get $B^2 = A$, as required.

Finally, assume that a $C \in B(H)$ commutes with A, i.e. $AC = CA$. We must show that $BC = CB$. Since C commutes with A, we may easily show that C commutes with B_n too, that is, $CB_n x = B_n Cx$ (for all n and all x). On the one hand, we clearly see that $B_n Cx \to BCx$. On the other hand, invoking the (sequential) continuity of C, we have that $CB_n x \to CBx$. By uniqueness of the strong limit, we get

$$BCx = CBx, \ \forall x \in H,$$

as desired.

(3) If $A = 0$, then $B = 0$ will do. So if $A \neq 0$, considering $T = \frac{A}{\|A\|}$ gives $0 \leq T \leq 1$. Then, apply what we have already done above.

(4) *The proof of uniqueness here, although not being complicated, is not as direct as one is used to with other theorems.*

We have already shown that $B^2 = A$. Assume that there is another positive $C \in B(H)$ such that $C^2 = A$. We must show that $Bx = Cx$ for all $x \in H$. Observe first that A plainly commutes with C. By Question (2), C commutes with B as well, i.e. $BC = CB$. This tells us that

$$(B + C)(B - C) = B^2 - C^2 = A - A = 0.$$

So, if we let $x \in H$ and set $y = (B - C)x$, then

$$< By, y > + < Cy, y >=< (B+C)y, y >=< (B+C)(B-C)x, y >= 0.$$

Because both B and C are positive, we obtain (cf. Exercise 5.1.6)

$$< By, y >=< Cy, y >= 0.$$

By Question (2) again, $B \geq 0$ has a square root which we denote by D, say. That is, $D^2 = B$. Therefore,

$$\|Dy\|^2 = < Dy, Dy > = < D^2y, y > = < By, y > = 0$$

and so $Dy = 0$. This implies that $By = D^2y = D(0) = 0$.

Using also a square root of C, we may similarly show that $Cy = 0$. Consequently,

$$\|Bx - Cx\|^2 = < (B - C)x, (B - C)x > = < (B - C)y, x > = 0.$$

Accordingly, $B = C$, i.e. we have proven that the positive A can *only have one* positive square root, marking the end of the proof.

Solution 5.2.27. Assume that $A \in B(H)$ is positive. Hence, there is a positive $B \in B(H)$ such that $B^2 = A$. Assume that there is another positive $C \in B(H)$ such that $A = C^2$ and so $B^2 = C^2$. We ought to show that $B = C$.

First, it is clear that

$$CA = C^3 = AC.$$

Hence C commutes with B as well (why?). This gives

$$(B - C)B(B - C) + (B - C)C(B - C) = (B^2 - C^2)(B - C) = 0.$$

As $B, C \geq 0$ and $B - C$ is *self-adjoint*, then $(B - C)B(B - C)$ and $(B - C)C(B - C)$ are both positive and so

$$(B - C)B(B - C) = (B - C)C(B - C) = 0.$$

Thereupon,

$$(B - C)B(B - C) - (B - C)C(B - C) = 0,$$

that is

$$(B - C)^3 = 0.$$

Whence

$$(B - C)^4 = 0.$$

Now, if $T \in B(H)$ is self-adjoint, then $\|T^2\| = \|T\|^2$. Since T^2 is self-adjoint, we get $\|T^4\| = \|T\|^4$.

Consequently,

$$0 = \|(B - C)^4\| = \|B - C\|^4,$$

that is $B = C$, as required.

Solution 5.2.28.

(1) Since A is positive, it admits a unique positive square root, which we denote by P (that is $P^2 = A$). Since B commutes with A, it commutes with P as well.

Let $x \in H$. We may write (remembering that positive operators are necessarily self-adjoint)

$$< ABx, x >=< P^2Bx, x >=< PBx, Px >=< BPx, Px >\geq 0$$

as B is positive. Therefore, $AB \geq 0$.

Since A and B are positive, both $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ exist and are well-defined. Since A and B also commute, AB is positive and it makes sense then to define $(AB)^{\frac{1}{2}}$. If we come to show that

$$(A^{\frac{1}{2}}B^{\frac{1}{2}})^2 = AB,$$

then by the uniqueness of the square root, the desired result follows.

Now since A and B commute, so do their square roots and we have

$$(A^{\frac{1}{2}}B^{\frac{1}{2}})^2 = A^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}} = A^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}} = AB.$$

The proof is complete.

(2) Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then both A and B are positive.

We may also check that

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 6 \end{pmatrix},$$

i.e. AB is not positive because it is not even self-adjoint and

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix} = BA.$$

(3) Since A, B and AB are all positive operators, they are all self-adjoint. Accordingly,

$$BA = B^*A^* = (AB)^* = AB,$$

that is A and B commute.

Solution 5.2.29. Since $KA = AK$ and A is self-adjoint, it follows that $AK^* = K^*A$. Hence $AK^*K = K^*KA$. Therefore $A^{\frac{1}{2}}K^*K = K^*KA^{\frac{1}{2}}$ as $A \geq 0$.

Now, let $x \in H$. By the Generalized Cauchy-Schwarz Inequality, we may write

$$< K^*AKx, x >^2 = < AK^*Kx, x >^2 \leq < Ax, x >< AK^*Kx, K^*Kx > .$$

But,

$$< AK^*Kx, K^*Kx > = < A^{\frac{1}{2}}K^*Kx, A^{\frac{1}{2}}K^*Kx > = \|A^{\frac{1}{2}}K^*Kx\|^2 = \|K^*KA^{\frac{1}{2}}x\|^2.$$

Because $\|K^*K\| \leq 1$, we obtain

$$\|K^*KA^{\frac{1}{2}}x\|^2 \leq \|A^{\frac{1}{2}}x\|^2 = < A^{\frac{1}{2}}x, A^{\frac{1}{2}}x > = < Ax, x >$$

so that

$$< K^*AKx, x >^2 \leq < Ax, x >^2,$$

completing the proof.

Solution 5.2.30.

(1) Let $x \in H$. Since $0 \leq A \leq B$, we have for all $x \in H$

$$0 \leq < Ax, x > \leq < Bx, x > \Longleftrightarrow 0 \leq < \sqrt{A}x, \sqrt{A}x > \leq < \sqrt{B}x, \sqrt{B}x >$$

and so (for all x)

$$0 \leq \|\sqrt{A}x\|^2 \leq \|\sqrt{B}x\|^2.$$

So, by Theorem 3.1.69, we know that $\sqrt{A} = K\sqrt{B}$ for some contraction $K \in B(H)$. Since $\sqrt{A}$ is self-adjoint, it follows that $K\sqrt{B}$ too is self-adjoint, i.e. $K\sqrt{B} = \sqrt{B}K^*$. Since $\sqrt{B} \geq 0$, by the Reid Inequality we obtain:

$$< \sqrt{A}x, x > = < \sqrt{B}K^*x, x > \leq < \sqrt{B}x, x >,$$

that is,

$$\sqrt{A} \leq \sqrt{B},$$

as required.

(2) As before, we know that $\sqrt{A} = K\sqrt{B}$ for some contraction $K \in B(H)$. Since $\sqrt{A}$ is invertible (as A is), it follows that $I = (\sqrt{A})^{-1}K\sqrt{B}$, i.e. the self-adjoint $\sqrt{B}$ is left invertible. By taking adjoints, we see that $\sqrt{B}$ is also right invertible. Thus, B is invertible and

$$(\sqrt{B})^{-1} = (\sqrt{A})^{-1}K = K^*(\sqrt{A})^{-1}$$

by the self-adjointness of both $(\sqrt{B})^{-1}$ and $(\sqrt{A})^{-1}$.

Finally, let $x \in H$. Then (since K^* too is a contraction)

$$< B^{-1}x, x > = \|(\sqrt{B})^{-1}x\|^2 = \|K^*(\sqrt{A})^{-1}x\|^2 \leq \|(\sqrt{A})^{-1}x\|^2 = < A^{-1}x, x >,$$

as needed.

Solution 5.2.31. Since $AB = BA$ and $A, B \geq 0$, we have $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$. Hence $\sqrt{A}\sqrt{B} \geq 0$. Therefore,

$$A + B \leq A + 2\sqrt{A}\sqrt{B} + B = (\sqrt{A} + \sqrt{B})^2.$$

Since $\sqrt{A} + \sqrt{B} \geq 0$, we get

$$\sqrt{A + B} \leq \sqrt{A} + \sqrt{B},$$

establishing half of the result.

Finally, to prove the other inequality, reason similarly using $(\sqrt{A} - \sqrt{B})^2 \geq 0$...

Solution 5.2.32.

(1) We need only verify that $I - A^2$ is a positive operator. Let $x \in H$. We have

$$\begin{aligned} < (I - A^2)x, x > \geq 0 &\Longleftrightarrow < x, x > - < A^2x, x > \geq 0 \\ &\Longleftrightarrow < A^2x, x > \leq \|x\|^2 \\ &\Longleftrightarrow < Ax, Ax > = \|Ax\|^2 \leq \|x\|^2. \end{aligned}$$

But by hypothesis, $\|A\| \leq 1$ which leads to

$$\|Ax\|^2 \leq \|A\|^2\|x\|^2 \leq \|x\|^2.$$

Therefore, $I - A^2 \geq 0$.

(2) We only prove U_+ is unitary (the proof for U_- is very akin). Since A is self-adjoint, one has

$$U_+^* = (A + i(I - A^2)^{\frac{1}{2}})^* = A - i(I - A^2)^{\frac{1}{2}}.$$

Since A and $I - A^2$ commute, so do A and $(I - A^2)^{\frac{1}{2}}$ and so

$$\begin{aligned} U_+U_+^* &= (A + i(I - A^2)^{\frac{1}{2}})(A - i(I - A^2)^{\frac{1}{2}}) \\ &= A^2 - iA(I - A^2)^{\frac{1}{2}} + i(I - A^2)^{\frac{1}{2}}A + I - A^2 \\ &= I. \end{aligned}$$

Similarly, one shows that $U_+^*U_+ = I$

Solution 5.2.33. We already know that any $A \in B(H)$ may be written as $A = \operatorname{Re} A + i \operatorname{Im} A$, that is, every $A \in B(H)$ may be expressed as a linear combination of *two self-adjoint* operators.

Now, suppose that $B \in B(H)$ is self-adjoint. WLOG, we may assume that $\|B\| \leq 1$ (otherwise, you know what you should do!). By Exercise 5.1.32, $B \pm i(I - B^2)^{\frac{1}{2}}$ are unitary operators and clearly

$$B = \frac{1}{2}[B + i(I - B^2)^{\frac{1}{2}}] + \frac{1}{2}[B - i(I - B^2)^{\frac{1}{2}}],$$

so that each self-adjoint operator may be expressed as a linear combination of *two unitary* operators, and this leads to the fact that any $A \in B(H)$ may be written as a linear combination of four unitary operators.

Solution 5.2.34.

(1) "$\Leftarrow$": Let $x \in H$. Then

$$0 \leq < KBx, Bx >=< Ax, Bx >=< BAx, x >,$$

that is, $BA \geq 0$.

(2) "$\Rightarrow$": Since $BA \geq 0$, it follows that BA is self-adjoint, i.e. $AB = BA$. As a consequence, $\ker A$ reduces A and B, and the restriction of A to $\ker A$ is the zero operator on $\ker A$. Hence, we can assume that A is injective. Therefore, because $\ker B \subset \ker A = \{0\}$, we see that B^{-1} is self-adjoint and **densely defined** (i.e. defined on a dense domain). Set $K_0 = AB^{-1}$. Then K_0 is densely defined and

$$||K_0(Bx)|| = ||AB^{-1}Bx|| = ||Ax|| \leq ||Bx||, \forall x \in H,$$

signifying that K_0 is a contraction with a unique contractive extension K to the whole H. Since

$$< K_0(Bx), Bx >=< Ax, Bx >=< BAx, x >\geq 0$$

for all $x \in H$, we see that K is positive as well. Clearly

$$KBx = K_0(Bx) = Ax$$

for all $x \in H$, and this completes the proof.

Solution 5.2.35. Since $AB \geq 0$, we know that (why?) $\sqrt{A} = K\sqrt{B}$ for some positive contraction $K \in B(H)$ and $K\sqrt{B} = \sqrt{B}K$. Hence

$$A = K\sqrt{B}K\sqrt{B} = K^2B.$$

So for all $x \in H$:

$$\|Ax\|^2 = \|K^2Bx\|^2 \leq \|Bx\|^2$$

or merely

$$< A^2x, x >=< Ax, Ax >= \|Ax\|^2 \leq \|Bx\|^2 =< B^2x, x >,$$

as required.

Solution 5.2.36.

(1) "$\Longrightarrow$": Assume that $T \geq 0$. By the Generalized Cauchy-Schwarz Inequality (applied to the vectors $(x,0)$ and $(0,y)$), we have

$$\left|< T\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} >\right|^2 \leq < T\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} >< T\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} > .$$

But $T = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}$ and so the previous inequality becomes after simplifications:

$$| < Cx, y > |^2 \leq < Ax, x >< By, y >,$$

valid obviously for all $x, y \in H$.

(2) "$\Longleftarrow$": Now, suppose that

$$| < Cx, y > |^2 \leq < Ax, x >< By, y >, \ \forall x, y \in H.$$

To show that T is positive, let $x, y \in H$ and observe that

$$< T\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} >=< Ax, x > + < C^*y, x > + < Cx, y > + < By, y > .$$

Since

$$< C^*y, x > + < Cx, y >= \overline{< Cx, y >} + < Cx, y >= 2\text{Re} < Cx, y >,$$

it follows that

$$\begin{aligned} < T\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > &=< Ax, x > +2\text{Re} < Cx, y > + < By, y > \\ &\geq 2 < Ax, x >^{\frac{1}{2}} < Bx, x >^{\frac{1}{2}} +2\text{Re} < Cx, y > \ \text{(why?)} \\ &\geq 2| < Cx, y > | + 2\text{Re} < Cx, y > \ \text{(by assumption)} \\ &\geq 2| < Cx, y > | - 2| < Cx, y > | \\ &= 0, \end{aligned}$$

marking the end of the proof.

Solution 5.2.37. Set

$$T = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

both defined on $H \oplus H$. Since $B, C \geq 0$, it easily follows that $T \geq 0$ as

$$\begin{aligned} < \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > &= < \begin{pmatrix} Bx \\ Cy \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} > \\ &= < Bx, x > + < Cy, y > \geq 0 \end{aligned}$$

for all $x, y \in H$. It is also clear that the square root of T is given by

$$\sqrt{T} = \begin{pmatrix} \sqrt{B} & 0 \\ 0 & \sqrt{C} \end{pmatrix}.$$

Since by assumption $BA = AC$, we get

$$TS = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & BA \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix} = ST.$$

Now, as $T \geq 0$, then we obtain $\sqrt{T}S = S\sqrt{T}$. This means that

$$\begin{pmatrix} \sqrt{B} & 0 \\ 0 & \sqrt{C} \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{B} & 0 \\ 0 & \sqrt{C} \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & \sqrt{B}A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A\sqrt{C} \\ 0 & 0 \end{pmatrix},$$

i.e. $\sqrt{B}A = A\sqrt{C}$, as required.

Solution 5.2.38. First, recall that

$$[A, B] = AB - BA.$$

(1) Let B be a self-adjoint contraction. By Exercise 5.1.32, $U = B + i\sqrt{I - B^2}$ is unitary and $B =$Re $U = \frac{U+U^*}{2}$.

$$\begin{aligned}
\|AB - BA\| &= \left\| A\left(\frac{U+U^*}{2}\right) - \left(\frac{U+U^*}{2}\right)A \right\| \\
&= \frac{1}{2}\|AU - UA + AU^* - U^*A\| \\
&\leq \frac{1}{2}\|AU - UA\| + \frac{1}{2}\|AU^* - U^*A\| \\
&= \frac{1}{2}\|AU - UA\| + \frac{1}{2}\|U(AU^* - U^*A)U\| \\
&= \frac{1}{2}\|AU - UA\| + \frac{1}{2}\|(UAU^* - A)U\| \\
&= \frac{1}{2}\|AU - UA\| + \frac{1}{2}\|UA - AU\| \\
&= \|AU - UA\| \\
&= \|(A - UAU^*)U\| \\
&= \|A - UAU^*\| \\
&\leq \max(\|A\|, \|UAU^*\|) \text{ (Exercises 5.1.8 \& 5.1.13)} \\
&= \|A\|,
\end{aligned}$$

establishing the result.

(2) Let B be self-adjoint. The inequality clearly holds for $B = 0$, so assume that $\|B\| > 0$. Hence $\frac{B}{\|B\|}$ remains self-adjoint and besides, it is a contraction. Therefore, the result of the previous question applies and yields
$$\left\|A\frac{B}{\|B\|} - \frac{B}{\|B\|}A\right\| \leq \|A\|,$$
that is,
$$\|AB - BA\| \leq \|A\|\|B\|,$$
as required.

(3) Let $B \in B(H)$. Define on $H \oplus H$
$$\tilde{A} = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} \mathbf{0} & B \\ B^* & \mathbf{0} \end{pmatrix},$$
where the $\mathbf{0}$ is the zero operator on H. Observe that $\tilde{B}$ is self-adjoint (even if B is not one), and that $\tilde{A}$ is self-adjoint because A is one! Hence, by the previous question we know that
$$\|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\| \leq \|\tilde{A}\|\|\tilde{B}\|.$$
But,
$$\tilde{A}\tilde{B} - \tilde{B}\tilde{A} = \begin{pmatrix} \mathbf{0} & AB - BA \\ AB^* - B^*A & \mathbf{0} \end{pmatrix}.$$
Also, we have
$$\left\|\begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix}\right\| = \left\|\begin{pmatrix} \mathbf{0} & C \\ D & \mathbf{0} \end{pmatrix}\right\| = \max(\|C\|, \|D\|).$$
Hence (why?)
$$\|\tilde{A}\| = \|A\| \text{ and } \|\tilde{B}\| = \|B\|$$
With all these observations, we infer that
$$\begin{aligned}\|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\| &= \max(\|AB - BA\|, \|AB^* - B^*A\|) \\ &= \max(\|AB - BA\|, \|(AB^* - B^*A)^*\|) \\ &= \max(\|AB - BA\|, \|BA - AB\|) \\ &= \|AB - BA\|,\end{aligned}$$
so that finally we get
$$\|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}\| \leq \|\tilde{A}\|\|\tilde{B}\| \Longleftrightarrow \|AB - BA\| \leq \|A\|\|B\|,$$
and this completes the proof.

Solution 5.2.39. Write $T = A + iB$ where $A, B \in B(H)$ are self-adjoint with $A = \operatorname{Re} T$ and $B = \operatorname{Im} T$ as is known to readers. Then clearly

$$T^2 = A^2 - B^2 + i(AB + BA).$$

So, if $T^2 = 0$, then

$$A^2 - B^2 + i(AB + BA) = 0 \Longrightarrow \begin{cases} A^2 = B^2, \\ AB = -BA. \end{cases}$$

Hence, if $A \geq 0$ (a similar argument works when $B \geq 0$), then

$$AB = -BA \Longrightarrow A^2B = -ABA = BA^2 \Longrightarrow AB = BA.$$

Therefore, T is normal. Accordingly

$$\|T\|^2 = \|T^2\| = 0 \Longrightarrow T = 0,$$

as suggested.

Solution 5.2.40. The proof is carried out in two steps.

(1) Let $\dim H < \infty$. The proof uses a trace argument. First, assume that $A \geq 0$. Clearly, the nilpotence of T does yield $\operatorname{tr} T = 0$. Hence

$$0 = \operatorname{tr}(A + iB) = \operatorname{tr} A + i \operatorname{tr} B.$$

Since A and B are self-adjoint, we know that $\operatorname{tr} A, \operatorname{tr} B \in \mathbb{R}$. By the above equation, this forces $\operatorname{tr} B = 0$ and $\operatorname{tr} A = 0$. The positiveness of A now intervenes to make $A = 0$. Therefore, $T = iB$ and so T is normal. Thus, and as alluded above,

$$0 = \|T^n\| = \|T\|^n,$$

thereby, $T = 0$.

In the event $B \geq 0$, reason as above to obtain $T = A$ and so $T = 0$, as wished.

(2) Let $\dim H = \infty$. The condition $\operatorname{Re} T \geq 0$ is equivalent to $\operatorname{Re} < Tx, x > \geq 0$ for all $x \in H$. So if E is a closed invariant subspace of T, then the previous condition also holds for $T|E : E \to E$.

Now, we proceed to show that $T = 0$, i.e. we must show that $Tx = 0$ for all $x \in H$. So, let $x \in H$ and let E be the span of $x, Tx, \cdots, T^{n-1}x$ (that is, the orbit of x under the action of T). Hence E is a finite dimensional subspace of H (and so it is equally a Hilbert space). By the nilpotence assumption, we have

$$T^n x = 0,$$

from which it follows that E is invariant for T. So, by the first part of the proof (the finite dimensional case), we know that $T = 0$ on E whereby $Tx = 0$. As this holds for any x, it follows that $T = 0$ on H, as needed.

Solution 5.2.41. Since A is invertible, it is seen that B too is invertible. Indeed, by the invertibility of A, we get that of A^p or that of B^p. So, $CB^p = B^pC = I$ for a certain $C \in B(H)$, and hence $(CB^{p-1})B = B(B^{p-1}C) = I$, whereby B is invertible.

Since p and q are relatively prime numbers, Bezout's theorem in arithmetic says that $up + vq = 1$ for some integers u and v (only one of them is negative). WLOG, suppose that u is the negative integer. Now, $A^p = B^p$ yields $A^{up} = B^{up}$, and $A^q = B^q$ implies that $A^{vq} = B^{vq}$. Therefore, $A^{up}A^{vq} = B^{up}B^{vq}$

$$A = A^{up+vq} = B^{up+vq} = B,$$

as looked forward to.

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