Fractional Distance: The Topology of the Real Number Line with Applications to the Riemann Hypothesis

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Abstract

Recent analysis has uncovered a broad swath of rarely considered real numbers called real numbers in the neighborhood of infinity. Here, we extend the catalog of the rudimentary analytical properties of all real numbers by defining a set of fractional distance functions on the real number line and studying their behavior. The main results are (1) to prove with modest axioms that some real numbers are greater than any natural number, (2) to develop a technique for taking a limit at infinity via the ordinary Cauchy definition reliant on the classical epsilon-delta formalism, and (3) to demonstrate an infinite number of non-trivial zeros of the Riemann zeta function in the neighborhood of infinity. We define numbers in the neighborhood of infinity with a Cartesian product of Cauchy equivalence classes of rationals. We axiomatize the arithmetic of such numbers, prove the operations are well-defined, and then make comparisons to the similar axioms of a complete ordered field. After developing the many underlying foundations, we present a basis for a topology.

Contents

Symbols

∞	Infinity
$\mathbb{R}$	Real Numbers $(-\infty, \infty)$
$\mathbb{R}^+$	Positive-Definite Real Numbers $(0, \infty)$
$\overline{\mathbb{R}}$	Affinely Extended Real Numbers $[-\infty, \infty]$
x	Coordinate Chart on $\mathbb{R}$
$d(x, y)$	Euclidean Metric
$\mathbb{C}$	Complex Numbers $x + iy$
$\mathbb{N}$	Natural Numbers
$\mathbb{Q}$	Rational Numbers
$\mathbb{Z}$	Integers
$\mathbb{W}$	Whole Numbers
$\mathbb{R}_0$	Real Numbers Less Than Some Natural Number
$\mathbb{R}_\infty$	$\mathbb{R} \setminus \mathbb{R}_0$
AB	Line Segment
X	Point
$\mathbf{AB}$	Line Segment With Algebraic Representation $[0, \infty]$
$\mathscr{X}$	Algebraic Representation of X "Curly X"
$\mathcal{X}$	Quantity of Fractional Distance "Script X"
$\mathcal{D}_{AB}(AX)$	Geometric Fractional Distance Function
$\mathcal{D}^\dagger_{AB}(AX)$	Algebraic Fractional Distance Function
$\mathcal{D}'_{AB}(AX)$	Algebraic Fractional Distance Function of the First Kind
$\mathcal{D}''_{AB}(AX)$	Algebraic Fractional Distance Function of the Second Kind
$\aleph_1, \widehat{\infty}$	Algebraic Infinity
$\aleph_{\mathcal{X}}$	Fraction of Algebraic Infinity $\mathcal{X} \times \aleph_1$
$\mathbb{T}$	Transfinitely Extended Real Numbers $(-\aleph_\infty, \aleph_\infty)$
$\mathbb{R}^0_\aleph$	Whole Neighborhood of the Origin
$\mathbb{R}^0_0$	Natural Neighborhood of the Origin
$\mathbb{R}^1_\aleph$	Maximal Whole Neighborhood of Infinity
$\mathbb{R}^1_0$	Maximal Natural Neighborhood of Infinity
$\left\{\mathbb{R}^{\mathcal{X}}_\aleph\right\}$	All Intermediate Whole Neighborhoods of Infinity $0 < \mathcal{X} < 1$
$\left\{\mathbb{R}^{\mathcal{X}}_0\right\}$	All Intermediate Natural Neighborhoods of Infinity $0 < \mathcal{X} < 1$
$\mathbb{R}^\cup_\aleph$	$\mathbb{R}^0_\aleph \cup \left\{\mathbb{R}^{\mathcal{X}}_\aleph\right\} \cup \mathbb{R}^1_\aleph$
$\mathbb{R}^\cup_0$	$\mathbb{R}^0_0 \cup \left\{\mathbb{R}^{\mathcal{X}}_0\right\} \cup \mathbb{R}^1_0$
$\mathbb{R}^{\mathcal{X}}_C$	$\mathbb{R}^{\mathcal{X}}_\aleph \setminus \mathbb{R}^{\mathcal{X}}_0$

$\{x_n\}$	Cauchy Sequence
$[x]$	Cauchy Equivalence Class
$C_{\mathbb{Q}}$	All Cauchy Sequences of Rational Numbers
$C_{\mathbb{Q}}^{\mathbf{AB}}$	Range of a Function of $[x], [y] \in C_{\mathbb{Q}}$
$\hat{P}$	Geometric Permutation Operator
$\hat{\mathcal{P}}_0([x]; \hat{0})$	Algebraic Permutation Operator
$\mathcal{R}$	Complete Ordered Field of Real Numbers $\{\mathbb{R}, +, \times, \leq\}$
$\mathbb{N}_{\mathcal{X}}$	Extended Natural Numbers in the Neighborhood of $\aleph_{\mathcal{X}}$
$\widehat{\mathbb{N}}$	Extended Naturals in the Maximal Neighborhood of Infinity
$\mathbb{N}_{\infty}$	$\mathbb{N} \cup \{\mathbb{N}_{\mathcal{X}}\} \cup \widehat{\mathbb{N}}$
E^x	Big Exponential Function
$\mathbb{F}$	Immeasurable or Non-Arithmatic Real Numbers
$\mathbb{F}_{\aleph}$	Cantor-Like Set $[0, \infty] \setminus \mathbb{R}_{\aleph}^{\cup}$
$\mathbb{F}_0$	Cantor-Like Set $[0, \infty] \setminus \mathbb{R}_0^{\cup}$
$\mathbb{F}_{\aleph}(n)$	n^{th} Leftmost Connected Subset of $\mathbb{F}_{\aleph}$
$\mathbb{F}_0(n)$	n^{th} Leftmost Connected Subset of $\mathbb{F}_0$
$\mathbb{F}_{\aleph}^{\mathcal{X}}$	Set Right-Adjacent to $\mathbb{R}_{\aleph}^{\mathcal{X}}$
$\mathbb{F}_0^{\mathcal{X}}$	Set Right-Adjacent to $\mathbb{R}_0^{\mathcal{X}}$
$\mathcal{F}(n)$	n^{th} Least Element of $\mathbb{F}_0$
$\mathcal{F}_{\mathcal{X}}$	Supremum of $\mathbb{R}_0^{\mathcal{X}}$
$\hat{T}$	Translation Operator
(L, R)	Dedekind Cut
S	Set
$\mathcal{T}$	Topology on a Set
$\mathcal{B}$	Basis for a Topology
$(S, \mathcal{T})$	Topological Space
$\mathbb{C}_0$	Complex Numbers in the Neighborhood of the Origin
$\mathbb{C}^{\cup}$	Arithmatic Subset of $\mathbb{C}$
$\mathfrak{D}(x)$	Dirichlet Series
$\zeta(z)$	Riemann Zeta Function

§1 Introduction

The original Euclidean definition of a real number [1] has given way over time to newer constructive definitions such as the Cauchy equivalence class suggested by Cantor [2], the Dedekind cut [3], and also axiomatic definitions, the most popular of which are the axioms of a complete ordered field based in Hilbert's axioms of geometry [4]. The main purpose of the present analysis is to compare and contrast geometric and algebraic constructions of the real numbers, and then to give a hybrid constructive-axiomatic definition which increases the mutual complements among the two notions of geometry and algebra.

The Euclid definition of $\mathbb{R}$ has its foundation in physical measurement. Throughout most of the history of mathematics, it was sufficient to give the Euclidean geometric conception of numbers as cuts in an infinite line, or "magnitudes" as Euclid is usually translated [1]. In modernity, the preoccupation of mathematics with algebra more so than quantity has stimulated the development of alternatives which are said to be "more rigorous" than Euclid. The main development of the present fractional distance analysis will be to present an alternative set of algebraic constructions and axioms which more thoroughly preserve the geometric notion that a number is a cut in an infinite line. We will show that Cantor's definition of $\mathbb{R}$ as the set of all Cauchy equivalence classes of rationals leaves something to be desired with respect to the underlying conception of $\mathbb{R}$ as an open-ended, infinite line $(-\infty, \infty)$. Namely, the equivalence class construction of $\mathbb{R}$, which is based on an assumed set of rational numbers $\mathbb{Q}$, precludes the existence of a neighborhood of infinity distinct from any neighborhood of the origin (outlined below), as does the similar Dedekind cut.

For a finite interval $x' \in [0, \frac{\pi}{2})$, we may use $x = \tan(x')$ to construct the interval $x \in [0, \infty)$ wherein everything is usually considered to be a real number. We will develop the notion of fractional distance to prove that if there exists a number at the Euclidean midpoint $x' = \frac{\pi}{4}$ of $[0, \frac{\pi}{2})$, then the bijectivity of the tangent function on $[0, \frac{\pi}{2})$ should require a real number at the Euclidean midpoint of $[0, \infty)$. A proof (Theorem 3.2.2) that there must exist such a number is the linchpin of everything in this analysis. Indeed, since Euler himself used this number [5–7], calling it $\frac{i}{2}$ in his own work, the fractional distance approach to $\mathbb{R}$ presented here should be considered ***a return to the old rather than a proposition for something new***. Such a number as $\frac{i}{2}$ will be said to be a number in the neighborhood of infinity because it will have non-zero "fractional distance" with respect to infinity. In that regard, we will say that every number having zero fractional distance with respect to infinity is a number in the neighborhood of the origin. We will show that the existence of the neighborhood of infinity is required to preserve Euclid's conception of a number as a cut in an infinite line. We will argue that any construction which preserves the concept of real numbers as cuts in an infinite line is necessarily

better than one which overwrites that concept.

Treatment of the neighborhood of infinity as a distinct numerical mode with separate behavior from the neighborhood of the origin is the direct motivator for everything new reported here. We will posit one very modest change to Cantor's Cauchy equivalence class construction such that it will more fully preserve the favorable notion that $\mathbb{R} = (-\infty, \infty)$. This notion is equivalent to granting that $\mathbb{R}$ has *the usual topology*. The modified equivalence class construction will give formal constructions for real numbers in the neighborhood of infinity rather than preclude their existence. With our new constructions and axioms given, we will present an analysis of $\mathbb{R}$ yielding unexpected properties which are non-trivial and exciting, and then we will give the formal topology.

In earlier work [8,9], we have demonstrated the existence of real numbers in the neighborhood of infinity, and we will do so again for the present analysis. Exceeding the previous work, here we will construct such numbers more or less directly from $\mathbb{Q}$. Then we will axiomatize the arithmetic of such numbers and study the consequences which follow.

The structure is as follows.

- Section Two: We give a simple Euclidean definition for real numbers. These geometric considerations set the stage for the algebraic considerations which follow.
- Section Three: We define and analyze a set of functions called fractional distance functions. These functions constitute the kernel of the analytical direction of the present work.
- Section Four: We give the properties of real numbers in the neighborhood of infinity. The ***formal algebraic construction*** of such numbers by Cauchy sequences is given therein.
- Section Five: We axiomatize a set of arithmetic operations for $\mathbb{R}$ and make a comparison with the similar field axioms. We find they are mostly the same but slightly different.
- Section Six: We prove some results with the present arithmetic axioms. Interestingly, we develop a technique by which it is possible to take a limit at infinity with the ordinary Cauchy prescription for limits: something that has been considered heretofore impossible.
- Section Seven: This section is dedicated most specifically to the topological and generally set theoretical properties of the real number line. The main thrust is to define a Cantor-like set on $\mathbb{R}$ and then to examine its consequences for the least upper bound property of connected sets.
- Section Eight: We apply the notions and consequences of fractional distance to the Riemann hypothesis. We show that the Riemann ζ function *does* have non-trivial zeros off the critical line.

§2 Mathematical Preliminary

§2.1 Real Numbers

In this section, the reader is invited to recall the distinction between the real numbers $\mathbb{R}$ and the real ordered number field $\mathcal{R} = \{\mathbb{R}, +, \times, \leq\}$: real numbers exist independently of their operations. Here, we define real numbers as cuts in the real number line pending a more formal, complementary definition by Cauchy sequences in Section 4, and by Dedekind cuts in Section 7. By defining a line, giving it a label "real," defining cuts in a line, and then defining real numbers as cuts in the real number line, we make a rigorous definition of real numbers sufficient for applications at any level of rigor. Specifically, the definition given in this section underpins the Cauchy and Dedekind definitions given later.

Generally, the definition of real numbers given in the present section is equivalent to the Euclidean magnitude defined in Euclid's *Elements*. Fitzpatrick, the translator of Euclid's original Greek in Reference [1], points out that Euclid's analysis was deliberately restricted to that which may be measured with a physical compass and straight edge: what are called the constructible numbers. Euclid surely was well aware, however, that the real number line is of immeasurable, non-constructible length, and that non-constructible numbers exist. The main motivator for the new formalism presented here is that we would like to consider both measurable and immeasurable magnitudes, or constructible and non-constructible numbers, exceeding those which can be defined in the canonical Cauchy and Dedekind approaches [2, 3].

Definition 2.1.1 A line is a 1D Hausdorff space parameterizable by the identity map on an unbounded scalar. The interval representation of a line is $(-\infty, \infty)$. In other words, the connected interval $(-\infty, \infty)$ is an infinite line.

Definition 2.1.2 A number line is a line equipped with a chart x and the Euclidean metric

$$d(x, y) = \big|y - x\big| \quad .$$

Definition 2.1.3 The real number line is a number line given the label "real."

Definition 2.1.4 If x is a cut in a line, then

$$(-\infty, \infty) = (-\infty, x] \cup (x, \infty) \quad .$$

Definition 2.1.5 A real number $x \in \mathbb{R}$ is a cut in the real number line.

Axiom 2.1.6 Real numbers are such that

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x \neq y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad \big|y - x\big| > \frac{1}{n} \quad .$$

Neither infinitesimals nor numbers having infinitesimal parts are real numbers.

Axiom 2.1.7 Real numbers are represented in algebraic interval notation as

$$\mathbb{R} = (-\infty, \infty) \quad .$$

In other words, $x \in \mathbb{R}$ if x is both less than infinity and greater than minus infinity. The connectedness of $\mathbb{R}$ is explicit in the interval notation.

Remark 2.1.8 In Section 4.2, we will supplement Axiom 2.1.7 by giving a definition in terms of Cauchy equivalence classes. Axiom 2.1.7 is often considered as lacking sufficient rigor, but the Cauchy definition will remedy any so-called insufficiencies of the broad generality of Axiom 2.1.7.

Definition 2.1.9 $\mathbb{R}_0$ is a subset of all real numbers

$$\mathbb{R}_0 = \left\{ x \in \mathbb{R} \;\middle|\; (\exists n \in \mathbb{N})[-n < x < n] \right\} \quad .$$

Here, we define $\mathbb{R}_0$ as the set of all $x \in \mathbb{R}$ such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$. We call this the set of real numbers less than some natural number (where absolute value is implied.) These numbers are said to lie within the neighborhood of the origin.

Definition 2.1.10 $\mathbb{R}_\infty$ is a subset of all real numbers with the property

$$\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0 \quad .$$

§2.2 Affinely Extended Real Numbers

To prove in Section 3.2 that $\mathbb{R}_\infty$ is not the empty set, namely that there are real numbers larger than every natural number, we will make reference to "line segments" beyond the simpler construction called "a line." Most generally, a line with two different endpoints A and B is a called a line segment AB. We will use notation such that $AB \equiv [a, b]$ where $[a, b]$ is an interval of numbers. Nowhere will we require that the endpoints must be real numbers so the interval $[a, b] = [-\infty, \infty]$ will conform to the definition of a line segment. The real line $\mathbb{R}$ together with two endpoints $\{\pm\infty\}$ is called the affinely extended real number line $\overline{\mathbb{R}} = [-\infty, \infty]$. The present section lays the foundation for an analysis of general line segments in Section 2.3 by first giving some properties of $\overline{\mathbb{R}}$.

Definition 2.2.1 For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have the properties

$$\lim_{x \to 0^{\pm}} \frac{1}{x} = \text{diverges} \quad , \quad \text{and} \quad \lim_{n \to \infty} \sum_{k=1}^{n} k = \text{diverges} \quad .$$

Definition 2.2.2 Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$, we have the properties

$$\lim_{x \to 0^{\pm}} \frac{1}{x} = \pm\infty \quad , \quad \text{and} \qquad \lim_{n \to \infty} \sum_{k=1}^{n} k = \infty \quad .$$

The limit as x approaches zero shall be referred to as "the limit definition of infinity." We avoid the appearance of self-reference noting that $n \to \infty$ means "as n increases without bound."

Axiom 2.2.3 The infinite element ∞ is such that

$$\infty - \infty = \text{undefined} \quad , \quad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Definition 2.2.4 The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} \quad .$$

This set is defined in interval notation as

$$\overline{\mathbb{R}} = [-\infty, \infty] \quad .$$

Remark 2.2.5 If $x_n > 0$ with $\{x_n\}$ being a monotonic sequence, the ∞ symbol is such that if $x_n \in \mathbb{R}$ and

$$\lim_{n \to \infty} x_n = \text{diverges} \quad ,$$

then for the same $x_n \in \overline{\mathbb{R}}$ we have

$$\lim_{n \to \infty} x_n = \infty \quad .$$

Definition 2.2.6 An affinely extended real number $x \in \overline{\mathbb{R}}$ is $\pm\infty$, or it is a cut in the affinely extended real number line:

$$[-\infty, \infty] = [-\infty, x] \cup (x, \infty] \quad .$$

Theorem 2.2.7 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 2.2.4.

§2.3 Line Segments

In this section, we review what is commonly understood regarding Euclidean line segments [1]. We begin to develop the relationship between points in a line segment and cuts in a line. During the analyses which follow in the

remainder of this work, we will closely examine the differences between cuts and points as a proxy for the fundamental relationship between algebra and geometry. Section 3.3 is dedicated specifically to these distinctions though they are treated throughout this text. The general principle of the distinction between cuts and points is the following. If x is a cut in a line, then

$$(-\infty, \infty) = (-\infty, x] \cup (x, \infty) \quad .$$

If x is a point in a line, then we have a tentative, preliminary understanding that

$$(-\infty, \infty) = (-\infty, x) \cup \{x\} \cup (x, \infty) \quad .$$

Definition 2.3.1 A line segment AB is a line together with two different endpoints $A \neq B$.

Definition 2.3.2 AB is a real line segment if and only if the endpoints A and B bound some subset of the real line $\mathbb{R} = (-\infty, \infty)$.

Definition 2.3.3 Much of the analysis presented here will depend on relationships between geometric and algebraic expressions. The $\equiv$ symbol will be used to denote symbolic equality between geometric and algebraic expressions.

Axiom 2.3.4 A real line segment AB is represented in interval notation as $AB \equiv [a, b]$ where a and b are any two affinely extended real numbers $a, b \in \overline{\mathbb{R}}$ such that $a < b$.

Definition 2.3.5 The Euclidean notation AB is called the geometric representation of a line segment. The interval notation $[a, b]$ is called the algebraic representation of a line segment.

Definition 2.3.6 The formal meaning of the relation $AB \equiv [a, b]$ is that a is the least number in the algebraic representation of A, b is the greatest number in the algebraic representation of B, and that every other number x in the algebraic representation of any point in AB has the property $a < x < b$.

Axiom 2.3.7 If AB and AC are two colinear line segments such that A is not between B and C, then

$$AB = AC \quad \iff \quad B = C \quad .$$

Axiom 2.3.8 Two line segments AB and CD are equal, meaning $AB = CD$, if and only if

$$\frac{AB}{CD} = \frac{CD}{AB} = 1 \quad .$$

Definition 2.3.9 **AB** is a special label given to the unique real line segment $AB \equiv [0, \infty]$. We have

$$AB = \mathbf{AB} \quad \Longleftrightarrow \quad AB \equiv [0, \infty] \ .$$

Definition 2.3.10 X is an interior point of AB if and only if

$$X \neq A \ , \ X \neq B \ , \quad \text{and} \quad X \in AB \ .$$

Axiom 2.3.11 If X is an interior point of AB, then

$$AB = AX + XB \ .$$

Axiom 2.3.12 Every geometric point X along a real line segment AB has one and only one algebraic interval representation $\mathscr{X}$. If $\mathscr{X}$ is the algebraic representation of X, then $X \equiv \mathscr{X}$, and $\mathscr{X}$ is a unique subset of $[a, b] \equiv AB$.

Remark 2.3.13 Axioms 2.3.4 and 2.3.12 establish that line segments and points in line segments have algebraic representations

$$AB \equiv [a, b] \ , \quad \text{and} \quad AB \ni X \equiv \mathscr{X} \subset [a, b] \ .$$

It will be a main result of the fractional distance analysis to show that the infinite length of a line segment such as $\mathbf{AB} \equiv [0, \infty]$ will allow us to put more than one number into the algebraic representation $\mathscr{X}$ of a geometric point X. If a line segment has finite length $L \in \mathbb{R}_0$, we will show that there is at most one real number in the algebraic representation of one of its interior points. However, this constraint will vanish in certain cases of $\text{len}(AB)$.

Definition 2.3.14 The algebraic representation $\mathscr{X}$ of a geometric point X lying along a real line segment AB is

$$\mathscr{X} = [x_1, x_2] \ , \quad \text{where} \quad x_1, x_2 \in \overline{\mathbb{R}} \ .$$

The special (intuitive) case of $x_1 = x_2 = x$ gives

$$\mathscr{X} = [x, x] = \{x\} = x \ .$$

Here, we have expressed $\mathscr{X}$ with included endpoints x_1 and x_2. Most generally, however, an algebraic representation of a geometric point is a single number, or it is some interval of numbers, *i.e.*: all variations of (x_1, x_2), $(x_1, x_2]$, and $[x_1, x_2)$ are allowable algebraic representations of X. We do not require that $x_1 \neq x_2$ in all cases.

Remark 2.3.15 A point in a line segment has a representation as a set of numbers, possibly only one number, and it remains to identify the exact relationship between numbers (cuts) and geometric points. The key feature of

Definition 2.3.14 is that it allows, provisionally, a many-to-one relationship between cuts in lines (algebraic) and points in line segments (geometric). In Section 3.3, we will strictly prove that which has been suggested: the algebraic representation of $X \in AB$ is only constrained to be a unique real number for certain cases of AB with finite length.

Definition 2.3.16 If $X \equiv \mathscr{X} = [x_1, x_2]$ with $x_1 \neq x_2$, and if $x \in [x_1, x_2]$, then x is said to be a *possible* algebraic representation of X. If $x_1 = x_2 = x$, then x is said to be *the* algebraic representation of X. If x is the algebraic representation of X, then $x \equiv X$. If x is a possible representation of X, then $x \in X$, *i.e.*: if x is a possible algebraic representation of X, then

$$x \in \mathscr{X} = [x_1, x_2] \equiv X \quad .$$

This statement may be abbreviated as $x \in X$, while $x \equiv X$ specifies the case of $x_1 = x_2$. We will also write $x \in X$ when it is uncertain if x_1 is equal to x_2.

Definition 2.3.17 A point C is called a midpoint of a line segment AB if and only if

$$\frac{AC}{AB} = \frac{CB}{AB} = \frac{1}{2} \quad .$$

Alternatively, C is a midpoint of AB if and only if

$$AC = CB \quad , \quad \text{and} \qquad AC + CB = AB \quad .$$

Definition 2.3.18 Hilbert's discarded axiom [4] states the following: any four points $\{A, B, C, D\}$ of a line can always be labeled so that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.

Remark 2.3.19 Hilbert's discarded axiom is discarded not because it wrong, but rather because it is implicit in Hilbert's other axioms [4]. It is discarded by redundancy rather than invalidity.

Theorem 2.3.20 *All line segments have at least one midpoint.*

Proof. Let there be a line segment AB and two circles of equal radii centered on the points A and B. Let the radii be less than AB but great enough such that the circles intersect at exactly two points S and T. The geometric configuration shown in Figure 1 is guaranteed to exist by Hilbert's discarded axiom pertaining to $\{A, X_1, X_2, B\}$. By construction, it follows that

$$AS = AT = BS = BT \quad .$$

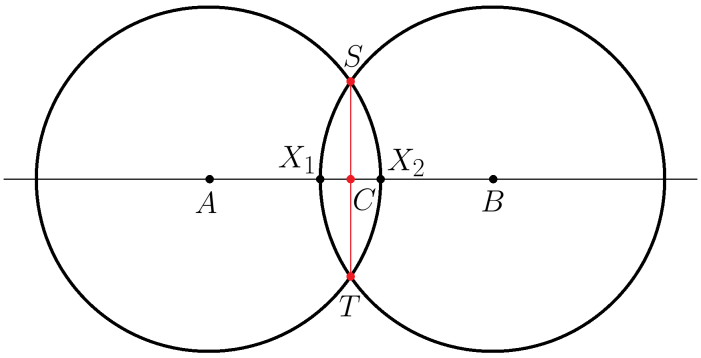

Figure 1: This figure proves that every line segment AB has one and only one midpoint.

Let the line segment ST intersect AB at C. By the Pythagorean theorem, C is a midpoint of AB because

$$AC^2 + CS^2 = AS^2 \quad , \quad \text{and} \qquad BC^2 + CS^2 = BS^2 \quad ,$$

together yield

$$AC = BC \quad .$$

C separates AB into two line segments so

$$AC + CB = AB \quad .$$

These two conditions, $AC = BC$ and $AC + CB = AB$, jointly conform to Definition 2.3.17, so C is a midpoint of an arbitrary line segment AB.

Example 2.3.21 Theorem 2.3.20 regards an arbitrary line segment AB. Therefore, the theorem holds in the case of an arbitrary line segment AB. One might be afflicted, however, with the assumption that it is not possible to define two such intersecting circles centered on the endpoints of an arbitrary line segment such as $\mathbf{AB} \equiv [0, \infty]$. To demonstrate how the arbitrary case of any line segment AB covers the specific case of $\mathbf{AB}$, suppose $AB \equiv [0, \frac{\pi}{2}]$, and let $x' \in \mathscr{X}$ be a number in the algebraic representation of $X \in AB$. We say that $[0, \frac{\pi}{2}]$ is the algebraic representation of AB charted in x'. Let x be such that

$$x = \tan(x') \quad ,$$

so that x and x' are two charts related by a conformal transformation. Using

$$\tan(0) = 0 \quad , \quad \text{and} \qquad \tan\left(\frac{\pi}{2}\right) = \infty \quad ,$$

where the latter follows from Definition 2.2.2, it follows that $[0, \infty]$ is the algebraic representation of AB charted in x. Therefore, $AB = \mathbf{AB}$ with respect to the x chart.

Hilbert's discarded axiom guarantees the existence of two points $X_1 \in AB$ and $X_2 \in AB$ with algebraic representations $\mathscr{X}_1'$ and $\mathscr{X}_2'$ such that, for example,

$$x_1' = \frac{\pi}{6} \in \mathscr{X}_1' \quad , \quad \text{and} \qquad x_2' = \frac{\pi}{3} \in \mathscr{X}_2' \quad .$$

If the radius of the circle centered on A is AX_2 and the radius of the circle centered on B is BX_1, then it is guaranteed that these circles will intersect at two points S and T, as in Figure 1. Since $AB = \mathbf{AB}$ in the x chart, it is required that $X_1 \in \mathbf{AB}$ and $X_2 \in \mathbf{AB}$. Therefore, circles centered on the endpoints of $\mathbf{AB}$ with radii AX_2 and BX_1 will intersect at exactly two points. ***The chart on the line segment cannot affect the line segment's basic geometric properties!*** It is unquestionable that the points X_1 and X_2 exist and are well-defined in the x' chart, and it is not possible to disrupt the geometric configuration by introducing a second chart onto AB. A chart can no more disrupt the geometric configuration than erasing an island from a map might make the physical island disappear from the sea. X_1 and X_2 do not cease to exist simply because we define a conformal chart $x = \tan(x')$. If they ceased to exist, then that would violate Hilbert's discarded axiom. This example demonstrates that Theorem 2.3.20 is valid even for the specific case of the infinite line segment $AB = \mathbf{AB}$.

Theorem 2.3.22 *All line segments have one and only one midpoint.*

Proof. For proof by contradiction, suppose C and D are two different midpoints of a line segment AB. C and D are midpoints of AB so we may derive from Definition 2.3.17

$$AC = CB = \frac{AB}{2} \quad , \quad \text{and} \qquad AD = DB = \frac{AB}{2} \quad .$$

It follows that $AC = AD$. By Axiom 2.3.7, therefore, $C = D$, and we invoke a contradiction having assumed that C and D are different. ∎

§3 Fractional Distance

§3.1 Fractional Distance Functions

If there are two circles with equal radii whose centers are separated by an infinite distance, then what numerical radii less than infinity will allow the circles to intersect at exactly two points? To answer this question, we will introduce fractional distance functions. We will use these functions to demonstrate the existence of real numbers in the neighborhood of infinity.

Definition 3.1.1 For any point X on a real line segment AB, the geometric fractional distance function $\mathcal{D}_{AB}$ is a continuous bijective map

$$\mathcal{D}_{AB}(AX) : AB \to [0, 1] \quad ,$$

which takes $AX \subseteq AB$ and returns real numbers. This function returns AX as a fraction of AB. Emphasizing the geometric construction, the geometric fractional distance function $\mathcal{D}_{AB}$ is defined as

$$\mathcal{D}_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{AX}{AB} & \text{for} \quad X \neq A,\ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

The quotient of two real line segments is defined as a real number.

Remark 3.1.2 The domain of $\mathcal{D}_{AB}(AX)$ is defined as subsets of real line segments. This allows $AX = AA$ which would be excluded from a domain of real line segments because AA does not have two different endpoints.

Theorem 3.1.3 *For any point $X \in AB$, the bijective geometric fractional distance function $\mathcal{D}_{AB}(AX) : AB \to R$ has range $R = [0, 1]$.*

Proof. Assume $\mathcal{D}_{AB}(AX) < 0$. Then one of the lengths in the fraction must be negative and we invoke a contradiction with the length of a line segment defined as a positive number (Definition 2.1.2). If $\mathcal{D}_{AB}(AX) > 1$, then $AX > AB$, and we invoke a contradiction by the implication $AX \nsubseteq AB$. We have excluded from R all numbers less than zero and greater than one. Since $\mathcal{D}_{AB}(AX)$ is a continuous function taking the values zero and one at the endpoints of its domain, the intermediate value theorem requires that the range of $\mathcal{D}_{AB}(AX) : AB \to R$ is $R = [0, 1]$.

Corollary 3.1.4 *All line segments have at least one midpoint.*

Proof. (Reproof of Theorem 2.3.20.) $\mathcal{D}_{AB}(AX)$ is a continuous function on the domain AB taking finite values zero and one at the endpoints of its domain. By the intermediate value theorem, there exists a point C in the domain AB for which $\mathcal{D}_{AB}(AC) = 0.5$. By Definition 2.3.17, C is a midpoint of AB.

Theorem 3.1.5 *Every midpoint of a line segment AB is an interior point of AB.*

Proof. If $X \in AB$ is not an interior point of AB, then $X = A$ or $X = B$. In each case respectively, the geometric fractional distance function returns

$$\mathcal{D}_{AB}(AA) = 0 \quad , \quad \text{or} \qquad \mathcal{D}_{AB}(AB) = 1 \quad .$$

A point C is a midpoint of AB if and only if

$$\mathcal{D}_{AB}(AC) = 0.5 \quad .$$

No midpoint can be an endpoint.

Remark 3.1.6 Given the geometric fractional distance function, it is not clear how to compute $\mathcal{D}_{AB}(AX)$ when X is an arbitrary interior point. By Definition 3.1.1, we know that the fraction $\frac{AX}{AB}$ is a real number but we have not yet developed any tools for finding the numerical value. The quotient notation required for computing fractional distance calls for an algebraic notion of distance.

Definition 3.1.7 $\mathcal{D}^{\dagger}_{AB}$ is the algebraic fractional distance function. It is an algebraic expression which totally replicates the behavior of the geometric fractional distance function $\mathcal{D}_{AB}$ on an arbitrary line segment $AB \equiv [a, b]$, *and* it has the added property that its numerical output is easily simplified. The algebraic fractional distance function $\mathcal{D}^{\dagger}_{AB}$ is constrained to be such that

$$\mathcal{D}^{\dagger}_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

for every point $X \in AB$.

Remark 3.1.8 In Definitions 3.1.9 and 3.1.11, we will define two kinds of algebraic fractional distance functions (FDFs). The purpose in defining two kinds of FDFs will be so that we may compare their properties, and then choose the one that exactly replicates the behavior of the geometric FDF $\mathcal{D}_{AB}$.

Definition 3.1.9 The algebraic FDF of the first kind

$$\mathcal{D}'_{AB}(AX) : AB \to [0, 1] \quad ,$$

is a map on subsets of real line segments

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} & \text{for} \quad X \neq A,\ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad ,$$

where

$$\frac{\|AX\|}{\|AB\|} = \frac{\text{len}[a, x]}{\text{len}[a, b]} \quad ,$$

and $[a, x]$ and $[a, b]$ are the line segments AX and AB expressed in interval notation.

Definition 3.1.10 The norm $\|AX\| = \text{len}[a, x]$ which appears in $\mathcal{D}'_{AB}(AX)$ is defined so that

$$\mathcal{D}'_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

Specifically, the length function is defined as the Euclidean distance between the endpoints of the algebraic representation. Per Definition 2.1.2, we have

$$\text{len}[a,b] = d(a,b) = \left|b-a\right| \quad .$$

Definition 3.1.11 The algebraic fractional distance function of the second kind

$$\mathcal{D}''_{AB}(AX) : [a,b] \to [0,1] \quad ,$$

is a map on intervals of the form

$$\mathcal{D}''_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\text{len}[a,x]}{\text{len}[a,b]} & \text{for} \quad X \neq A,\ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Remark 3.1.12 Take note of the main difference between the two algebraic FDFs. The first kind has a geometric domain

$$\mathcal{D}'_{AB}(AX) : AB \to \mathbb{R} \quad ,$$

but the second kind has an algebraic domain

$$\mathcal{D}''_{AB}(AX) : [a,b] \to \mathbb{R} \quad .$$

As a matter of consistency of notation, we have written $\mathcal{D}''_{AB}(AX)$ even when the notation $\mathcal{D}''_{AB}([a,x])$ might better illustrate that the domain of $\mathcal{D}''_{AB}$ is intervals rather than line segments. The reader is so advised.

Axiom 3.1.13 The ordering of $\mathbb{R}$ is such that for any $x, y \in \mathbb{R}$, if

$$x \in [x_1,x_2] = \mathscr{X} \equiv X \quad , \quad \text{and} \quad y \in [y_1,y_2] = \mathscr{Y} \equiv Y \quad ,$$

then

$$\mathcal{D}_{AB}(AX) > \mathcal{D}_{AB}(AY) \quad \Longrightarrow \quad x > y \quad .$$

Theorem 3.1.14 *The geometric fractional distance function $\mathcal{D}_{AB}$ is injective (one-to-one) on all real line segments.*

Proof. By Definition 3.1.1, the geometric FDF is

$$\mathcal{D}_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{AX}{AB} & \text{for} \quad X \neq A,\ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

For proof by contradiction, assume $\mathcal{D}_{AB}$ is not always injective. Then there exists some $X_1 \neq X_2$ such that

$$\frac{AX_1}{AB} = \frac{AX_2}{AB} \quad .$$

The range of $\mathcal{D}_{AB}$ is $[0, 1]$, and it is known that all such $0 \leq x \leq 1$ have an additive inverse element. This allows us to write

$$0 = \frac{AX_2}{AB} - \frac{AX_1}{AB} = \frac{AX_2 - AX_1}{AB} \quad \iff \quad AX_2 = AX_1 \quad .$$

Axiom 2.3.7 gives $AX = AY$ if and only if $X = Y$, so the implication $X_1 = X_2$ contradicts the assumed condition $X_1 \neq X_2$. The geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is injective on all real line segments. ∎

Remark 3.1.15 In Theorem 3.1.14, we have not considered specifically the case in which AB is a line segment of infinite length. There are many numbers x_1 and x_2 such that zero being equal to their difference divided by infinity does not imply that $x_1 = x_2$, *e.g.*:

$$0 = \frac{5 - 3}{\infty} \quad \not\iff \quad 5 = 3 \quad . \tag{3.1}$$

However, $\mathcal{D}_{AB}(AX)$ does not have numbers in its domain. The fraction in Equation (3.1) can never appear when computing $\frac{AX}{AB}$ because $\mathcal{D}_{AB}(AX)$ takes line segments or simply the point A (written as AA in abused line segment notation.)

To be clear, simplifying the expression $\mathcal{D}_{AB}(AX)$ in the general case requires some supplemental constraint like $AB = cAX$ for some scalar c. In the Euclidean program, constraints in this form might pertain to ratios of marks on a measuring stick. With a such a constraint in place, either algebraic or geometric, and by way of Axiom 2.3.8, we may evaluate the quotient as

$$\frac{AX}{AB} = \frac{cAB}{AB} = c \quad .$$

Without such auxiliary constraints, we have no general method for the evaluation of the quotient. Theorem 3.1.14 holds, however, because numbers such as the ∞ in the denominator of Equation (3.1) will be used only to compute $\mathcal{D}^{\dagger}_{AB}(AX)$ when we introduce the norm $\|AX\|$. The main feature distinguishing the algebraic FDF $\mathcal{D}^{\dagger}_{AB}$ from the geometric FDF $\mathcal{D}_{AB}$ is that the former allows us to compute the quotient in the general case with no requisite auxiliary constraints. Therefore, we might write $\mathcal{D}^{\dagger}_{AB}(AX; x)$ to show that is is a function of AX *and* a chart x on AB, or $\mathcal{D}^{\dagger}_{AB}([a, x]; x)$ as mentioned earlier. However, we will not write that explicitly. In the absence of words to the contrary, and if AB is a real line segment, then it should be assumed that the chart is the standard Euclidean coordinate.

Theorem 3.1.16 *The geometric fractional distance function $\mathcal{D}_{AB}$ is surjective (onto) on all real line segments.*

Proof. Given the range $R = [0, 1]$ proven in Theorem 3.1.3, proof follows from the notion of geometric fractional distance. ∎

Remark 3.1.17 Now that we have shown a few of the elementary properties of the geometric FDF, we will continue to do so and also examine the similar behaviors of the algebraic FDFs of the first and second kinds.

Conjecture 3.1.18 The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ is injective (one-to-one) on all real line segments. (This is proven in Theorem 6.1.4.)

Theorem 3.1.19 *The algebraic fractional distance function of the second kind $\mathcal{D}''_{AB}$ is not injective (one-to-one) on all real line segments.*

Proof. Recall that Definition 3.1.11 gives $\mathcal{D}''_{AB} : [a, b] \to [0, 1]$ as

$$\mathcal{D}''_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\text{len}[a, x]}{\text{len}[a, b]} & \text{for} \quad X \neq A, \ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Injectivity requires that

$$\mathcal{D}''_{AB}(AX) = \mathcal{D}''_{AB}(AY) \quad \iff \quad [a, x] = [a, y] \quad \iff \quad x = y \ .$$

Let $n, m \in \mathbb{N}$ be such that $n \neq m$ and also such that $n \in \mathscr{N} \equiv N$ and $m \in \mathscr{M} \equiv M$. We have

$$\mathcal{D}''_{\mathbf{AB}}(AN) = \frac{\text{len}[0, n]}{\text{len}[0, \infty]} = 0 \ , \quad \text{and} \qquad \mathcal{D}''_{\mathbf{AB}}(AM) = \frac{\text{len}[0, m]}{\text{len}[0, \infty]} = 0 \ .$$

Therefore, the algebraic FDF of the second kind is not injective on all real line segments because

$$\mathcal{D}''_{AB}(AN) = \mathcal{D}''_{AB}(AM) \quad \not\iff \quad n = m \ . \qquad \text{∎}$$

Remark 3.1.20 At this point, we can rule out $\mathcal{D}''_{AB}$ as the definition of $\mathcal{D}^{\dagger}_{AB}$ because the geometric FDF $\mathcal{D}_{AB}$ which constrains $\mathcal{D}^{\dagger}_{AB}$ is one-to-one. If $\mathcal{D}_{AB}$ is one-to-one on all real line segments, then so is $\mathcal{D}^{\dagger}_{AB}$.

Carefully note that the domain of the algebraic FDF of the first kind is line segments rather than algebraic intervals. We have

$$\mathcal{D}'_{AB}(AX) : AB \to [0, 1] \ , \quad \text{and} \qquad \mathcal{D}''_{AB}(AX) : [a, b] \to [0, 1] \ .$$

Taking for granted that we will prove the injectivity of $\mathcal{D}'_{AB}$ in Theorem 6.1.4, this distinction of domain—AB versus $[a, b]$—will prohibit the breakdown in the one-to-one property when a point $X \in AB$ can have many different numbers in its algebraic representation. An assumption that the domain of the algebraic FDF is an algebraic interval $[a, b]$ is likely a root cause of ***much pathology in modern analysis***.

Theorem 3.1.21 *The geometric fractional distance function $\mathcal{D}_{AB}$ is continuous everywhere on the domain* **AB**.

Proof. To prove that $\mathcal{D}_{AB}$ is continuous on $\mathbf{AB} \equiv [0, \infty]$, it will suffice to show that $\mathcal{D}_{\mathbf{AB}}$ is continuous at the endpoints and an interior point.

• (*Interior point*) A function $f(x)$ is continuous at an interior point x_0 of its domain $[a, b]$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad .$$

In terms of the geometric FDF, the statement that $\mathcal{D}_{\mathbf{AB}}$ is continuous at an interior point $X_0 \in \mathbf{AB}$ becomes

$$\lim_{X \to X_0} \mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{D}_{\mathbf{AB}}(AX_0) \quad .$$

Obviously, $\mathcal{D}_{\mathbf{AB}}$ satisfies the definition of continuity on the interior of **AB**.

• (*Endpoint A*) A function $f(x)$ is continuous at the endpoint a of its domain $[a, b]$ if and only if

$$\lim_{x \to a^+} f(x) = f(a) \quad .$$

We conform to this definition of continuity with

$$\lim_{X \to A^+} \mathcal{D}_{\mathbf{AB}}(AX) = \lim_{X \to A^+} \frac{AX}{\mathbf{AB}} = \frac{AA}{\mathbf{AB}} = \mathcal{D}_{\mathbf{AB}}(AA) \quad .$$

• (*Endpoint B*) A function $f(x)$ is continuous at the endpoint b of its domain $[a, b]$ if and only if

$$\lim_{x \to b^-} f(x) = f(b) \quad .$$

We conform to this definition with

$$\lim_{X \to B^-} \mathcal{D}_{\mathbf{AB}}(AX) = \lim_{X \to B^-} \frac{AX}{\mathbf{AB}} = \frac{\mathbf{AB}}{\mathbf{AB}} = \mathcal{D}_{\mathbf{AB}}(\mathbf{AB}) \quad .$$

The geometric FDF is continuous everywhere on its domain. ∎

Theorem 3.1.22 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ is not continuous everywhere on the domain* **AB**.

Proof. A function $f(x)$ with domain $x \in [a, b]$ is continuous at b if

$$\lim_{x \to b^-} f(x) = f(b) \quad .$$

In terms of $\mathcal{D}'_{AB}$, the statement that $\mathcal{D}'_{\mathbf{AB}}$ is continuous at B becomes

$$\lim_{X \to B} \mathcal{D}'_{\mathbf{AB}}(AX) = \mathcal{D}'_{\mathbf{AB}}(\mathbf{AB}) = 1 \quad .$$

Evaluation yields

$$\lim_{X \to B} \mathcal{D}'_{\mathbf{AB}}(AX) = \lim_{x \to \infty} \frac{\text{len}[0, x]}{\text{len}[0, \infty]} = \lim_{x \to \infty} x \frac{1}{\infty} = \lim_{x \to \infty} 0 \neq 1 = \mathcal{D}'_{\mathbf{AB}}(AB) \quad .$$

The algebraic FDF of the first kind is not continuous everywhere on all real line segments. ∎

Remark 3.1.23 In Theorem 3.1.22, we have shown that the limit approaches zero rather than the unit value required for $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AB)$ to agree with $\mathcal{D}_{\mathbf{AB}}(AB)$. However, we may also write this limit as

$$\lim_{x \to \infty} \frac{1}{\infty} x = \lim_{\substack{x \to \infty \\ y \to \infty}} \frac{x}{y} = \lim_{y \to \infty} \infty \frac{1}{y} = \frac{\infty}{\infty} = \text{undefined} \quad .$$

Perhaps, then, it would be better to write simply

$$\lim_{x \to \infty} \frac{x}{\infty} = \frac{\infty}{\infty} = \text{undefined} \neq 1 \quad .$$

In any case, we have shown that an elementary evaluation does not produce the correct limit at infinity. Therefore, we should also examine the Cauchy definition of the limit relying on the ε–δ formalism.

Theorem 3.1.24 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ does not converge to a Cauchy limit at infinity.*

Proof. According to the Cauchy definition of the limit of $f : D \to R$ at infinity, we say that

$$\lim_{x \to \infty} f(x) = l \quad ,$$

if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in D \quad ,$$

we have

$$0 < \left| x - \infty \right| < \delta \quad \Longrightarrow \quad \left| f(x) - l \right| < \varepsilon \quad .$$

Since ε and δ are implicitly real-valued, there is no $\delta > \infty$. Thus, $\mathcal{D}'_{AB}(AX)$ fails the Cauchy criterion for convergence to a limit at infinity. ∎

Remark 3.1.25 In general, the above Cauchy definition of a limit fails for any limit at infinity because there is never a δ greater than infinity. Usually, this issue is worked around with the metric space definition of a limit at infinity, but it is a ***main result of this analysis*** that we will develop a technique for taking a limit at infinity with the normal Cauchy prescription. This result appears in Section 6.1.

Remark 3.1.26 The algebraic FDF $\mathcal{D}^{\dagger}_{AB}$ exists by definition. It is a function which has every behavior of the geometric FDF $\mathcal{D}_{AB}$ and also adds the ability to compute numerical ratios between the lengths of any two real line segments. Numbers being generally within the domain of algebra, the geometric FDF returns a fraction that we have no general way to simplify. Since it is hard to conceive of an irreducible analytical form for the algebraic FDF other than $\mathcal{D}'_{AB}$ and $\mathcal{D}''_{AB}$, it is somewhat paradoxical that neither of them replicate the global behavior of the algebraic FDF $\mathcal{D}^{\dagger}_{AB}$. However, after developing some more material, we will show in Section 6.1 that $\mathcal{D}^{\dagger}_{AB}$ is $\mathcal{D}'_{AB}$ after all. We will prevent an unwarranted assumption about infinity from sneakily propagating into the present analysis. Then we will fix the discontinuity of $\mathcal{D}'_{AB}$ which we have demonstrated in Theorems 3.1.22 and 3.1.24.

Theorem 3.1.27 *If x is a real number in the algebraic representations of both $X \in AB$ and $Y \in AB$, then $X = Y$.*

Proof. If $X \neq Y$, then

$$\mathcal{D}^{\dagger}_{AB}(AX) \neq \mathcal{D}^{\dagger}_{AB}(AY) \quad .$$

If $x \in X$ and $x \in Y$, then it is possible to make cuts at X and Y such that

$$\mathcal{D}^{\dagger}_{AB}(AX) = \frac{\text{len}[a, x]}{\text{len}[a, b]} = \mathcal{D}^{\dagger}_{AB}(AY) \quad .$$

This contradiction requires $X = Y$. ∎

§3.2 Comparison of Real and Natural Numbers

The main result of this section is to prove via analysis of FDFs that there exist real numbers greater than any natural number. Consequently, $\mathbb{R}_{\infty} = \mathbb{R} \setminus \mathbb{R}_0$ cannot be the empty set.

Definition 3.2.1 Every interval has a number at its center. The number at the center of an interval $[a, b]$ is defined as the average of a and b if an average can be computed. Likewise, the number in the center is the unique number c such that

$$[a, b] = [a, c] \cup (c, b] \quad , \quad \text{and} \qquad \text{len}[a, c] = \text{len}[c, b] \quad .$$

Theorem 3.2.2 *There exists a unique real number halfway between zero and infinity.*

Proof. By Theorem 2.3.22 and Definition 2.3.17, there exists one midpoint C of every line segment AB such that

$$\mathcal{D}_{AB}(AC) = 0.5 \quad .$$

Recalling that we have defined $\mathcal{D}_{AB}(AX) = \mathcal{D}^{\dagger}_{AB}(AX)$ for all $X \in AB$, and recalling that $\mathbf{AB} \equiv [0, \infty]$, it follows that

$$\mathcal{D}^{\dagger}_{\mathbf{AB}}(AC) = 0.5 \quad .$$

Using $C \equiv \mathscr{C} = [c_1, c_2]$, Axiom 2.3.11 and Definition 2.3.14 require

$$\mathbf{AB} = AC + CB \quad \iff \quad [0, \infty] = [0, c_1) \cup \mathscr{C} \cup (c_2, \infty] \quad .$$

It follows that

$$\mathscr{C} \subset \mathbb{R} \quad .$$

Every possible number that can be in the algebraic representation of the point C is a real number. If $c_1 = c_2 = c$, then $c \in \mathbb{R}$ is the unique real number halfway between zero and infinity. If $c_1 \neq c_2$, then, by Definition 3.2.1, the number at the center of $[c_1, c_2]$ is the unique real number halfway between zero and infinity. ∎

Remark 3.2.3 How can $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AC) = 0.5$ when Definition 3.1.9 gives

$$\mathcal{D}'_{\mathbf{AB}}(AC) = \frac{\text{len}[0, c]}{\infty} \quad ?$$

The prevailing assumption about infinity is

$$x \in \mathbb{R} \quad \iff \quad \frac{x}{\infty} = 0 \quad . \tag{3.2}$$

If Equation (3.2) is true, then either (i) there exists a line segment without a midpoint, or (ii) the geometric and algebraic fractional distance functions do not agree for every X in an arbitrary AB.

Every line segment does have a midpoint (Theorem 2.3.22), and our fractional distance functions are defined to always agree (Definition 3.1.7). Therefore, Equation (3.2), which is a statement dependent on the assumed properties of ∞, must be reformulated. In Section 4.1, we will define notation for subsets of $\mathbb{R}$ consisting of all numbers having fractional distance $\mathcal{X}$ with respect to $\mathbf{AB}$, meaning that $\frac{x}{\infty} = \mathcal{X}$. The sets will be labeled $\mathbb{R}^{\mathcal{X}}_{\aleph}$ most generally with $0 < \mathcal{X} < 1$, but it will follow that $\mathbb{R}^{0}_{\aleph}$ is the set of all real numbers having zero fractional distance with respect to $\mathbf{AB}$. We know that $\mathbb{R}_0 \subset \mathbb{R}^{0}_{\aleph}$, but it shall remain to be determined whether or not there are real numbers greater

than any natural number yet still having zero fractional distance with respect to **AB**. In Section 7.4, we will closely examine whether or not such numbers ought to exist.

While we will postpone the definition of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ to Section 4.1, and while the formal construction of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ by equivalence classes of Cauchy sequences will not appear until Section 4.2, here we will go ahead and answer the question, "How can $\mathcal{D}_{\mathbf{AB}}^{\dagger}(AC) = 0.5$ when Definition 3.1.9 gives

$$\mathcal{D}'_{\mathbf{AB}}(AC) = \frac{\text{len}[0, c]}{\infty} \quad ?"$$

The answer is that Equation (3.2) must be reformulated as

$$x \in \mathbb{R}_{\aleph}^{0} \quad \iff \quad \frac{x}{\infty} = 0 \quad ,$$

if we are to avoid harsh contradictions in the definitions of our FDFs. Regarding Theorem 3.2.2 and the present question which follows, the real numbers in the algebraic representation of the geometric midpoint of **AB** shall be

$$x \in \mathbb{R}_{\aleph}^{0.5} \quad \iff \quad \frac{x}{\infty} = 0.5 \quad .$$

In addition to motivating the soon-to-be-defined $\mathbb{R}_{\aleph}^{\mathcal{X}}$ notation, the present remark illustrates the reasoning behind allowing geometric points to be represented as entire intervals $X \equiv \mathscr{X}$. The reason is that many real numbers divided by infinity give zero but only the geometric left endpoint of **AB** will have vanishing fractional distance. For instance, if $x \in \mathbb{R}_{\aleph}^{0.5}$ and n is a natural number having zero fractional magnitude with respect to infinity, then

$$\frac{x + n}{\infty} = \frac{x}{\infty} + \frac{n}{\infty} = 0.5 + 0 = 0.5 \quad .$$

Obviously, $x \in \mathbb{R}_{\aleph}^{0.5}$ is not a unique number though the midpoint C is a unique point.

Definition 3.2.4 If $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is the set of all numbers whose fractional distance with respect to **AB** is $\mathcal{X}$, and if $0 < \mathcal{X} < 1$, then $\aleph_{\mathcal{X}}$ is the number in the center of the interval representation of $\mathbb{R}_{\aleph}^{\mathcal{X}}$.

Remark 3.2.5 The reader is invited to recall that Euler often employed the letter i to refer to an infinitely large integer, and that Euler made use of the number $\frac{i}{2}$ for proofs in his most seminal works [5–7]. Therefore, we introduce nothing new with the $\aleph_{\mathcal{X}}$ notation because $\frac{i}{2} \sim \aleph_{0.5}$.

Main Theorem 3.2.6 *Some elements of* $\mathbb{R}$ *are greater than every element of* $\mathbb{N}$.

Proof. Let **AB** have a midpoint C so that $\mathcal{D}_{\mathbf{AB}}(AC) = 0.5$. Then every real number $c \in [c_1, c_2] \equiv C$ is greater than any $n \in \mathbb{N}$ because $\frac{n}{\infty} = 0$ implies $n \in \mathscr{A} \equiv A$ through the definition $\mathcal{D}_{\mathbf{AB}}(AA) = 0$. $\mathcal{D}_{\mathbf{AB}}$ is one-to-one, so, by Axiom 3.1.13 giving for $x \in X$ and $y \in Y$

$$\mathcal{D}_{AB}(AX) > \mathcal{D}_{AB}(AY) \quad \Longrightarrow \quad x > y \ ,$$

we find that every $c \in \mathscr{C} \subset \mathbb{R}$ is greater than every $n \in \mathbb{N}$. Generally, every $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ is greater than any natural number whenever $\mathcal{X} > 0$.

Corollary 3.2.7 $\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0$ *is not the empty set.*

Proof. Definition 2.1.9 defines $\mathbb{R}_0$ as the subset of $\mathbb{R}$ whose elements are less than some element of $\mathbb{N}$. We have proven in Main Theorem 3.2.6 that some elements of $\mathbb{R}$ are not in $\mathbb{R}_0$. It follows that $\mathbb{R}_\infty \neq \varnothing$.

§3.3 Comparison of Cuts in Lines and Points in Line Segments

In this section, we will make clarifications regarding the cases in which an interior point of a line segment can or cannot be uniquely identified with a single real number. Namely, we distinguish cases in which $X \equiv x$ and those in which $X \equiv \mathscr{X} = [x_1, x_2]$ with $x_1 \neq x_2$.

Theorem 3.3.1 *If AB is a real line segment of finite length $L \in \mathbb{R}_0$, then every point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. Let $a, b \in \mathbb{R}_0$ and $AB \equiv [a, b]$. The algebraic FDF $\mathcal{D}_{AB}^{\dagger}$ is defined to behave exactly as the geometric FDF $\mathcal{D}_{AB}$, so, therefore, $\mathcal{D}_{AB}^{\dagger}$ must be one-to-one (injective). By Definition 2.3.14, every point in a real line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \ .$$

Therefore, the present theorem will be proven if we show that $x_1 = x_2$ for all $X \in AB$ with $L \in \mathbb{R}_0$. To initiate proof by contradiction, assume $x_1, x_2 \in \mathbb{R}_0$ and $x_1 \neq x_2$. (The validity of this condition follows from $L \in \mathbb{R}_0$.) Then

$$\min[\mathcal{D}_{AB}^{\dagger}(AX)] = \frac{\text{len}[a, x_1]}{\text{len}[a, b]} = \frac{x_1 - a}{b - a} \ ,$$

and

$$\max[\mathcal{D}_{AB}^{\dagger}(AX)] = \frac{\text{len}[a, x_2]}{\text{len}[a, b]} = \frac{x_2 - a}{b - a} \ .$$

The one-to-one property of $\mathcal{D}_{AB}^{\dagger}$ requires that

$$\frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a} \quad \Longleftrightarrow \quad x_1 = x_2 \ .$$

This contradicts the assumption $x_1 \neq x_2$. The theorem is proven. 🍃

Theorem 3.3.2 *If AB is a real line segment of infinite length $L = \infty$, then no point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. By Definition 2.3.14, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

It follows that

$$\min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \infty]} = \frac{x_1}{\infty} \quad ,$$

Now suppose that $x_0 \in \mathbb{R}_0^+$ where the superscript "+" indicates the positive-definite subset. Further suppose $z = x_1 + x_0$ so that $z > x_1$. Then

$$\frac{\text{len}[0, z]}{\text{len}[0, \infty]} = \frac{z}{\infty} = \frac{x_1 + x_0}{\infty} = \frac{x_1}{\infty} = \min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] \quad .$$

Invoking the single-valuedness of bijective functions, we find that

$$\min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \max[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{x_2}{\infty} \quad \Longrightarrow \quad x_1 < z \leq x_2 \quad .$$

Therefore, $x_1 \neq x_2$, and the theorem is proven. 🍃

Example 3.3.3 This example illustrates some of the underlying machinations associated with the many-to-one relationship between numbers and points in an infinitely long line segment. If we separate an endpoint from a closed interval of numbers, we may write

$$[a, b] = \{a\} \cup (a, b] \quad .$$

To separate an endpoint from a line segment, we write

$$AB = A + AB \quad .$$

If A has an algebraic representation $\mathscr{A}$ such that $\text{len}(\mathscr{A}) > 0$, then the only way that we can leave the length of AB unchanged after removing A is for AB to have infinite length. Given $\text{len}(\mathscr{A}) > 0$, observe that

$$\|AB\| - \text{len}(\mathscr{A}) = \|AB\| \quad \Longleftrightarrow \quad \|AB\| = \infty \quad .$$

Remark 3.3.4 Theorems 3.3.1 and 3.3.2 do not cover all cases of $\text{len}(AB) = L$. For instance, four coarse bins of L are

- $L \in \mathbb{R}_0$
- $L \in \mathbb{R}^0_{\aleph} \setminus \mathbb{R}_0$ (L larger than any $n \in \mathbb{N}$ yet not so large that $\frac{L}{\infty} > 0$.)

- $L \in \mathbb{R}_\infty \setminus \mathbb{R}^0_\aleph$ (This is also written $L \in \{\mathbb{R}^{\mathcal{X}}_\aleph\} \cup \mathbb{R}^1_\aleph$ when $\{\mathbb{R}^{\mathcal{X}}_\aleph\}$ is the set of all $\mathbb{R}^{\mathcal{X}}_\aleph$ such that $0 < \mathcal{X} < 1$, as in Section 4.1.)
- $L = \infty$

We have not considered the two intermediate cases of finite L, the lesser of which is finite $L \in \mathbb{R}^0_\aleph \setminus \mathbb{R}_0$. Since we have not yet introduced any numbers with which to describe this case, and since we will not decide $\mathbb{R}^0_\aleph \setminus \mathbb{R}_0 = \varnothing$ until Section 7.4, we cannot at this time prove the result regarding the multi-valuedness of points in line segments having $L \in \mathbb{R}^0_\aleph \setminus \mathbb{R}_0$. The limit of the third case as $L \in \{\mathbb{R}^{\mathcal{X}}_0\} \cup \mathbb{R}^1_0$ is proven to be many-to-one in Theorem 6.2.1.

§4 The Neighborhood of Infinity

§4.1 Intermediate Neighborhoods of Infinity

In this section, we will develop notation useful for describing real numbers whose fractional magnitude with respect to infinity is greater than zero.

Definition 4.1.1 The number $\aleph_{\mathcal{X}}$ is defined to have the property

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \mathcal{X} \quad .$$

Equivalently, if $\aleph_{\mathcal{X}} \in \mathscr{X} \equiv X \in \mathbf{AB}$, then

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \quad .$$

Remark 4.1.2 We have shown in Theorem 3.3.2 that there are many real numbers in the algebraic representation of $X \in \mathbf{AB}$. When X is not an endpoint of $\mathbf{AB}$, $\aleph_{\mathcal{X}}$ can be thought of the as the number in the center of the interval $(x_1, x_2) = \mathscr{X} \equiv X$. Definition 3.2.1 defines the number in the center of $\mathscr{X}$ as the average of x_1 and x_2 if the average is computable, but, as yet, we have no way to determine the numbers that bound the algebraic representation of X. This also makes it impossible to compute the length function. However, we will determine and analyze the bounds of the algebraic representation of $X \in \mathbf{AB}$ in Section 7.5. For the special cases of $\aleph_0$ and $\aleph_1$, we should not think of them as being in the centers of the intervals $\mathscr{A} \equiv A$ and $\mathscr{B} \equiv B$. Instead, $\aleph_0$ is the least number in $\mathscr{A} \equiv A \in \mathbf{AB}$, and $\aleph_1$ is the greatest number in $\mathscr{B} \equiv B \in \mathbf{AB}$.

Definition 4.1.3 For $0 < \mathcal{X} < 1$, $\mathbb{R}^{\mathcal{X}}_\aleph$ is a subset of positive real numbers $\mathbb{R}^+$ such that

$$\mathbb{R}^{\mathcal{X}}_\aleph = \left\{ \aleph_{\mathcal{X}} + b \;\middle|\; |b| \in A \in \mathbf{AB},\ \mathcal{D}_{\mathbf{AB}}(AA) = 0 \right\} \quad .$$

The set $\mathbb{R}^{\mathcal{X}}_\aleph$ is called the whole neighborhood of $\aleph_{\mathcal{X}}$. The set $\{\mathbb{R}^{\mathcal{X}}_\aleph\}$ of all $\mathbb{R}^{\mathcal{X}}_\aleph$, meaning the union of $\mathbb{R}^{\mathcal{X}}_\aleph$ for every $0 < \mathcal{X} < 1$, is called the set of

all intermediate whole neighborhoods of $\mathbb{R}$. We will also call $\mathbb{R}_{\aleph}^{\mathcal{X}}$ the whole neighborhood of numbers that are $100 \times \mathcal{X}\%$ of the way down the real number line. (These conventions ignore the negative branch of $\mathbb{R}$.)

Definition 4.1.4 Given $0 < \mathcal{X} < 1$, it will also be useful to define a set $\mathbb{R}_0^{\mathcal{X}} \subseteq \mathbb{R}_{\aleph}^{\mathcal{X}}$ such that

$$\mathbb{R}_0^{\mathcal{X}} = \left\{ \aleph_{\mathcal{X}} + b \;\middle|\; b \in \mathbb{R}_0 \right\} \quad .$$

The set $\mathbb{R}_0^{\mathcal{X}}$ is called the natural neighborhood of $\aleph_{\mathcal{X}}$ because we have constrained b to be less than some $n \in \mathbb{N}$. $\{\mathbb{R}_0^{\mathcal{X}}\}$ is the union of $\mathbb{R}_0^{\mathcal{X}}$ for every $0 < \mathcal{X} < 1$.

Definition 4.1.5 Every number of the form $x = \aleph_{\mathcal{X}} + b$ has a big part $\aleph_{\mathcal{X}}$ and a little part b. It is understood that $b < \aleph_{\mathcal{X}}$ for any $\mathcal{X} > 0$. We define notations

$$\mathrm{Big}(\aleph_{\mathcal{X}} + b) = \aleph_{\mathcal{X}} \quad , \quad \text{and} \qquad \mathrm{Lit}(\aleph_{\mathcal{X}} + b) = b \quad .$$

Remark 4.1.6 We have omitted from Definitions 4.1.3 and 4.1.4 the cases of $\mathcal{X} = 0$ and $\mathcal{X} = 1$, though they do follow more or less directly. The main issue is that we must restrict the sign of b to keep the elements of the set within the totally real interval $[0, \infty) \subset \mathbb{R}$. The little part b is non-negative for $\mathcal{X} = 0$, and it is negative-definite for $\mathcal{X} = 1$.

The difference between the natural neighborhoods $\mathbb{R}_0^{\mathcal{X}}$ and the whole neighborhoods $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is that b is not restricted to $\mathbb{R}_0$ in the latter. In Definition 4.1.4, we did not give the condition on b in terms of the absolute value, as in Definition 4.1.3, because $\mathbb{R}_0$ contains negative numbers while $b \in A \in \mathbf{AB} \equiv [0, \infty]$ is strictly non-negative. The main purpose in defining distinct sets $\{\mathbb{R}_0^{\mathcal{X}}\}$ and $\{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$ is this: we know that there exist numbers larger than any $b \in \mathbb{R}_0$ (Main Theorem 3.2.6), but we do not know if all such numbers have greater than zero fractional magnitude with respect to $\mathbf{AB}$. We will revisit this issue in Section 7.4. In the meantime, we will be careful to treat $\mathbb{R}_0^{\mathcal{X}}$ and $\mathbb{R}_{\aleph}^{\mathcal{X}}$ as distinct sets which may or may not be equal.

Definition 4.1.7 The whole neighborhood of the origin is

$$\mathbb{R}_{\aleph}^0 = \left\{ x \;\middle|\; x \in \mathscr{A} \equiv A \in \mathbf{AB} \right\} \quad ,$$

and the natural neighborhood of the origin is

$$\mathbb{R}_0^0 = \left\{ x \;\middle|\; x \in \mathbb{R}_0,\ x \geq 0 \right\} \quad ,$$

Remark 4.1.8 Note that $\mathbb{R}_0 \nsubseteq \mathbb{R}_0^0 \subseteq \mathbb{R}_{\aleph}^0$ because $\mathbb{R}_0$ contains positive and negative numbers, as per Definition 2.1.9.

Definition 4.1.9 A real number x is said to be in the neighborhood of the origin if and only if

$$x \in X \quad , \quad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AX) = 0 \quad .$$

All such numbers are said to be $x \in \mathbb{R}^0_{\aleph}$. Every real number not in the neighborhood of the origin is said to in the neighborhood of infinity. A positive real number x is said to be in the neighborhood of infinity if and only if

$$x \in X \quad , \quad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AX) \neq 0 \quad .$$

Remark 4.1.10 Definition 2.1.10 states that $\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0$. Therefore, if $\mathbb{R}^0_{\aleph} \setminus \mathbb{R}^0_0 \neq \varnothing$, meaning that there do exist real numbers greater than any natural number yet not great enough to have non-zero fractional distance with respect to **AB**, then the set $\mathbb{R}_\infty$ will contain numbers in the neighborhood of the origin *and* numbers in the neighborhood of infinity. To avoid ambiguity, we will not use the symbol $\mathbb{R}_\infty$. Instead, we will mostly use the detailed $\mathbb{R}^{\mathcal{X}}_{\aleph}$ and $\mathbb{R}^{\mathcal{X}}_0$ set enumerations given in the present section. With this scheme of distinct whole and natural neighborhoods, we have left room judiciously for numbers in the neighborhood of the origin that are still larger than any natural number. In other work [8, 9], we have used the semantic convention that every number in the neighborhood of the origin is less than some natural number, and that meant that $\mathbb{R}_0$ was the set of all real numbers in the neighborhood of the origin. The present convention, however, is better suited to the fuller analysis given here. The reader should carefully note that the present neighborhood of the origin $\mathbb{R}^0_{\aleph}$ includes all numbers which have zero fractional distance along the real number line, even if some of those numbers are larger than any $n \in \mathbb{N}$.

Definition 4.1.11 The δ-neighborhood of a number $x \in \mathbb{R}$ is an interval $(x-\delta, x+\delta)$ or some closed or half-open permutation thereof. While there is no inherent constraint on the magnitude of δ, here we will take "δ-neighborhood" to imply $\delta \in \mathbb{R}_0$. We will use the convention that the Ball function defines an open δ-neighborhood as

$$\text{Ball}(x, \delta) = (x - \delta, x + \delta) \quad .$$

Definition 4.1.12 The δ-neighborhood of an interior point $X \in AB$ is a line segment YZ where

$$\left|\mathcal{D}_{AB}(AX) - \mathcal{D}_{AB}(AY)\right| = \left|\mathcal{D}_{AB}(AX) - \mathcal{D}_{AB}(AZ)\right| = \delta \quad .$$

Remark 4.1.13 Without regard to the δ-neighborhood of any point or number, we have defined neighborhoods of fractional distance with the geometric FDF, as in Definition 4.1.9. If $\mathcal{D}_{\mathbf{AB}}(AX) = 0$, then the numbers in the algebraic representation of X are said to be in the neighborhood of the origin.

They are said to be in the neighborhood of infinity otherwise. Neither of these neighborhoods, neither that of the origin nor that of infinity, are defined formally as δ-neighborhoods, though such a definition may be inferred. In advance of the following definition for $\mathbb{R}_{\aleph}^{\mathcal{X}}$ (Definition 4.1.14), recall that Definition 3.2.4 gave $\aleph_{\mathcal{X}}$ as the number in the center of the interval $\mathbb{R}_{\aleph}^{\mathcal{X}} = (a, b)$.

Definition 4.1.14 An alternative definition for $\mathbb{R}_{\aleph}^{\mathcal{X}}$ valid in the neighborhood of infinity, specifically for $0 < \mathcal{X} < 1$, is

$$\mathbb{R}_{\aleph}^{\mathcal{X}} = \left\{ \aleph_{\mathcal{X}} \pm b \;\middle|\; b \in \mathbb{R}_{\aleph}^{0} \right\} \quad .$$

This definition is totally equivalent to Definition 4.1.3.

§4.2 Equivalence Classes for Intermediate Natural Neighborhoods of Infinity

Euclid's definition of $\mathbb{R}$ is inherently a geometric one based on the measurement of quantity. The purpose of Cantor's definition by Cauchy equivalence classes [2,10–12] is to give an algebraic definition based on rationals. In this section, we will append the algebraic Cauchy definition to the Euclidean definition given in Section 2.1. This totally algebraic hybrid construction will not unduly exclude the neighborhood of infinity from $\mathbb{R}$. We have shown that if every number in the interval $(-\infty, \infty)$ is to be a real number, then there must exist numbers such as $\aleph_{0.5}$ which are greater than any natural number. In its ordinary incarnation, however, the Cauchy definition of $\mathbb{R}$ contradicts the axiom that $\mathbb{R} = (-\infty, \infty)$ because it precludes the existence of numbers larger than any natural number. In this section, therefore, we will modify the Cauchy definition so that it will support the underlying geometric construction and facilitate the algebraic construction of numbers in the neighborhood of infinity. We will only construct the natural neighborhoods here because the equality or inequality of $\mathbb{R}_{0}^{\mathcal{X}}$ and $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is not treated until Section 7.4.

Definition 4.2.1 The rational numbers $\mathbb{Q}$ are an Archimedean number field satisfying all of the well-known field axioms given in Section 5.4.

Definition 4.2.2 A sequence $\{x_n\}$ is a Cauchy sequence if and only if

$$\forall \delta \in \mathbb{Q} \quad \exists m, n, N \in \mathbb{N} \quad \text{s.t.} \quad m, n > N \quad \Longrightarrow \quad \left| x_n - x_m \right| < \delta \quad .$$

Definition 4.2.3 We say a relation is an equivalence relation if and only if (i) S is a set, (ii) every $x \in S$ is related to x, meaning the relation is reflexive, (iii) for every $x, y \in S$, the relation of x to y implies the relation of y to x, meaning the relation is symmetric, and (iv) for every $x, y, z \in S$, the relation of x to y and the relation of y to z together imply the relation of x to z, meaning the relation is transitive. The equivalence class of $x \in S$, namely the set of all objects which are related to x by an equivalence relation, is denoted

$[x]$. At times we will write $[x] = [\{x_n\}]$ or $[x] = [(x_n)]$ to emphasize that the equivalence relation is among Cauchy sequences where $\{x_n\}$ and (x_n) have the same meaning.

Definition 4.2.4 $C_{\mathbb{Q}}$ is the set of all Cauchy sequences of rational numbers.

Remark 4.2.5 Usually the Cauchy construction of $\mathbb{R}$ is formulated as, "Every $x \in \mathbb{R}$ is some Cauchy equivalence class $[x] \subset C_{\mathbb{Q}}$," but here we will take a slightly different approach.

Axiom 4.2.6 Every $x \in \mathbb{R}$ may be constructed algebraically as (i) the value of a function $f([\mathcal{X}], [b])$ where $[\mathcal{X}], [b] \in C_{\mathbb{Q}}$, or (ii) a partition of that function's range.

Axiom 4.2.7 Every $x \in \mathbb{R}_0 \subset \mathbb{R}$ is a Cauchy equivalence class of rationals $x = [x] \subset C_{\mathbb{Q}}$ and also a Dedekind partition of $\mathbb{Q}$ in canonical form $x = (L, R)$. (Dedekind cuts are defined in Section 7.5.)

Remark 4.2.8 Axiom 4.2.7 grants that the reals are constructed by Cauchy equivalence classes or Dedekind partitions (cuts) in the most canonical sense ***if one takes the complementary axiom that every real number is less than some natural number***. We do not take that axiom, so we specify $x \in \mathbb{R}_0$ as the object of relevance.

Remark 4.2.9 Cantor's Cauchy construction of $\mathbb{R}$, like the Dedekind construction, is said to be "rigorous" because it begins with the rationals $\mathbb{Q}$. However, before one may assume the existence of $\mathbb{Q}$, one must define zero because $0 \in \mathbb{Q}$, but $0 \notin \mathbb{N}$. Therefore, to be rigorous, one simply may not assume $\mathbb{Q}$ as a consequence of $\mathbb{N}$. To introduce zero, we will introduce a line segment AB and define zero as the least number in the algebraic representation of the geometric point A. It is true that this present approach can be criticized as being "not rigorous" because we have assumed AB in the same way that others assume $\mathbb{Q}$, but the present construction is "more rigorous" because it bumps that which is assumed down to a more primitive level, *i.e.*: Euclid's principles of geometry [1]. Once $\mathbb{Q}$ is anchored to A, we may use $C_{\mathbb{Q}}$ to infer the existence of $\mathbf{AB}$ from the limit definition of infinity (Definition 2.2.2). With 0 as the least number in the interval $[0, \aleph_1]$, the identity $0 = \aleph_0$ follows from the symmetry of $\mathbf{AB}$ under permutations of the labels of its endpoints: $\mathbf{AB} \equiv [\aleph_0, \aleph_1]$. No small part is required to describe the greatest number in the algebraic representation of $B \in \mathbf{AB}$, so no small part can be required for the least number in the algebraic representation of $A \in \mathbf{AB}$.

Definition 4.2.10 The symbol $\hat{0}$ is an instance of the number zero with the

instruction not to do any of zero's absorptive operations. The absorptive operations of zero are

$$0 + x = x \quad , \quad \text{and} \qquad 0 \cdot x = 0 \quad .$$

Expressions containing $\hat{0}$ are not to be simplified by either of these operations.

Axiom 4.2.11 For every Cauchy sequence $\{x_n\}$ in the equivalence class $[x] \subset C_{\mathbb{Q}}$, there exists another Cauchy sequence $\{\hat{0} + x_n\} = \{x_n\}$. This is to say

$$\{x_n\} \in [x] \quad \iff \quad \{\hat{0} + x_n\} \in [x] \quad ,$$

or that, equivalently, there exists an additive identity element for every $x \in \mathbb{Q}$.

Example 4.2.12 With Axiom 4.2.11, we have associated every element of $C_{\mathbb{Q}}$ with the endpoint A of the real line segment $\mathbf{AB}$. This is done because every $x \in \mathbb{Q}$ has zero fractional magnitude with respect to infinity. Now we may mingle the geometric and algebraic notations to write

$$\{x_n\} = \{\hat{0} + x_n\} \equiv \{A + x_n\} \in [A + x] \quad .$$

By extending the line segment in consideration from $\mathbf{AB} \equiv [0, \infty]$ to $ZB \equiv [-\infty, \infty]$, the number zero is now in the center of A which is an interior point of ZB. Therefore, we may give an algebraic construction by Cauchy equivalence classes for all

$$\mathbb{R}_0^{\mathcal{X}} = \{\aleph_{\mathcal{X}} + b \mid b \in \mathbb{R}_0\} \quad ,$$

by changing the interior point attached to the sequences in the equivalence classes. For any interior point $X \in \mathbf{AB}$, there is an equivalence class $[X + x]$ such that

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \quad , \quad [x] = b \in \mathbb{R}_0 \qquad \Longrightarrow \qquad [X + x] \equiv [\aleph_{[\mathcal{X}]} + x] = \aleph_{\mathcal{X}} + b \quad .$$

In this notation, the comma is a logical "and" ($\wedge$), so the implication follows if both conditions on the left are true. Note well, the number $\mathcal{X}$ indicating that $\aleph_{\mathcal{X}}$ has $100 \times \mathcal{X}\%$ fractional distance with respect to $\mathbf{AB}$ is an equivalence class $\mathcal{X} = [\mathcal{X}] \subset C_{\mathbb{Q}}$ with no requisite geometric part because $0 < \mathcal{X} < 1$ implies $\mathcal{X} \in \mathbb{R}_0$. Note that the symmetry of ZB about the number at the center of the algebraic representation of its midpoint gives a further requirement for $0 = \aleph_0$.

Definition 4.2.13 In the following definitions, the sign of x is restricted appropriately for the neighborhood of the origin and the maximal neighborhood of infinity. $C_{\mathbb{Q}}^{\mathbf{AB}}$ is the sum of $C_{\mathbb{Q}}$ with the set of all $X \in \mathbf{AB}$. Specifically,

$$C_{\mathbb{Q}}^{\mathbf{AB}} = \{X\} + C_{\mathbb{Q}} = \{X + [b] \mid X \in \mathbf{AB},\ [b] \subset C_{\mathbb{Q}}\} \quad .$$

Since it is considered desirable to give a totally algebraic construction, we may give an equivalent definition as the range of a function $f([\mathcal{X}], [x]) = \aleph_{[\mathcal{X}]} + [x]$ where $[\mathcal{X}], [x] \in C_{\mathbb{Q}}$:

$$C_{\mathbb{Q}}^{\mathbf{AB}} = \{\aleph_{[\mathcal{X}]} + [x] \mid [\mathcal{X}], [x] \subset C_{\mathbb{Q}},\ 0 \leq [\mathcal{X}] \leq 1\} \quad .$$

In this second convention, $x \geq 0$ if $\mathcal{X} = 0$, and $x < 0$ if $\mathcal{X} = 1$. This is required for the elements of $C_{\mathbb{Q}}^{\mathbf{AB}}$ to be in $[0, \infty)$.

Definition 4.2.14 Every $\aleph_{\mathcal{X}} \in \mathbb{R}_{\aleph}^{\mathcal{X}} \subset \mathbb{R}$ is a Cauchy equivalence class $\aleph_{\mathcal{X}} = [\aleph_{\mathcal{X}}] = \aleph_{[\mathcal{X}]} \subset C_{\mathbb{Q}}^{\mathbf{AB}}$, where $\aleph_{\mathcal{X}} \in \mathbb{R}$ implies $0 \leq \mathcal{X} < 1$ so that $\mathcal{X} = [\mathcal{X}] \subset C_{\mathbb{Q}}$.

Remark 4.2.15 The $X + [b]$ notation in Definition 4.2.13 is not inherently well-defined because the summed quantities usually appear on opposite sides of the $\equiv$ relation. Thus, the equality of the hybrid definition for $C_{\mathbb{Q}}^{\mathbf{AB}}$ and its algebraic definition must be formalized.

Definition 4.2.16 The sum of a geometric point with an element of $C_{\mathbb{Q}}$ shall be defined as the sum of two equivalence classes. The equivalence class of an interior geometric point $X \in \mathbf{AB}$ is the equivalence class of the number in the center of its algebraic representation $\mathscr{X} \equiv X$. For an interior point or an endpoint, we have

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \quad \Longrightarrow \quad [X] \equiv [\aleph_{\mathcal{X}}] = \aleph_{[\mathcal{X}]} = \aleph_{\mathcal{X}} \ .$$

According to the usual algebra of equivalence classes, we have

$$[X] + [b] = [X + b] \ .$$

Axiom 4.2.17 Every $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ is a Cauchy equivalence class $x = \aleph_{[\mathcal{X}]} + [b] = [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$. $\mathrm{Big}(x)$ is defined by $[\mathcal{X}] \in C_{\mathbb{Q}}$, and $\mathrm{Lit}(x)$ is defined by $[b] \in C_{\mathbb{Q}}$. As in Definition 4.1.5, x is defined as the sum of its big and little parts. In other words, without inventing the object $C_{\mathbb{Q}}^{\mathbf{AB}}$, we have the equivalent condition that every $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ is an ordered pair of Cauchy equivalence classes of rationals

$$x = \big([\mathcal{X}], [b]\big) \in C_{\mathbb{Q}} \times C_{\mathbb{Q}} \ ,$$

where $C_{\mathbb{Q}} \times C_{\mathbb{Q}}$ is the Cartesian product. In this case, we take

$$\big([\mathcal{X}], [b]\big) = \aleph_{\mathcal{X}} + b \ .$$

Every element of $C_{\mathbb{Q}}^{\mathbf{AB}}$ has a representation as an ordered pair $(\mathcal{X}, b)$.

Remark 4.2.18 Axiom 4.2.17 is totally compliant with the requirement of Axiom 4.2.6 that all real numbers can be constructed as the values of a function or partitions of the function's range. Any partition of the range is equally a partition of the ordered pairs in the Cartesian product (Section 7.5).

Example 4.2.19 This example gives a Cauchy equivalence class definition of $\aleph_{\mathcal{X}}$, as in Definition 4.2.14. Suppose $0 \leq x \leq 1$ and that

$$x = [x] = \big[\{x_n\}\big] = \big[\{x_1, x_2, x_3, \dots\}\big] \ .$$

It follows that

$$\aleph_x = \aleph_{[x]} = [\aleph_x] = \left[\left\{\aleph_x^n\right\}\right] = \left[\aleph_{\{x_n\}}\right] = \left[\left\{\aleph_{x_1}, \aleph_{x_2}, \aleph_{x_3}, \dots\right\}\right] \quad ,$$

where we have moved the iterator n into the superscript position at one of the intermediate steps.

Theorem 4.2.20 *If X and Y are two interior points of* $\mathbf{AB}$*, then two Cauchy equivalence classes $[X + x]$ and $[Y + y]$ are equivalent if and only if $X = Y$ and $x = y$.*

Proof. By Definition 4.2.16, we have $[X + x], [Y + y] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$. Axiom 4.2.17 grants that every element of $C_{\mathbb{Q}}^{\mathbf{AB}}$ can be expressed as an ordered pair

$$\aleph_{\mathcal{X}} + b = ([\mathcal{X}], [b]) \in C_{\mathbb{Q}} \times C_{\mathbb{Q}} \quad .$$

By the definition of the equivalence class, every element of $C_{\mathbb{Q}}$ is such that

$$[x] = [y] \quad \iff \quad x = y \quad ,$$

so the same must be true for the ordered pairs:

$$\big([\mathcal{X}], [x]\big) = \big([\mathcal{Y}], [y]\big) \quad \iff \quad \big(\aleph_{\mathcal{X}}, x\big) = \big(\aleph_{\mathcal{Y}}, y\big) \quad .$$

The equivalence class of X is uniquely determined by the equivalence class of $\aleph_{\mathcal{X}}$ (Definition 4.2.16), so it follows that $X = Y$ if and only if $[X] = [Y]$. The theorem is proven. ∎

§4.3 The Maximal Neighborhood of Infinity

The main purpose of this section is to treat the properties of real numbers $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ for the special case of $\mathcal{X} = 1$. Again, the reader must note that formally $\mathbb{R}_{\aleph}^{1} \not\subset \{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$ due to the restriction $0 < \mathcal{X} < 1$ given by Definition 4.1.3. Whenever $\mathbb{R}_{0}^{\mathcal{X}}$ or $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is taken to mean $\mathcal{X} = 0$ or $\mathcal{X} = 1$, referring the neighborhood of the origin and the maximal neighborhood of infinity respectively, we will always make an explicit statement indicating $0 \leq \mathcal{X} \leq 1$.

Definition 4.3.1 The whole maximal neighborhood of infinity is

$$\mathbb{R}_{\aleph}^{1} = \left\{\aleph_1 - b \;\middle|\; b \in \mathbb{R}_{\aleph}^{0}\right\} \quad .$$

Remark 4.3.2 We have defined $\aleph_1$ as the greatest number in the algebraic representation $\mathscr{B}$ of $B \in \mathbf{AB} \equiv [0, \infty]$. Therefore, $\aleph_1$ is an infinite element not in the real numbers. As the arithmetic of ∞ is usually defined, if we set $\aleph_1 = \infty$, then it would follow that $\infty - b = \infty$ and $\mathbb{R}_{\aleph}^{1} \cap \mathbb{R} = \varnothing$. This is not the desired behavior, so we will make special notation custom tailored to deliver what is desired.

Definition 4.3.3 ∞ is called geometric infinity or simply infinity.

Definition 4.3.4 $\widehat{\infty}$ is called algebraic infinity. It shall be called infinity hat as well.

Definition 4.3.5 Additive absorption is a property of ∞ such that all $x \in \mathbb{R}$ are additive identities of ∞. The additive absorptive property is

$$\infty \pm x = \infty \quad .$$

Multiplicative absorption is a property of $\pm\infty$ such that all non-zero $x \in \mathbb{R}$ are multiplicative identities of $\pm\infty$. The multiplicative absorptive property is

$$\infty \cdot x = \begin{cases} \infty & \text{for} \quad x > 0 \\ -\infty & \text{for} \quad x < 0 \end{cases} \quad .$$

Remark 4.3.6 Note that infinity and zero are both multiplicative absorbers, while zero's additive absorptive property is such that zero gets absorbed. Indeed, the contradiction inherent to mutual multiplicative absorption may be identified as a reason contributing to the canonical non-definition of the $0 \cdot \infty$ operation.

Definition 4.3.7 The symbol $\widehat{\infty}$ refers to an infinite element

$$\pm\left|\widehat{\infty}\right| = \lim_{x \to 0^{\pm}} \frac{1}{x} \quad , \quad \text{and} \qquad \left|\widehat{\infty}\right| = \lim_{n \to \infty} \sum_{k=1}^{n} k \quad ,$$

together with an instruction not to perform the additive or multiplicative operations usually imbued to infinite elements.

Remark 4.3.8 What we have done in Definition 4.3.7 is exactly what we have done with $\hat{0}$ in Definition 4.2.10. In the case of $\hat{0}$, it was not in any way strange to entertain the notion that one might simply choose not to do the absorptive operations of zero, and neither should the present convention for $\widehat{\infty}$ be considered in any way strange or ill-defined. In Section 4.4, we will construct an infinite element—what might be called an instance of infinity—stripped of its absorptive operations by considering the invariance of **AB** under the permutations of the labels of its endpoints. As between Sections 4.1 and 4.2, we will define some objects in the present section to facilitate a formal construction in Section 4.4.

Theorem 4.3.9 *The two open intervals* $(-\infty, \infty)$ *and* $(-\widehat{\infty}, \widehat{\infty})$ *are identically equal. In other words, the real number line may be expressed identically as* $\mathbb{R} = (-\widehat{\infty}, \widehat{\infty})$ *or* $\mathbb{R} = (-\infty, \infty)$.

Proof. For $a, b \in \mathbb{R}^+$, it may be taken for granted that

$$(-a, b) = (-|a|, |b|) \quad .$$

It follows, therefore, that this is true for $a, b \in \overline{\mathbb{R}}^+$. Then, per Definition 4.3.7,

$$\pm\left|\widehat{\infty}\right| = \pm\infty \quad \Longrightarrow \quad \mathbb{R} = (-\widehat{\infty}, \widehat{\infty}) \quad . \qquad \text{🍃}$$

Example 4.3.10 This example demonstrates the arithmetic constraints that would have to be placed on the limit definition of infinity if it was said to define $\widehat{\infty}$ rather than $|\widehat{\infty}|$, as in Definition 4.3.7. This example also demonstrates the general motivation for such notation by demonstrating the large burden that would be imposed if the absolute value bars were absent in Definition 4.3.7. In its limit incarnation, the additive absorptive property of ∞ is demonstrated as

$$a + \infty = a + \lim_{x \to 0} \frac{1}{x} = \lim_{x \to 0} \frac{ax + 1}{x} = \text{diverges} = \infty \quad .$$

Therefore, if the limit were said to define $\widehat{\infty}$, then the hat's arithmetic constraint "don't simplify this expression by absorption" would mean to keep a out of the limited expression. Similarly, multiplicative absorption is demonstrated as

$$a \cdot \infty = a \cdot \lim_{x \to 0} \frac{1}{x} = \lim_{x \to 0} \frac{a}{x} = \text{diverges} = \infty \quad .$$

In either absorptive case, the limit expression diverges in $\mathbb{R}$, and no contradiction is obtained by keeping a out of the expression to avoid it being "absorbed."

The utility in adding the hat to infinity is that it supports the notion that a number lying x units of Euclidean distance away from the least number $0 = \aleph_0$ in the algebraic representation of $A \in \mathbf{AB}$ should, under permutation of the labels of the endpoints of $\mathbf{AB}$, be mapped to another number x' lying x units of distance away from the greatest number $\aleph_1$ in the algebraic representation of $B \in \mathbf{AB}$. By suppressing the additive absorption, we let $x' = \aleph_1 - x = \widehat{\infty} - x \neq \infty$. Per Definition 4.3.1, this number is $x' \in \mathbb{R}^1_{\aleph}$. By suppressing the multiplicative absorption of $\widehat{\infty}$, we introduce notation by which it is possible to complement Definition 4.1.1 with the statement

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \mathcal{X} \quad \Longleftrightarrow \quad \aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad .$$

In the former part this treatise, we have demonstrated a requirement for numbers such as x' and $\aleph_{\mathcal{X}}$, and $\widehat{\infty}$ is a notation for an infinite element tailored to this requirement. Indeed, where ***algebra is called the study of mathematical symbols and the rules for manipulating them***, algebraic infinity $\widehat{\infty}$ is a perfectly ordinary algebraic object, and well defined.

Definition 4.3.11 For any $\mathcal{X}, \mathcal{Y} \in \mathbb{R}$, the symbol $\aleph_{\mathcal{X}}$ has the properties

$$\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad , \quad \text{and} \qquad \aleph_{\mathcal{X}} \cdot \aleph_{\mathcal{Y}} = \aleph_{(\mathcal{X}\aleph_{\mathcal{Y}})} = \aleph_{\aleph_{(\mathcal{X}\mathcal{Y})}} \quad .$$

Definition 4.3.12 In terms of $\widehat{\infty}$, the whole maximal neighborhood of infinity is defined as

$$\mathbb{R}^1_{\aleph} = \left\{ \widehat{\infty} - b \;\middle|\; b \in \mathbb{R}^0_{\aleph},\ b \neq 0 \right\} \quad .$$

Definition 4.3.13 The maximal natural neighborhood of infinity is defined as

$$\mathbb{R}^1_0 = \left\{ \widehat{\infty} - b \;\middle|\; b \in \mathbb{R}^+_0 \right\} \quad .$$

§4.4 Equivalence Classes for the Maximal Natural Neighborhood of Infinity

We could easily construct $\mathbb{R}^1_0$ following the prescription in Section 4.2. There, we introduced zero as the least number in the algebraic representation of $A \in \mathbf{AB} \equiv [0, \infty]$, and then we made the extension to an arbitrary interior point by considering A as the midpoint of $ZB \equiv [-\infty, \infty]$. However, we could have left A as an endpoint and then extended the construction to the other endpoint B to define the maximal neighborhood of infinity via a symmetry argument. For breadth, here we will use a similar symmetry argument to take a slightly different approach to the Cauchy construction of the maximal neighborhood infinity. The material in the present section will constitute an independent motivation for the intermediate neighborhoods, separate from the main fractional distance approach. We will generate a non-absorbing infinite element $\widehat{\infty}$, and then we will define the $\aleph_{\mathcal{X}}$ as its fractional parts.

In Section 4.2, we defined a real number as an ordered pair of Cauchy equivalence classes of rationals: one for the big part and one for the small part. This approach required that we assume the $\aleph$ notation before we can define an equivalence class $[\aleph_{\mathcal{X}}] = \aleph_{[\mathcal{X}]} = \aleph_{\mathcal{X}}$. We were very well motivated to assume numbers in this form, particularly by Main Theorem 3.2.6 proving that some real numbers are larger than any real number, and by Theorem 3.2.2 proving that there exists at least one real number having 50% fractional magnitude with respect to **AB**. However, it remains that $\aleph_{\mathcal{X}}$ is inherently foreign to what is called real analysis. Therefore, in the present section, we will give an alternative construction for $\mathbb{R}^1_0$ based on the geometric invariance of line segments under the permutations of the labels of their endpoints. The numbers in the maximal neighborhood of infinity are defined according to ∞: a number not at all foreign to real analysis. Then, with $|\widehat{\infty}| = \infty$ defined as in the previous section, and with a formal construction given here for the maximal neighborhood of infinity, we will use $\widehat{\infty}$ as an independent constructor for $\aleph_{\mathcal{X}}$ and the intermediate neighborhoods.

Axiom 4.4.1 A Euclidean line segment AB [1] is invariant under permutations of the labels of its endpoints, *e.g.*: $AB = BA$.

Definition 4.4.2 Define a geometric permutation operator $\hat{P}$ such that

$$\hat{P}(AB) = BA \quad .$$

Remark 4.4.3 In this section, we will construct $\mathbb{R}_0^1$ from the operation of $\hat{P}$ on Cauchy equivalence classes of rational numbers, *e.g.*: $\hat{P}([x])$. To do so, we must develop the induced operation of $\hat{P}$ on the algebraic interval representation $[a,b] \equiv AB$. (It is a pleasant coincidence that the equivalence class bracket notation is exactly consistent with the abused notion for a closed one-point interval $[x,x] = [x]$.) As in Section 4.2, our departure from the usual Cauchy construction of $\mathbb{R}$ begins with an acknowledgment that $0 \in \mathbb{Q}$ does not follow from $\mathbb{N}$. Again, we introduce zero as the least number in the algebraic representation of $A \in AB$, and then we infer the existence of **AB** from the limit definition of infinity. Next, we assume zero is an additive identity element of every $n \in \mathbb{N}$ to obtain

$$\frac{m}{n} \in \mathbb{Q} \quad \Longrightarrow \quad \frac{m}{n} = \frac{0+m}{n} = \frac{0}{n} + \frac{m}{n} = 0 + \frac{m}{n} \quad .$$

Finally, we will put the hat on $\hat{0}$ to remind us not to simplify the expression. The elements of $C_{\mathbb{Q}}$ now have an explicit interpretation as Euclidean magnitudes measured relative to the origin of $\mathbb{R}$. Specifically, $\frac{m}{n}$ is an abstract element of $\mathbb{Q}$, but $\hat{0} + \frac{m}{n}$ is the rational length of a real line segment whose left endpoint has zero as the least number in its algebraic representation. Though this may be inferred directly, it follows from Definition 4.2.16 giving $[A] = [\aleph_0]$ and the symmetry requirement for $\aleph_0 = 0$.

Definition 4.4.4 The Euclidean chart x on **AB** is such that $\min(x \in A) = 0$ and $\max(x \in B) = \aleph_1$, regardless of the permutation of the labels of the endpoints. In other words, the ordering of real numbers is such that numbers nearer to B are always greater than those nearer to A.

Definition 4.4.5 Define an operator $\hat{\mathcal{P}}_0([x];\hat{0})$ which formalizes the notion of $\hat{P}([x])$. Per Definition 4.4.2, the domain of $\hat{P}$ is not $C_{\mathbb{Q}}$, so we introduce a special algebraic permutation operator $\hat{\mathcal{P}}_0([x];\hat{0})$ dual to $\hat{P}$ which formally operates on equivalence classes. The definition is

$$\hat{\mathcal{P}}_0 : \hat{0} + C_{\mathbb{Q}} \to \widehat{\infty} - C_{\mathbb{Q}} \quad ,$$

where

$$\hat{0} + C_{\mathbb{Q}} = \left\{ \hat{0} + [x] \;\middle|\; [x] \subset C_{\mathbb{Q}} \right\} \quad , \quad \text{and} \qquad \widehat{\infty} - C_{\mathbb{Q}} = \left\{ \widehat{\infty} - [x] \;\middle|\; [x] \subset C_{\mathbb{Q}} \right\} \quad .$$

Example 4.4.6 This example demonstrates the working of $\hat{P}$ and $\hat{\mathcal{P}}_0$ to give a formal construction of $\mathbb{R}_0^1$ by Cauchy sequences of rational numbers. Suppose $b \in \mathbb{R}_0$ is a well-defined equivalence class of rationals lying within the algebraic representation $\mathscr{A}$ of $A \in \mathbf{AB}$. Now operate on **AB** with $\hat{P}$ so that

$$\hat{P}(\mathbf{AB}) = \mathbf{BA} \quad .$$

The permutation of the labels of the endpoints has not changed the geometric position of b along the line segment. Definition 4.4.4 requires that the orientation of the Euclidean coordinate along the line segment has been reversed, so, therefore, we no longer have the property $b = [x] \subset C_{\mathbb{Q}}$ for the following reason. Every rational number is less than some natural number, and all such numbers have zero fractional distance with respect to **AB**. Before operating with $\hat{P}$, b was in the algebraic representation of the point A, but, by operating with the geometric permutation operator $\hat{P}$, b becomes a number in the algebraic representation of B. The FDFs are defined such that

$$\mathcal{D}_{\mathbf{AB}}(\mathbf{AB}) = \mathcal{D}^{\dagger}_{\mathbf{AB}}(\mathbf{AB}) = 1 \quad ,$$

so now b must have unit fractional magnitude with respect to **AB**. Every $[x] \subset C_{\mathbb{Q}}$ has zero fractional magnitude, so, if $[b] \not\subset C_{\mathbb{Q}}$, what number has b become? That number is given by

$$\hat{\mathcal{P}}_0(\hat{0}, [b]) = \widehat{\infty} - [b] \quad .$$

Under permutation of the labels of the endpoints of a line segment, a number having Euclidean distance $[b] \subset C_{\mathbb{Q}}$ from one endpoint becomes another number having the same distance relative to the other endpoint.

Remark 4.4.7 We take it for granted that if there exists a real number x separated by distance L from the least number in the algebraic representation of the endpoint A of an arbitrary real line segment $AB \equiv [a, b]$—with x interior in the sense that $x \in (a, b)$—then it is guaranteed by the geometric mirror symmetry of all line segments that there must exist another real number separated from the endpoint B by the same distance L. If we bestowed $\widehat{\infty}$ with the property of additive absorption, then there would be no such number because $\hat{0} + x \to x$, but $\widehat{\infty} - x \to \widehat{\infty}$. Similarly, if there exists a real number lying one third of the way from A to B, then there must exist another real number lying one third of the way from B to A. This follows from the cut-in-a-line definition of $\mathbb{R}$ given by Definition 2.1.5. For the case of **AB**, it will be impossible to express these third fraction numbers if $\widehat{\infty}$ has the property of multiplicative absorption. Since the third numbers *must* exist, $\aleph_{\mathcal{X}}$ *does* exist. Therefore, the existence of an instance of infinity devoid of any absorptive properties is absolutely granted and required if the mirror symmetry of a geometric line segment is to be preserved in its interpretation as an algebraic interval of numbers.

Our thesis is that we should preserve the underlying geometric construction of $\mathbb{R}$ without invoking a contradictory algebraic construction. Under this thesis, $\widehat{\infty}$ is forced into existence. Often times, the position is taken that infinity is absolutely absorptive due to the limit definition of infinity and the attendant absorptive properties of limits (Example 4.3.10). As an indirect consequence of such reasoning, the mirror symmetry of line segments must be rejected in the algebraic realm of mathematics. But why should it be preferred that the algebraic construction overrides the geometric construction? Is it not equally

valid to override the algebraic construction with the geometric one? Considering the history of mathematics, it is, in the opinion of this writer, far more appropriate to preserve the geometric construction at all costs. It is very easy to do so when the symbol $\widehat{\infty}$ is given by the limit definition of infinity as

$$\lim_{x\to 0^{\pm}} \frac{1}{x} = \pm\left|\widehat{\infty}\right| \quad ,$$

without $\widehat{\infty}$ itself being interchangeably equal with the limit expression. Furthermore, this scheme is such that ***the algebraic and geometric concepts are complementary*** without requiring that one override the other.

In Definition 4.2.17, we gave the definition of $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ in terms of ordered pairs of elements of $C_{\mathbb{Q}}$. The purpose of the present alternative treatment for the maximal neighborhood $\mathbb{R}_0^1$ is not to replace that definition but to complement it with a different equivalence class construction for the maximal neighborhood: one from which the constructions of the intermediate neighborhoods may be extracted. In this present section, we have used the permutation operator $\hat{P}$ which is quite similar to the implicit translation operator by which we were able to attach elements of $C_{\mathbb{Q}}$ to different interior points of **AB** in Section 4.2. The main utility in developing the idea of a number in the neighborhood of infinity as the operation of $\hat{\mathcal{P}}_0$ on an equivalence class of rationals is that it independently generates the requirement for an infinite element lacking the usual absorptive properties of infinity. With $\widehat{\infty}$ granted, it gives a separate means by which we may construct the $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ without invoking the direct ordered pair definition: the $\aleph_{\mathcal{X}}$ in such numbers are the fractions of the non-absorbing infinite element $\widehat{\infty}$.

Axiom 4.4.8 Every $x \in \mathbb{R}_0^1$ is defined as the output of $\hat{\mathcal{P}}_0$ operating on an element of $C_{\mathbb{Q}}$. This is the Cauchy equivalence class construction of real numbers in the maximal natural neighborhood of infinity.

Remark 4.4.9 If we wish to construct $\mathbf{AB} \equiv [0, \widehat{\infty}]$ from $AB \equiv [0, \frac{\pi}{2}]$, as we constructed $\mathbf{AB} \equiv [0, \infty]$ in Example 2.3.21, then we need to make rigorous the relationship between $[0, \infty]$ and $[0, \widehat{\infty}]$. This was accomplished in Theorem 4.3.9 proving that $(-\infty, \infty) = (-\widehat{\infty}, \widehat{\infty})$. The absolute value, or the magnitude, of $\widehat{\infty}$ is the same as that of ∞, so the algebraic intervals $[x, \infty]$ and $[x, \widehat{\infty}]$ must be the same. Though we cannot directly construct $[0, \widehat{\infty}]$ from $[0, \frac{\pi}{2}]$, we may indirectly construct it by using the limit definition of infinity (Definition 2.2.2) to write

$$\lim_{\theta\to\frac{\pi}{2}^-} \tan(\theta) = \lim_{\theta\to\frac{\pi}{2}^-} \frac{\sin(\theta)}{\cos(\theta)} = \lim_{\substack{x\to 0\\ y\to 1}} \frac{y}{x} = \left|\widehat{\infty}\right| \quad .$$

Now we may infer the existence of conformal $\mathbf{AB} \equiv [0, \widehat{\infty}]$ from the assumed interval $[0, \frac{\pi}{2}]$ as a corollary of Theorem 4.3.9.

Example 4.4.10 In this example, we complement the separate definitions for ∞ and $\widehat{\infty}$ heretofore given. We will show, for example, how they might be more fully conceptually distinguished as two mutually distinct kinds of infinite elements with markedly different qualia beyond their separate technical definitions. We will offer these qualia as an example only; we will not alter the technical definitions with the supplemental considerations proposed here. To that end, it is sometimes claimed, without proof, that one cannot include endpoints at the ends of $\mathbb{R} = (-\infty, \infty)$ because the notion of an endpoint contradicts the notion of the infinite geometric extent of a line extending infinitely far in both directions. Infinite geometric extent is the main principle that we will look at in this example.

Suppose geometric infinity ∞ is a number which cannot be written as the included or unincluded endpoint of an interval without contradicting the notion of the infinite geometric extent of a number line. An unincluded endpoint may always be included via compactification, contrary to the claim cited above, so we might distinguish geometric infinity from algebraic by letting the former be non-compactifiable. Consider Definition 2.1.2 which defines a number line as a 1D metric space in the Euclidean metric

$$d(x, y) = \big|y - x\big| \quad .$$

If we included geometric infinity as an endpoint, then we could invoke the invariance of line segments under permutations of the labels their endpoints to demonstrate a contradiction. Given

$$(x, y) = (x_0, y_0) \quad , \quad \text{and} \qquad (\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = (\infty - x_0, \infty - y_0) \quad ,$$

not only do the points lose their unique identity when attached to B instead of A, but if we put $(\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0))$ into the Euclidean metric, then we get

$$d(\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = \big|\infty - y_0 - \big(\infty - x_0\big)\big| = \big|\infty - \infty\big| = \text{undefined} \quad .$$

Clearly, this does not gel well with our intention to define a number line as a line equipped with a metric. The line is supposed to have some metrical distance between any two numbers, but, now, under the permutation of the labels A and B, we find two numbers that don't even have vanishing distance between them. The distance has become undefined, though this does not follow from the invariance of Euclidean line segments under such permutations.

Algebraic infinity is a number which avoids all of these problems. Under permutation, we have

$$(x, y) = (x_0, y_0) \quad , \quad \text{and} \qquad (\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = (\widehat{\infty} - x_0, \widehat{\infty} - y_0) \quad .$$

Jumping ahead to the arithmetic of such numbers axiomatized in Section 5.2, we find exactly what is expected:

$$d(\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = \big|\widehat{\infty} - y_0 - \big(\widehat{\infty} - x_0\big)\big| = \big|x_0 - y_0\big| = d(x_0, y_0) \quad ,$$

The only issue which remains is to revisit is the construction for $\mathbf{AB} \equiv [0, \infty]$ that we have given by a conformal chart $x = \tan(x')$ on the line segment $AB \equiv [0, \frac{\pi}{2}]$ whose endpoints unquestioningly exist in any frame of standard analysis. For this, we might propose a semantic convention to distinguish the geometric infinite element ∞ from the algebraic one $\widehat{\infty}$. Let algebraic infinity be such that it can be embedded in a larger space, but let geometric infinity be such that it is totally maximal and cannot be embedded in something larger than itself. For example, the interval $[0, \frac{\pi}{2}] \subset [-\pi, \pi]$ is such that the conformal chart which sends $\frac{\pi}{2}$ to an infinite element implicitly places that element within the parent interval $[-\pi, \pi]$. The convention proposed here would require that the infinite element to which $\frac{\pi}{2}$ is conformally mapped must be algebraic infinity $\widehat{\infty}$. If we take the convention that geometric infinity ∞ is always totally geometrically maximal, then that would forbid its existence on the interior of the interval $[-\pi, \pi]$ which contains points to the right of $\frac{\pi}{2}$. In a formal adoption of the distinctions made here, one would examine the merits of a supplemental transfinite ordering relation $\widehat{\infty} < \infty$.

Remark 4.4.11 Definition 4.3.7 gives

$$\left|\widehat{\infty}\right| = \infty = \lim_{x \to 0} \frac{1}{x} \quad \not\Longrightarrow \quad \widehat{\infty} = \lim_{x \to 0} \frac{1}{x} \ .$$

Due to the transitivity of the equivalence relation, we must be careful about whether $\aleph_1$ is equal to geometric infinity or algebraic. If we take the convention that geometric infinity ∞ is imbued with the notion of infinite geometric extent such that an infinite line cannot have an endpoint there, as in Example 4.4.10, then we should not let $\aleph_1$ be defined by ∞ when it is said to be the greatest number in the algebraic representation of the endpoint $B \in \mathbf{AB}$. Due to the possibility of constructing $\mathbf{AB}$ from any other line segment by one chart transformation or another, $\mathbf{AB}$ ought to be taken as $[0, \widehat{\infty}] = [0, \aleph_1]$ in the absence of explicit words to the contrary.

Definition 4.4.12 The symbol $\aleph_1$ is an alternative notation for algebraic infinity. We have

$$\aleph_1 = \widehat{\infty} \ , \quad \text{and} \qquad \aleph_1 \neq \infty \ .$$

Theorem 4.4.13 *The whole maximal neighborhood of infinity $\mathbb{R}^1_{\aleph}$ is a subset of the real numbers.*

Proof. Taking for granted that $x \in \mathbb{R}^1_{\aleph}$ does not have any infinitesimal part, which is obvious, it suffices to show the compliance with Definition 2.1.5: a real number $x \in \mathbb{R}$ is a cut in the real number line. Compliance follows directly from Definition 4.3.12 giving

$$\mathbb{R}^1_{\aleph} = \left\{ \widehat{\infty} - b \;\middle|\; b \in \mathbb{R}^0_{\aleph},\ b \neq 0 \right\} \ ,$$

because we clearly have

$$(-\infty, \infty) = (-\infty, \widehat{\infty} - b] \cup (\widehat{\infty} - b, \infty) \quad .$$

Even though we do not yet have an equivalence class construction for $b \in \mathbb{R}^0_{\aleph} \setminus \mathbb{R}_0$, it is obvious that $\widehat{\infty} - b$ is a cut in the real number line because b, whatever its algebraic construction, is such that it is positive and has zero fractional magnitude with respect to **AB**. (The intuitive ordering assumed in this theorem is formalized in Axiom 5.2.18.)

Corollary 4.4.14 *All numbers $x \in \{\mathbb{R}^{\mathcal{X}}_{\aleph}\}$ are real numbers.*

Proof. The ordering of $\mathbb{R}$ given by Axiom 3.1.13 is such that $0 < \mathcal{X} < 1$ guarantees

$$(0, \infty) = (0, \aleph_{\mathcal{X}} \pm b] \cup (\aleph_{\mathcal{X}} \pm b, \infty) \quad .$$

Definition 2.1.5 is satisfied trivially. The theorem is proven.

Remark 4.4.15 As a final aside in this section, note the curious condition under which algebraic infinity $\aleph_1$ has its foundation in the geometric properties of a line segment, while geometric infinity ∞ has its foundation in the limit of an algebraic expression. The reciprocity among these two constructions of an infinite element might indicate some deeply fundamental issues extending beyond the semantic convention of our having chosen to call one infinite element geometric and the other algebraic. We will not proceed along that analytical direction, but the reciprocity of the cross-sampling of the concepts is interesting and tantalizing.

§5 Arithmetic

§5.1 Operations for Infinite Elements

Here we give arithmetic operations for $\infty, \widehat{\infty} \notin \mathbb{R}$ to support the axioms for real numbers $x \in \mathbb{R}$ with non-zero big parts to appear in Section 5.2.

Remark 5.1.1 All of the contradictions which forbid additive and multiplicative inverses for ∞ stem from its limit definition. Should we bestow, then, these inverses on $\widehat{\infty} = \aleph_1$? To the extent that the notion of fractional distance requires that $100\% - 100\% = 0\%$ and $100\%/100\% = 1$, the answer is yes. We should not expect any contradictions related to inverses for $\widehat{\infty}$ because $\widehat{\infty} \neq \infty$. The limit definition is out of scope. Similarly, all of the contradictions which disallow a definition for the operation $0 \cdot \infty$ are rooted in the limit definition of infinity, but $0 \cdot \widehat{\infty} = \aleph_0 = 0$ follows as a special case of $\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty}$.

Axiom 5.1.2 $\aleph_1$ is such that

$$\aleph_1 - \aleph_1 = 0 \quad , \quad \text{and} \qquad \frac{\aleph_1}{\aleph_1} = 1 \quad .$$

Axiom 5.1.3 The operations for $\infty \neq \aleph_1$ with $b \in \mathbb{R}_0^+$ are

$$\begin{aligned} \infty \pm b &= \infty \\ \infty \pm \big(-b\big) &= \infty \\ -\big(\pm\infty\big) &= \mp\infty \\ \infty \cdot b &= \infty \\ \frac{\infty}{b} &= \infty \\ \frac{b}{\infty} &= 0 \quad . \end{aligned}$$

The addition and multiplication operations are commutative here.

Axiom 5.1.4 We give the following supplemental operations for zero and ∞:

$$\begin{aligned} \infty + 0 = 0 + \infty &= \infty \\ \infty \cdot 0 = 0 \cdot \infty &= \text{undefined} \\ \frac{\infty}{0} &= \text{undefined} \\ \frac{0}{\infty} &= 0 \quad . \end{aligned}$$

Axiom 5.1.5 The operations for $\widehat{\infty} = \aleph_1$ with $b \in \mathbb{R}_0^+$ are

$$\begin{aligned} \widehat{\infty} \pm b &= \pm b + \widehat{\infty} \\ \widehat{\infty} \pm \big(-b\big) &= \widehat{\infty} \mp b \\ -\big(\pm\widehat{\infty}\big) &= \mp\widehat{\infty} \\ \widehat{\infty} \cdot b = b \cdot \widehat{\infty} &= \aleph_b \\ \frac{\widehat{\infty}}{b} &= \aleph_{(b^{-1})} \\ \frac{b}{\widehat{\infty}} &= 0 \quad . \end{aligned}$$

Axiom 5.1.6 We give the following supplemental axioms for zero and $\widehat{\infty}$:

$$\begin{aligned} \widehat{\infty} + 0 = 0 + \widehat{\infty} &= \widehat{\infty} \\ \widehat{\infty} \cdot 0 = 0 \cdot \widehat{\infty} &= 0 \\ \frac{0}{\widehat{\infty}} &= 0 \end{aligned}$$

$$\frac{\widehat{\infty}}{0} = \text{undefined} \quad .$$

Remark 5.1.7 The most important facet of Axiom 5.1.6 is the $0 \cdot \widehat{\infty}$ operation contrary to the undefined $0 \cdot \infty$ operation (Axiom 5.1.4). This is required to preserve the notion of fractional distance: zero times 100% is 0%. To facilitate this definition, it will be required that we define division as a separate operation distinct from multiplication by an inverse. This will be one of the major distinctions of the axioms in Section 5.2 from the well-known field axioms. We demonstrate the principle in Example 5.1.8.

Example 5.1.8 This example gives a common argument in favor of the non-definition of a product between an infinite element and zero. Then we will show how the contradiction is avoided if we do not grant an assumed associativity among multiplication and division. The $0 \cdot \widehat{\infty} = 0$ operation given by Axiom 5.1.6 requires that we avoid such contradictions.

Suppose $c \in \mathbb{R}_{\aleph}^{0}$ so that

$$\frac{c}{\widehat{\infty}} = 0 \quad .$$

Now suppose $0 \cdot \widehat{\infty}$ is a defined operation so that

$$z = 0 \cdot \widehat{\infty} \quad .$$

Substitute $\frac{c}{\widehat{\infty}} = 0$ and use the $\frac{\widehat{\infty}}{\widehat{\infty}} = 1$ property of Axiom 5.1.2 to obtain by association of multiplication and division the expression

$$z = 0 \cdot \widehat{\infty} = \frac{c}{\widehat{\infty}} \cdot \widehat{\infty} = c \cdot \frac{\widehat{\infty}}{\widehat{\infty}} = c \quad .$$

This shows that $0 \cdot \widehat{\infty}$ is not a well-defined operation because $z = c$ is not a unique output. When we define division as a third operation beyond multiplication and addition, however, we should not assume associativity among the distinct divisive and multiplicative operations, and neither will we axiomatize it in Section 5.2. Without assumed associativity among the terms, we cannot show that z fails to be a well-defined output of the product $0 \cdot \widehat{\infty}$. In that case, we will assume there is no problem with the definition $0 \cdot \widehat{\infty} = 0$.

Axiom 5.1.9 For any non-negative $x \in \mathbb{R}$, we have

$$x^{\widehat{\infty}} = \begin{cases} \infty & \text{for} \quad x > 1 \\ 1 & \text{for} \quad x = 1 \\ 0 & \text{for} \quad 0 \leq x < 1 \end{cases} \quad .$$

The product of an infinite number of finite numbers greater than one is absolutely absorptive.

Remark 5.1.10 By choosing the $x^{\widehat{\infty}} = \infty$ convention in Axiom 5.1.9 rather than the alternative $x^{\widehat{\infty}} = \widehat{\infty}$, we sidestep the notion that two different numbers x and y raised to the same power might both land precisely at $\aleph_1$. For $x \neq y$, we would not expect, in general, that $x^{\aleph_1} = y^{\aleph_1} = \aleph_1 = \widehat{\infty}$. In the remainder of this section, we will further motivate the $x^{\widehat{\infty}} = \infty$ operation.

Theorem 5.1.11 *For $k \neq 1$, we have $\widehat{\infty}^k \neq \widehat{\infty}$.*

Proof. To prove this theorem, it will suffice to prove that

$$\widehat{\infty}^2 = \widehat{\infty} \cdot \widehat{\infty} = \aleph_1 \cdot \aleph_1 \neq \aleph_1 \quad .$$

Definition 4.3.11 requires that for any $x \in \mathbb{R}$, the symbol $\aleph_x$ is such that

$$\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad , \quad \text{and} \qquad \aleph_{\mathcal{X}} \cdot \aleph_{\mathcal{Y}} = \aleph_{(\mathcal{X}\aleph_{\mathcal{Y}})} = \aleph_{\aleph_{(\mathcal{X}\mathcal{Y})}} \quad .$$

Choose $x = \aleph_{\mathcal{X}}$ such that $0 < \mathcal{X} < 1$. Then

$$\aleph_{\aleph_{\mathcal{X}}} = \aleph_{\mathcal{X}} \cdot \widehat{\infty} = \mathcal{X} \cdot \widehat{\infty} \cdot \widehat{\infty} = \mathcal{X} \cdot \widehat{\infty}^2 \quad .$$

If $\widehat{\infty}^2 = \widehat{\infty}$, however, then we could write

$$\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} = \mathcal{X} \cdot \widehat{\infty}^2 \qquad \Longrightarrow \qquad \aleph_{\mathcal{X}} = \aleph_{\aleph_{\mathcal{X}}} \quad .$$

Since $\aleph_{\mathcal{X}} \in \mathbb{R}$, it cannot be equal to the number $\aleph_{\aleph_{\mathcal{X}}} \notin \mathbb{R}$ which has much greater than unit fractional distance with respect to infinity. This contradiction proves the theorem. ∎

Theorem 5.1.12 *The operation $x^{\widehat{\infty}} = \widehat{\infty}$ is not well-defined.*

Proof. Assume $0 < \mathcal{X} < 1$ and consider two expressions

$$x^{\aleph_{\mathcal{X}}+b} = x^{\aleph_{\mathcal{X}}} x^b = \left(x^{\mathcal{X}}\right)^{\widehat{\infty}} x^b = \widehat{\infty} x^b \quad ,$$

and

$$x^{\aleph_{\mathcal{X}}+b} = x^{\aleph_{\mathcal{X}}} x^b = \left(x^{\widehat{\infty}}\right)^{\mathcal{X}} x^b = \widehat{\infty}^{\mathcal{X}} x^b \quad .$$

By Theorem 5.1.11, we have $x^b \widehat{\infty} \neq x^b \widehat{\infty}^{\mathcal{X}}$ when $0 < \mathcal{X} < 1$. This proves the theorem. ∎

Remark 5.1.13 Note that the contradiction derived in Theorem 5.1.12 is avoided in the convention of Axiom 5.1.9:

$$x^{\aleph_{\mathcal{X}}+b} = \left(x^{\mathcal{X}}\right)^{\widehat{\infty}} x^b = \infty x^b = \infty \quad , \quad \text{and} \qquad x^{\aleph_{\mathcal{X}}+b} = \left(x^{\widehat{\infty}}\right)^{\mathcal{X}} x^b = \infty^{\mathcal{X}} x^b = \infty \quad .$$

Here, we have relied on the usual understanding that the multiplicative absorptive property of ∞ is such all powers of ∞ are identically equal to ∞. This exceeds the definition of absorption given in Definition 4.3.5 such that ∞ absorbs $x \in \mathbb{R}$, but it is standard to set all powers of ∞ equal to ∞.

§5.2 Arithmetic Axioms for Real Numbers in Natural Neighborhoods

When one defines $\mathbb{R}$ such that the set $\mathcal{R} = \{\mathbb{R}, +, \times, \leq\}$ conforms the field axioms (Section 5.4), it is a natural progression to prove that Cauchy equivalence classes satisfy those axioms. We do *not* presently presume that $\mathbb{R}$ is such that $\mathcal{R}$ obeys the field axioms, so we will not make any such proofs. Instead, we will list the axiomatized arithmetic operations obeyed by real numbers whose little parts are less than some natural number. For disambiguation with the well-known "field axioms," the axioms given in this section are called "the arithmetic axioms." In Section 5.3, we will make proofs of certain operations given in the arithmetic axioms, and we will give examples. In Section 5.5, we will define the operations in terms of the numbers' underlying equivalence classes. All of the axioms given here pertain only to the natural neighborhoods $\mathbb{R}_0^{\mathcal{X}}$. When we give the treatment leading to $\mathbb{R}_{\aleph}^{\mathcal{X}} \setminus \mathbb{R}_0^{\mathcal{X}} = \varnothing$ (conjectured in Section 7.4), these axioms will be fairly comprehensive. However, when we impose connectedness on $\mathbb{R}$ in Section 7.5, we will find that these axioms are not *totally* comprehensive.

The equivalence class constructions given in Section 4 were only for natural neighborhoods, and here we will follow with the axiomatized arithmetic for the elements of those neighborhoods. Almost everything about the field axioms shall be preserved in the natural neighborhoods. The major exception is that we will not enforce the global closure of $\mathbb{R}$ under its operations. Among the other departures from the field axioms will be the identification of ***division as an operation separate from its usual definition in terms of multiplication by an inverse***. Closure is nice for group theoretical applications, but it is not needed for most applications in arithmetic. For example, the set of integers $\{3, 4, 5\}$ is not closed under integer addition, and yet it remains a perfectly sound algebraic structure with which one may do summation mathematics in the usual way. If one were to claim, "Non-closure doesn't break arithmetic because $\{3, 4, 5; +\}$ is a subset of $\{\mathbb{R}; +\}$, which is an algebraic group as defined by the field axioms," then we could make an easy rebuttal by defining a set $\mathbb{T} \supset \mathbb{R}$ as

$$\mathbb{T} = \left\{ x \;\middle|\; -\aleph_\infty < x < \aleph_\infty \right\} \quad .$$

Then the present convention for non-closed $\{\mathbb{R}; +\}$ defined with the Euclidean magnitude and supplemental arithmetic axioms is such that $\{\mathbb{R}; +\}$ is a subset of the closed additive group of 1D transfinitely continued real numbers $\{\mathbb{T}; +\}$.

Axiom 5.2.1 All $\mathbb{R}_0$ numbers obey the well-known axioms of a complete ordered field: Axioms 5.4.3, 5.4.5, and 5.4.8.

Remark 5.2.2 To make a distinction between the intermediate neighborhoods of infinity and the maximal neighborhood, in this section we will use the symbol $\widehat{\infty}$ rather than the symbol $\aleph_1$ associated with $\{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$ as $\aleph_{\mathcal{X}}$. However,

the reader should note that the arithmetic of the maximal neighborhood follows from the arithmetic of the intermediate neighborhoods as a special case of $\aleph_{\mathcal{X}}$ with $\mathcal{X} = 1$.

Axiom 5.2.3 Addition is commutative and associative. There exists an additive identity element 0 and an additive inverse x^{-1} for every $x \in \mathbb{R}$. The operations for $+$ are given as follows when $a, b, x, y \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$.

$+$	0	$y \in \mathbb{R}_0$	$\left(\aleph_{\mathcal{Y}} + a\right) \in \mathbb{R}_0^{\mathcal{X}}$	$\left(\widehat{\infty} - \lvert a\rvert\right) \in \mathbb{R}_0^1 \cup \widehat{\infty}$
0	0	y	$\aleph_{\mathcal{Y}} + a$	$\widehat{\infty} - \lvert a\rvert$
x	x	$x + y$	$\aleph_{\mathcal{Y}} + \left(a + x\right)$	$\widehat{\infty} - \left(\lvert a\rvert - x\right)$
$\left(\aleph_{\mathcal{X}} + b\right)$	$\aleph_{\mathcal{X}} + b$	$\aleph_{\mathcal{X}} + \left(b + y\right)$	$\aleph_{(\mathcal{X}+\mathcal{Y})} + \left(b + a\right)$	$\aleph_{(\mathcal{X}+1)} + \left(b - \lvert a\rvert\right)$
$\left(\widehat{\infty} - \lvert b\rvert\right)$	$\widehat{\infty} - \lvert b\rvert$	$\widehat{\infty} - \left(\lvert b\rvert - y\right)$	$\aleph_{(1+\mathcal{Y})} - \left(\lvert b\rvert - a\right)$	$\aleph_2 - \left(\lvert b\rvert + \lvert a\rvert\right)$

Remark 5.2.4 The most important property given by Axiom 5.2.3 is

$$\aleph_{\mathcal{X}} + \aleph_{\mathcal{Y}} = \aleph_{(\mathcal{X}+\mathcal{Y})} \quad .$$

This equality follows from the geometric notion of addition. If, for instance, $\aleph_{\mathcal{X}}$ is a number with 10% fractional distance along **AB**, and $\aleph_{\mathcal{Y}}$ is a number with 20% fractional distance, then it follows that their sum is a number with 30% fractional distance along **AB**. Axiom 5.2.3 makes clear that $\mathbb{R}$ does not satisfy the usual understanding that the reals are closed under their operations. Any number $\aleph_{\mathcal{X}} + b$ with $\mathcal{X} > 1$ is not a real number, *e.g.*: the sum of two positive numbers with 99% fractional magnitude is not a real number. No x with big part $\aleph_{1.98}$ can be $x \in \mathbb{R}$.

Theorem 5.2.5 *All real numbers $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ have an additive inverse.*

Proof. The number x^{-1} is the additive inverse of x if and only if

$$x + x^{-1} = x^{-1} + x = 0 \quad .$$

The statement of the theorem requires that (i) $x = \aleph_{\mathcal{X}} + b$, (ii) $0 < \mathcal{X} < 1$, and (iii) $b \in \mathbb{R}_0$. Assume that x^{-1} has the form $\aleph_{(\mathcal{X}^{-1})} + b^{-1}$. The definition of the additive inverse requires

$$1 = \left(\aleph_{\mathcal{X}} + b\right) + \left(\aleph_{(\mathcal{X}^{-1})} + b^{-1}\right) = \aleph_{(\mathcal{X}+\mathcal{X}^{-1})} + \left(b + b^{-1}\right) \quad .$$

Equating the big and little parts of this expression, we obtain two requirements

$$\aleph_{(\mathcal{X}+\mathcal{X}^{-1})} = \aleph_0 \quad \iff \quad \mathcal{X} + \mathcal{X}^{-1} = 0 \quad \iff \quad \mathcal{X}^{-1} = -\mathcal{X} \ ,$$

and

$$b + b^{-1} = 0 \quad \iff \quad b^{-1} = -b \ .$$

For every $[\mathcal{X}], [b] \subset C_{\mathbb{Q}}$, there exists a $[-\mathcal{X}], [-b] \subset C_{\mathbb{Q}}$, so, therefore, every $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ has an additive inverse.

Axiom 5.2.6 Multiplication is commutative and associative, and it is distributive over addition. Among $\mathbb{R}_0$ numbers, it is associative with division (which shall not be defined as multiplication by an inverse), but ***multiplication is not associative with division in general***. There exists a multiplicative identity $1 \neq 0$ for every $x \in \mathbb{R}$. There exists a multiplicative inverse for every $x \in \mathbb{R}_0$, but ***there does not exist a multiplicative inverse for all real numbers***. The operations for $\{\cdot\} = \{\times\}$ are given as follows when $a, b \in \mathbb{R}_0$, $x, y \in \mathbb{R}_0^+$, and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$.

$\times$	0	∓ 1	$y \in \mathbb{R}_0^+$	$\left(\aleph_{\mathcal{Y}} + a\right) \in \mathbb{R}_0^{\mathcal{X}}$	$\left(\widehat{\infty} - \lvert a\rvert\right) \in \mathbb{R}_0^1 \cup \widehat{\infty}$
0	0	0	0	0	0
± 1	0	-1	$\pm y$	$\aleph_{(\pm\mathcal{Y})} \pm a$	$\pm\widehat{\infty} \mp \lvert a\rvert$
x	0	$\mp x$	xy	$\aleph_{(x\mathcal{Y})} + ax$	$\aleph_x - \lvert a\rvert x$
$\left(\aleph_{\mathcal{X}} + b\right)$	0	$\aleph_{(\mp\mathcal{X})} \mp b$	$\aleph_{(\mathcal{X}y)} + by$	$\aleph_{(\aleph_{\mathcal{X}}\mathcal{Y} + a\mathcal{X} + b\mathcal{Y})} + ba$	$\aleph_{(\aleph_{\mathcal{X}} - \lvert a\rvert\mathcal{X} + b)} - b\lvert a\rvert$
$\left(\widehat{\infty} - \lvert b\rvert\right)$	0	$\mp\widehat{\infty} \pm \lvert b\rvert$	$\aleph_y - \lvert b\rvert y$	$\aleph_{(\aleph_{\mathcal{Y}} + a - \lvert b\rvert\mathcal{Y})} - \lvert b\rvert a$	$\aleph_{(\widehat{\infty} - \lvert a\rvert - \lvert b\rvert)} + \lvert ba\rvert$

Remark 5.2.7 The most important property given in Axiom 5.2.6 is

$$\pm\aleph_{\mathcal{X}} = \aleph_{(\pm\mathcal{X})} \ .$$

This operation follows from

$$\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad \implies \quad \pm\aleph_{\mathcal{X}} = \pm\left(\mathcal{X} \cdot \widehat{\infty}\right) = \left(\pm\mathcal{X}\right) \cdot \widehat{\infty} = \aleph_{(\pm\mathcal{X})} \ .$$

This shows that multiplication is axiomatically associative.

Theorem 5.2.8 *Real numbers in the intermediate natural neighborhoods of infinity $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ do not have a multiplicative inverse.*

Proof. A number x^{-1} is the multiplicative inverse of $x \in \mathbb{R}$ if and only if

$$x \cdot x^{-1} = x^{-1} \cdot x = 1 \ .$$

The statement of the theorem requires that (i) $x = \aleph_{\mathcal{X}} + b$, (ii) $0 < \mathcal{X} < 1$, and (iii) $b \in \mathbb{R}_0$. Axiom 5.2.6 grants that multiplication is distributive over addition, so the definition of the multiplicative inverse requires

$$\big(\aleph_{\mathcal{X}} + b\big)x^{-1} = \aleph_{(\mathcal{X}x^{-1})} + bx^{-1} = 1 \quad .$$

Equating the big and little parts of this expression, we obtain two requirements

$$\aleph_{(\mathcal{X}x^{-1})} = \aleph_0 \quad \Longleftrightarrow \quad \mathcal{X}x^{-1} = 0 \quad \Longleftrightarrow \quad x^{-1} = 0 \quad ,$$

and

$$bx^{-1} = 1 \quad \Longleftrightarrow \quad x^{-1} = \frac{1}{b} \quad .$$

This contradicts the requirement $b \in \mathbb{R}_0$, so, therefore, $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ does not have a multiplicative inverse.

Remark 5.2.9 Certain of the products in Axiom 5.2.6 rely on Axiom 5.2.3. For instance, the value in the lower right corner of the multiplication table is computed as

$$\begin{aligned}
\big(\widehat{\infty} - |b|\big)\big(\widehat{\infty} - |a|\big) &= \widehat{\infty} \cdot \widehat{\infty} - |b|\widehat{\infty} - |a|\widehat{\infty} + |ba| \\
&= \aleph_1 \cdot \aleph_1 - |a|\aleph_1 - |b|\aleph_1 + |ba| \\
&= \aleph_{(\aleph_1)} - \aleph_{|a|} - \aleph_{|b|} + |ba| \\
&= \aleph_{(\aleph_1)} - \big(\aleph_{|a|} + \aleph_{|b|}\big) + |ba| \\
&= \aleph_{\widehat{\infty}} + \aleph_{(-|a|-|b|)} + |ba| \\
&= \aleph_{(\widehat{\infty}-|a|-|b|)} + |ba| \quad .
\end{aligned}$$

Furthermore, it follows from Axioms 5.2.3 and 5.2.6 that

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad .$$

This difference is the primary operation behind the original ideation for a non-absorptive infinite element. If a and b are two numbers at distances a and b respectively from the endpoint 0 of the interval $[0, \infty]$, then their difference $a - b$ must be equal (up to a sign) to the difference of two numbers lying at distances a and b from the endpoint ∞ of the same interval.

Example 5.2.10 The purpose of this example is to demonstrate that even while numbers greater than $\widehat{\infty}$ do not exist in real analysis, expressions implying the existence of such are numbers are generally not considered contradictory. Consider the quadratic equation

$$ax^2 + bx + c = 0 \quad , \quad \text{where} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

For every case in which $4ac > b^2$, the number x does not exist in real analysis, and yet it is never claimed that the quadratic formula is contradictory. Instead,

we claim that there must exist an imaginary number $i \notin \mathbb{R}$ with the property $i = \sqrt{-1}$. Therefore, the principle of fractional distance should support a conclusion that there exist transfinite numbers $x \notin \mathbb{R}$ with the property $x > \widehat{\infty}$.

We have seen the existence of such numbers implied previously when examining algebraic infinity as the endpoint of a line segment embedded in a line extending infinitely far in both directions. If we use $x = \tan(x')$ to define $\mathbf{AB} \equiv [0, \widehat{\infty}]$ on $AB \equiv [0, \frac{\pi}{2}]$, and if a number is a cut in a line as per Definition 2.1.5, then there should exist non-real, transfinite numbers which are cuts in an infinite line to the right of $x = \widehat{\infty}$ in the algebraic representation of the point B.

Remark 5.2.11 When the field axioms give the arithmetic operations of $\mathbb{R}$, the difference operations follow from the sum operations as the addition of a product with -1. The $\div$ operations usually follow from the $\times$ operations as multiplication by an inverse. Presently, we may define the difference operations accordingly, but we may not do so for the quotient operations. As demonstrated in Example 5.1.8, the preservation of the respective geometric notions of the algebraic operations requires that $\{+, \times, \div\}$ is a set of three distinct arithmetic operations among which there is not universal associativity. Obviously, this is a major distinction of the present axioms from the field axioms. However, Axiom 5.2.1 grants that all $x \in \mathbb{R}_0$ obey the usual field axioms, so there is an implicit axiom regarding a limited associativity of $\{\times, \div\}$ in the neighborhood of the origin. We will make the explicit with Axiom 5.2.12.

Axiom 5.2.12 Division and multiplication are mutually associative for any $x \in \mathbb{R}_0$. That is, all factors which are elements of $\mathbb{R}_0$ may be moved into or out of quotients and products in the usual way, even if those quotients and products contain $x \notin \mathbb{R}_0$.

Axiom 5.2.13 The operations for $\div$ are given as follows when $a, b \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$. There exists a divisive identity $1 \neq 0$ for every $x \in \mathbb{R}$. It is the same as the multiplicative identity. There exists at least one divisive inverse for every non-zero $x \in \mathbb{R}$. In this table, the row value is the numerator, and the column value is the denominator.

$\div$	0	$y \in \mathbb{R}_0$	$(\aleph_{\mathcal{Y}} + a) \in \mathbb{R}_0^{\mathcal{X}}$	$(\widehat{\infty} - \lvert a \rvert) \in \mathbb{R}_0^1 \cup \widehat{\infty}$
0	nan	0	0	0
x	nan	$\frac{x}{y}$	0	0
$(\aleph_{\mathcal{X}} + b)$	nan	$\aleph_{(\mathcal{X}y^{-1})} + \frac{b}{y}$	$\frac{\mathcal{X}}{\mathcal{Y}}$	$\mathcal{X}$
$(\widehat{\infty} - \lvert b \rvert)$	nan	$\aleph_{(y^{-1})} - \frac{\lvert b \rvert}{y}$	$\frac{1}{\mathcal{Y}}$	1

Example 5.2.14 This example demonstrates that the quotient operations given by Axiom 5.2.13 are well-defined. (This is proven rigorously in Main Theorem 5.5.12.) An operation is well-defined if it generates a unique output. It is obvious in Axiom 5.2.13 that each operation has one and only one output, but it is foreign to the usual understanding of the arithmetic of real numbers that the operands giving the unique resultants are not themselves unique. Consider

$$\frac{\aleph_{\mathcal{X}} + b}{\aleph_{\mathcal{Y}} + a} = \frac{\mathcal{X}}{\mathcal{Y}} \quad .$$

If multiplication was associative with division, and vice versa, then we could multiply both sides by $\aleph_{\mathcal{Y}} + a$ to obtain a contradiction in the form

$$\begin{aligned} \frac{\aleph_{\mathcal{X}} + b}{\aleph_{\mathcal{Y}} + a} \cdot \left(\aleph_{\mathcal{Y}} + a\right) &= \frac{\mathcal{X}}{\mathcal{Y}} \cdot \left(\aleph_{\mathcal{Y}} + a\right) \\ \aleph_{\mathcal{X}} + b \cdot \frac{\aleph_{\mathcal{Y}} + a}{\aleph_{\mathcal{Y}} + a} &= \aleph_{\left(\frac{\mathcal{Y}}{\mathcal{X}}\mathcal{X}\right)} + \frac{\mathcal{X}}{\mathcal{Y}} a \\ \aleph_{\mathcal{X}} + b &= \aleph_{\mathcal{X}} + \frac{\mathcal{X} a}{\mathcal{Y}} \quad . \end{aligned}$$

This is false whenever $b \neq \frac{\mathcal{X}a}{\mathcal{Y}}$, but it is not possible to show this contradiction without assuming associativity among $\{\times, \div\}$.

Example 5.2.15 This example demonstrates another immediate contradiction should we assume associativity among multiplication and division. Axiom 5.2.13 gives

$$\frac{\aleph_{\mathcal{Y}}}{\aleph_{\mathcal{X}}} = \frac{\mathcal{Y}}{\mathcal{X}} \quad , \quad \text{and} \qquad \frac{1}{\aleph_{\mathcal{X}}} = 0 \quad .$$

If we bestow the associativity, then

$$\frac{\aleph_{\mathcal{Y}}}{\aleph_{\mathcal{X}}} = \aleph_{\mathcal{Y}} \cdot \frac{1}{\aleph_{\mathcal{X}}} = \aleph_{\mathcal{Y}} \cdot 0 = 0 \neq \frac{\mathcal{Y}}{\mathcal{X}} \quad .$$

Definition 5.2.16 A divisive identity is a number e satisfying $x \div e = x$. The divisive identity element of $\mathbb{R}$ is $1 \in \mathbb{R}_0$.

Theorem 5.2.17 *All real numbers $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ have a non-unique divisive inverse.*

Proof. If x^{-1} is the divisive inverse of x, then $x \div x^{-1} = 1$. By Axiom 5.2.13, any two $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ having equal big parts are mutual divisive inverses. ∎

Axiom 5.2.18 The ordering of $\mathbb{R}$ is given as follows when $a, b, c, d, x, y \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$. For the table, it is granted that

$$a > b \quad , \quad c > d > 0 \quad , \quad x > y \quad , \quad \text{and} \qquad \mathcal{X} > \mathcal{Y} \quad .$$

This table is such that the row identity is on the left of the given relation and the column identity is on the right.

$\leq$	$y \in \mathbb{R}_0$	$(\aleph_{\mathcal{Y}} + b) \in \mathbb{R}_0^{\mathcal{Y}}$	$(\aleph_{\mathcal{X}} + b) \in \mathbb{R}_0^{\mathcal{X}}$	$(\widehat{\infty} - \lvert d \rvert) \in \mathbb{R}_0^1$	$\widehat{\infty}$
x	$>$	$<$	$<$	$<$	$<$
$(\aleph_{\mathcal{X}} + a)$	$>$	$>$	$>$	$<$	$<$
$(\widehat{\infty} - \lvert c \rvert)$	$>$	$>$	$>$	$<$	$<$

§5.3 Limit Considerations Regarding the Arithmetic Axioms

We have not directly defined infinity hat with the limit definition of infinity. Instead, we have defined $\widehat{\infty}$ to have the same absolute value as ∞. They are both the unincluded endpoint of the interval $[0, \mathcal{I})$ where $\mathcal{I} \notin \mathbb{R}$ is such that $\mathcal{I} = \sup(\mathbb{R})$. Although we began fractional distance analysis with the notion of $\mathbf{AB} \equiv [0, \infty]$ and then proved $[0, \infty) = [0, \widehat{\infty})$ (Theorem 4.3.9), the proposed introduction of an auxiliary convention (Example 4.4.10) relating geometric infinity ∞ to infinite geometric extent rather than the divergence of a limit in $\mathbb{R}$ would allow us to say, informally, and as a matter tangential, hypothetical ideation only, that $[0, \widehat{\infty}) \neq [0, \infty)$ and $[0, \widehat{\infty}] \neq [0, \infty]$. In the convention of Example 4.4.10, ∞ as an endpoint, either included or unincluded, will contradict the notion of infinite geometric extent. In general, we have only introduced this convention as a thinking device, and there is no reason to directly forbid the usual extended real interval $\overline{\mathbb{R}} = [-\infty, \infty]$. Rather, we have only shown that it is better to write $\overline{\mathbb{R}} = [-\widehat{\infty}, \widehat{\infty}]$ because it doesn't suggest the non-existence of the neighborhood of infinity. An interval is a set of numbers equipped with an order relation but not necessarily any operations, and infinity only differs from infinity hat in its operations, so the two intervals $[0, \infty]$ and $[0, \widehat{\infty}]$ must be the same in the present conventions. Although we have not defined $\widehat{\infty}$ directly with the limit definition of ∞ (Definition 2.2.2), having instead deduced its existence independently from (i) the bijectivity of the tangent function on an appropriate neighborhood of the origin and (ii) the geometric invariance of line segments under permutations of the labels of their endpoints, it remains that the magnitude of $\widehat{\infty}$ is given by the limit definition: $|\widehat{\infty}| = \infty$. Since the identity of real numbers is identically their magnitude, and since it is only two alternative sets of arithmetic axioms which separate ∞ and $\widehat{\infty}$, in this section we will study the compliance of the arithmetic axioms with the limit definition of infinity.

Example 5.3.1 Although the limit definition of ∞ is said to be its identical definition (Definition 2.2.2), we cannot always substitute the limit definition of

infinity to directly compute all expressions involving geometric infinity. Consider the use of the limit definition to write

$$\infty - \infty = \left(\lim_{x\to 0} \frac{1}{x}\right) - \left(\lim_{y\to 0} \frac{1}{y}\right) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y-x}{xy} \quad .$$

Generally, this limit does not exist because, for example, we obtain different results on the lines $y = x$ and $y = 2x$. Presently, however, there is only one possible line: the real number line. Along the line $x = y$, we find

$$\infty - \infty = \left(\lim_{x\to 0} \frac{1}{x}\right) - \left(\lim_{x\to 0} \frac{1}{x}\right) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0 \quad .$$

This contradicts Axiom 2.2.3 which gives

$$\infty - \infty = \text{undefined} \quad .$$

To the contrary, if we examine $\widehat{\infty} - \widehat{\infty}$ under the ansatz that this expression may be computed with the limit definition along the line $x = y$, then we find

$$\widehat{\infty} - \widehat{\infty} = \left(\lim_{x\to 0} \frac{1}{x}\right) - \left(\lim_{x\to 0} \frac{1}{x}\right) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0 \quad .$$

This is exactly what is given in Axiom 5.1.2, so the ansatz is borne out. At least sometimes, we can use the limit definition of infinity to compute $\widehat{\infty}$'s operations.

Remark 5.3.2 Example 5.3.1 has demonstrated that although ∞ is directly defined with the limit definition of infinity, we cannot always use that definition to simplify ∞'s expressions. Although $\lim_{x\to 0} \frac{1}{x} - \lim_{y\to 0} \frac{1}{y}$ does not exist in the general case, we exclude $x \neq y$ because for any $y = f(x)$ we have

$$x \to \aleph_1 \quad \Longrightarrow \quad y \to f(\aleph_1) \quad ,$$

and we want a bijective $f : \mathbb{R} \to \mathbb{R}$. When $x = y$, the limit definition does not agree with the axiomatized operation $\infty - \infty$. Still, other identities for ∞ such as Axiom 5.1.3 giving $\frac{b}{\infty} = 0$ for $b \in \mathbb{R}_0$ *do* follow directly from the limit definition of geometric infinity. We have for $b \in \mathbb{R}_0$

$$\frac{b}{\infty} = \frac{b}{\lim_{x\to 0} \frac{1}{x}} = \lim_{x\to 0} \frac{b}{\frac{1}{x}} = \lim_{x\to 0} xb = 0 \quad .$$

In the present section, as in Example 4.3.10, we will take the hat on $\widehat{\infty}$ as a constraint on the freedom of algebraic manipulations involving the limit expression. Particularly, the non-absorptivity of $\widehat{\infty}$ allows us to combine limit expressions but forbids us moving any scalars into these limit expressions. The main purpose of the present section is to demonstrate cases of the validity of the ansatz that sometimes we can correctly compute expressions involving $\widehat{\infty}$ by making the direct substitution with the limit definition. Distinctions between the cases in which we can and cannot do so may be analytically significant.

Theorem 5.3.3 *The property of Axioms 5.2.3 and 5.2.6 giving for* $a, b \in \mathbb{R}_0^+$

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad ,$$

follows from the limit definition of infinity.

Proof. Proof follows from direct substitution of the limit definition of infinity (Definition 2.2.2). We have

$$\begin{aligned}
\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) &= \left[\left(\lim_{x\to 0}\frac{1}{x}\right) - b\right] - \left[\left(\lim_{x\to 0}\frac{1}{x}\right) - a\right] \\
&= \lim_{x\to 0}\left(\frac{1}{x} - b - \frac{1}{x} + a\right) \\
&= \lim_{x\to 0}\big(-b + a\big) \\
&= a - b \quad .
\end{aligned}$$

Theorem 5.3.4 *The property of Axioms 5.2.3 and 5.2.6 giving for* $a, b \in \mathbb{R}_0$ *and* $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$

$$\big(\aleph_{\mathcal{X}} + b\big) - \big(\aleph_{\mathcal{Y}} + a\big) = \aleph_{(\mathcal{X}-\mathcal{Y})} - a + b \quad ,$$

follows from the limit definition of infinity.

Proof. Proof follows from direct substitution of the limit definition of infinity (Definition 2.2.2). We have

$$\begin{aligned}
\big(\aleph_{\mathcal{X}} + b\big) - \big(\aleph_{\mathcal{Y}} + a\big) &= \big(\mathcal{X}\,\widehat{\infty} + b\big) - \big(\mathcal{Y}\,\widehat{\infty} + a\big) \\
&= \left[\mathcal{X}\left(\lim_{x\to 0}\frac{1}{x}\right) + b\right] - \left[\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right) + a\right] \\
&= \big(\mathcal{X} - \mathcal{Y}\big)\left(\lim_{x\to 0}\frac{1}{x}\right) - a + b \\
&= \big(\mathcal{X} - \mathcal{Y}\big)\widehat{\infty} - a + b \\
&= \aleph_{(\mathcal{X}-\mathcal{Y})} - a + b \quad .
\end{aligned}$$

Remark 5.3.5 Theorem 5.3.4 requires clarification because we might have written

$$\begin{aligned}
\big(\aleph_{\mathcal{X}} + b\big) - \big(\aleph_{\mathcal{Y}} + a\big) &= \big(\mathcal{X}\,\widehat{\infty} + b\big) - \big(\mathcal{Y}\,\widehat{\infty} + a\big) \\
&= \left[\left(\lim_{x\to 0}\frac{\mathcal{X}}{x}\right) + b\right] - \left[\left(\lim_{x\to 0}\frac{\mathcal{Y}}{x}\right) + a\right] \\
&= \left(\lim_{x\to 0}\frac{\mathcal{X} - \mathcal{Y}}{x}\right) - a + b
\end{aligned}$$

$$= \widehat{\infty} - a + b \quad .$$

Since $\widehat{\infty} = \aleph_1$, this would necessarily entail a contradiction because the condition $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) \leq 1$ forbids $\mathcal{X} - \mathcal{Y} = 1$. In the above algebraic manipulation, we have given at the second step

$$\aleph_{\mathcal{X}} = \mathcal{X}\,\widehat{\infty} = \lim_{x \to 0} \frac{\mathcal{X}}{x} \quad ,$$

but this contradicts Definition 4.3.7 requiring that $\widehat{\infty}$ does not have absorptive properties. Such a property is explicitly bestowed to the limit definition of infinity when we move the scalar $\mathcal{X}$ into the limit expression. Therefore, it is implicit in the axioms that scalar multipliers of $\widehat{\infty}$ must not be transferred by multiplicative association into the limit expression when substituting the limit definition of algebraic infinity $\widehat{\infty}$. In practice, this has little to no relevance because arithmetic follows from the arithmetic axioms rather than the limit definition of infinity. The purpose of the present section, rather, is to show that at least many of the axioms may be derived from the limit definition, and that ***the present axiomatic framework is very strong*** because many of its axioms are directly provable when we assume the usual associativities, commutativities, and distributivities constrained by the rules of non-absorptivity.

Theorem 5.3.6 *The property of Axioms 5.2.3 and 5.2.6 giving for* $a, b \in \mathbb{R}_0^+$ *and* $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) \leq 1$

$$\left(\aleph_{\mathcal{X}} + b\right) \cdot a = \aleph_{(\mathcal{X}a)} + ba \quad ,$$

follows from the limit definition of infinity.

Proof. Proof follows from direct substitution of the limit definition of infinity. We have

$$\begin{aligned}
\left(\aleph_{\mathcal{X}} + b\right) \cdot a &= \left(\mathcal{X}\,\widehat{\infty} + b\right) \cdot a \\
&= \left[\mathcal{X}\left(\lim_{x \to 0} \frac{1}{x}\right) + b\right] \cdot a \\
&= \mathcal{X}a\left(\lim_{x \to 0} \frac{1}{x}\right) + ba \\
&= \mathcal{X}a\widehat{\infty} + ba \\
&= \aleph_{(\mathcal{X}a)} + ba \quad .
\end{aligned}$$

Theorem 5.3.7 *The property of Axiom 5.2.6 (reliant on Axiom 5.2.3) giving for* $a, b \in \mathbb{R}_0$ *and* $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$

$$\left(\aleph_{\mathcal{X}} + b\right) \cdot \left(\aleph_{\mathcal{Y}} + a\right) = \aleph_{(\aleph_{(\mathcal{X}\mathcal{Y})} + a\mathcal{X} + b\mathcal{Y})} + ba \quad ,$$

follows from the limit definition of infinity.

Proof. Proof of the present theorem follows from direct substitution of the limit definition of infinity. We have

$$\begin{aligned}\left(\aleph_{\mathcal{X}}+b\right)\left(\aleph_{\mathcal{Y}}+a\right) &= \left(\mathcal{X}\,\widehat{\infty}+b\right)\left(\mathcal{Y}\,\widehat{\infty}+a\right)\\ &= \left[\mathcal{X}\left(\lim_{x\to 0}\frac{1}{x}\right)+b\right]\left[\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right)+a\right]\\ &= \mathcal{X}\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right)^{2}+a\mathcal{X}\left(\lim_{x\to 0}\frac{1}{x}\right)+b\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right)+ba \quad .\end{aligned}$$

If we wrote here

$$\widehat{\infty}\cdot\widehat{\infty}=\left(\lim_{x\to 0}\frac{1}{x}\right)^{2}=\lim_{x\to 0}\frac{1}{x^{2}}=\widehat{\infty} \quad ,$$

then that would not exactly violate Definition 4.3.7 because it shows infinity absorbing itself while Definition 4.3.5 gives the multiplicative absorptive property in terms of a composition between $\widehat{\infty}$ and $x\in\mathbb{R}$. However, moving the exponent into the limit violates Definition 4.3.11 requiring that

$$\widehat{\infty}\cdot\widehat{\infty}=\widehat{\infty}\cdot\aleph_{1}=\aleph_{\widehat{\infty}}\neq\aleph_{1}=\widehat{\infty} \quad .$$

Therefore, we finish the proof as

$$\begin{aligned}\left(\aleph_{\mathcal{X}}+b\right)\left(\aleph_{\mathcal{Y}}+a\right) &= \mathcal{X}\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right)\aleph_{1}+a\aleph_{\mathcal{X}}+b\aleph_{\mathcal{Y}}+ba\\ &= \aleph_{\left(\mathcal{X}\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right)\right)}+\aleph_{a\mathcal{X}}+\aleph_{b\mathcal{Y}}+ba\\ &= \aleph_{(\mathcal{X}\mathcal{Y}\cdot\widehat{\infty})}+\aleph_{(a\mathcal{X}+b\mathcal{Y})}+ba\\ &= \aleph_{(\aleph_{(\mathcal{X}\mathcal{Y})}+a\mathcal{X}+b\mathcal{Y})}+ba \quad .\end{aligned}$$

Theorem 5.3.8 *The property of Axiom 5.2.13 giving for $a,b\in\mathbb{R}_{0}$ and $0<\min(\mathcal{X},\mathcal{Y})\leq\max(\mathcal{X},\mathcal{Y})<1$*

$$\frac{\aleph_{\mathcal{X}}+b}{\widehat{\infty}}=\mathcal{X} \quad ,$$

follows from the limit definition of infinity.

Proof. We will use the property that $\mathcal{X}\in\mathbb{R}_{0}$ to allow us move it out of the quotient, as per Axiom 5.2.12. We have

$$\frac{\aleph_{\mathcal{X}}+b}{\widehat{\infty}}=\frac{\mathcal{X}\left(\lim_{x\to 0}\frac{1}{x}\right)}{\lim_{x\to 0}\frac{1}{x}}+\frac{b}{\lim_{x\to 0}\frac{1}{x}}=\mathcal{X}\frac{\lim_{x\to 0}\frac{1}{x}}{\lim_{x\to 0}\frac{1}{x}}+\lim_{x\to 0}xb=\mathcal{X}\lim_{x\to 0}1=\mathcal{X} \quad .$$

Remark 5.3.9 The property of Axiom 5.2.13 giving for $a, b \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$

$$\frac{a}{\aleph_{\mathcal{X}} + b} = 0 \quad ,$$

does not follow from the limit definition of infinity. If we wrote

$$\frac{a}{\aleph_{\mathcal{X}} + b} = \frac{a}{\mathcal{X}\left(\lim_{x\to 0} \frac{1}{x}\right) + b} = \frac{a}{\mathcal{X}} \cdot \frac{1}{\left(\lim_{x\to 0} \frac{1}{x}\right) + \frac{b}{\mathcal{X}}} \quad ,$$

then we would have no way to evaluate the quotient without bringing the denominator's $\frac{b}{\mathcal{X}}$ term into the limit expression. If we did, then the expected zero output would follow directly, but moving that term into the limit expression is not allowed because doing so would give $\widehat{\infty}$ an additive absorptive property.

Theorem 5.3.10 *The quotient of any $\mathbb{R}_0$ number divided by any number with a non-vanishing big part is identically zero.*

Proof. Suppose $x, b \in \mathbb{R}_0^+$ and $0 < \mathcal{X} < 1$, and that

$$\frac{x}{\aleph_{\mathcal{X}} + b} = z \quad .$$

Axiom 5.2.12 allows us to take $x \in \mathbb{R}_0$ out of the quotient, so we may write

$$\frac{1}{\aleph_{\mathcal{X}} + b} = \frac{z}{x} \quad .$$

The quotient is only well defined for $z = 0$. ∎

Theorem 5.3.11 *Quotients of the form $\mathbb{R}_0^{\mathcal{X}} \div \mathbb{R}_0^{\mathcal{Y}}$ are always equal to $\frac{\mathcal{X}}{\mathcal{Y}}$.*

Proof. By Theorem 5.3.10, we have

$$\frac{\aleph_{\mathcal{X}} + b}{\aleph_{\mathcal{Y}} + a} = \frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}} + a} + \frac{b}{\aleph_{\mathcal{Y}} + a} = \frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}} + a} \quad .$$

If $a = 0$, then

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}}} = \frac{\left|\aleph_{\mathcal{X}}\right|}{\left|\aleph_{\mathcal{Y}}\right|} = \frac{\mathcal{X} \lim_{x\to 0} \frac{1}{x}}{\mathcal{Y} \lim_{x\to 0} \frac{1}{x}} = \frac{\mathcal{X}}{\mathcal{Y}} \cdot \frac{\lim_{x\to 0} \frac{1}{x}}{\lim_{x\to 0} \frac{1}{x}} = \frac{\mathcal{X}}{\mathcal{Y}} \cdot \lim_{x\to 0} 1 = \frac{\mathcal{X}}{\mathcal{Y}} \quad .$$

To prove the present theorem in the general case of a, we will demonstrate a contradiction. Suppose $c \neq \frac{\mathcal{X}}{\mathcal{Y}}$, and that

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}} + a} = c \quad .$$

Further suppose that $\mathcal{X} < \mathcal{Y}$ so that we may assume $0 < c < 1$. Then c has a multiplicative inverse and

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{(c\mathcal{Y})} + ca} = 1 \quad .$$

Then

$$\lim_{a \to 0} \frac{\aleph_{\mathcal{X}}}{\aleph_{(c\mathcal{Y})} + ca} = \frac{\aleph_{\mathcal{X}}}{\aleph_{(c\mathcal{Y})}} = \frac{\mathcal{X}}{c\mathcal{Y}} = 1 \quad \Longleftrightarrow \quad c = \frac{\mathcal{X}}{\mathcal{Y}} \quad .$$

It follows that the small part of the denominator does not contribute to the quotient. The case of $\mathcal{X} > \mathcal{Y}$ follows from the case of $a = 0$. The theorem is proven. ∎

Example 5.3.12 This example demonstrates the associativity of multiplication and division for $\mathbb{R}_0$ numbers such as c. Consider the expression

$$c \cdot \frac{\aleph_{\mathcal{X}} + b}{\aleph_{\mathcal{Y}} + a} = c \cdot \frac{\mathcal{X}}{\mathcal{Y}} = \frac{c\mathcal{X}}{\mathcal{Y}} \quad .$$

If we move c into the quotient and perform the multiplication before the division, then

$$c \cdot \frac{\aleph_{\mathcal{X}} + b}{\aleph_{\mathcal{Y}} + a} = \frac{c \cdot \left(\aleph_{\mathcal{X}} + b\right)}{\aleph_{\mathcal{Y}} + a} = \frac{\aleph_{(c\mathcal{X})} + cb}{\aleph_{\mathcal{Y}} + a} = \frac{c\mathcal{X}}{\mathcal{Y}} \quad ,$$

demonstrates that the operation remains well defined with the special associative operations for $\mathbb{R}_0$.

Example 5.3.13 This example treats the negative exponent inverse notation. We have

$$\frac{x}{\aleph_{\mathcal{X}} + b} = 0 \quad \not\Longrightarrow \quad \frac{\aleph_{\mathcal{X}} + b}{x} = \frac{1}{0} \quad .$$

The usual "invert and multiply" rule for dividing by fractions relies on an assumed associativity between multiplication and division, so it cannot be used in certain cases of numbers with non-vanishing big parts. We have

$$\frac{\aleph_{\mathcal{X}} + b}{x} = \aleph_{\left(\frac{\mathcal{X}}{x}\right)} + \frac{b}{x} \quad , \quad \text{and} \quad \left(\frac{x}{\aleph_{\mathcal{X}} + b}\right)^{-1} = \frac{1}{\left(\frac{x}{\aleph_{\mathcal{X}} + b}\right)} = \text{undefined} \quad .$$

§5.4 Field Axioms

In earlier work on the neighborhood of infinity [8], we studied exclusively the maximal neighborhood of infinity using the symbol $\widehat{\mathbb{R}}$ to refer to what we have labeled $\mathbb{R}_0^1$ in the present conventions. To build numbers of the form $x = \widehat{\infty} - b$ in the set $\widehat{\mathbb{R}} \sim \mathbb{R}_0^1$, it was only required to suppress the additive absorption of $\widehat{\infty}$. The remaining multiplicative absorption resulted in certain

(undesirable?) mathematical artifacts which are presently eliminated by the total suppression of all absorptive properties for $\widehat{\infty}$. In this section, we will list those artifacts which are cured in the present conventions, and then we will examine that which remains yet still disagrees with the field axioms. After we have reviewed the issues that were cleared up, we will give a common statement of the field axioms together with the ordering and completeness axioms.

Remark 5.4.1 If $\widehat{\infty}$ retains multiplicative absorption as in previous work [8], then for $n, b \in \mathbb{N}$ we have

$$n\big(\widehat{\infty} - b\big) \leq \big(\widehat{\infty} - b\big) \quad .$$

This ordering relation is not supported by the geometric notion of multiplication. The product of any positive number x multiplied by a natural number n should be greater than or equal to x. The intuitive geometric arrangement of this operation is preserved when $\widehat{\infty}$ has neither additive nor multiplicative absorption. Another cured artifact is observed in the sums of numbers in the maximal neighborhood of infinity. Even without multiplicative absorption, the geometric notion of the difference is preserved with

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad ,$$

but the notion of the sum is not. With multiplicative absorption in place, adding two $\mathbb{R}_0^1$ numbers yields

$$\big(\widehat{\infty} - b\big) + \big(\widehat{\infty} - a\big) = 2\widehat{\infty} - \big(b + a\big) = \widehat{\infty} - \big(b + a\big) \quad . \tag{5.1}$$

The geometric notion of addition would require that the sum of two numbers just less than infinity would *not* be another number just less than infinity. This issue is cured in the present convention with the implicit transfinite ordering $\aleph_{0.9} + \aleph_{0.9} = \aleph_{1.8} \gg \aleph_1$.

The most undesirable artifact (most significant problem?) with allowing $\widehat{\infty}$ to retain multiplicative absorption was the loss of additive associativity. Subtracting $(\widehat{\infty} - c)$ from both sides of Equation (5.1) yields

$$\big[\big(\widehat{\infty} - b\big) + \big(\widehat{\infty} - a\big)\big] - \big(\widehat{\infty} - c\big) = \big[\widehat{\infty} - \big(b + a\big)\big] - \big(\widehat{\infty} - c\big) \quad .$$

Assuming the associative property of addition, we may arrange the LHS brackets to write

$$\begin{gathered}\big(\widehat{\infty} - b\big) + \big[\big(\widehat{\infty} - a\big) - \big(\widehat{\infty} - c\big)\big] = \big[\widehat{\infty} - \big(b + a\big)\big] - \big(\widehat{\infty} - c\big) \\ \widehat{\infty} + \big[c - \big(b + a\big)\big] = c - \big(b + a\big) \quad .\end{gathered}$$

Subtracting the $\mathbb{R}_0$ part from both sides yields the plain contradiction $\widehat{\infty} = 0$. This was avoided, originally, by revoking additive associativity in Reference [8]. In the present conventions, we avoid this result by taking multiplicative absorption away from infinity hat.

While it is permissible, in principle, to have notions of addition and multiplication which are not inherently geometric, it is highly undesirable for basic arithmetic if addition is not associative. Indeed, it is tantamount to arbitrary to say, "$\widehat{\infty}$ has one kind of absorption but not the other," so the present convention is better because it gives operations which are inherently geometric, *and* wherein addition has the highly desirable associative property.

Definition 5.4.2 A field is a set S together with the addition and multiplication operators which satisfies the addition and multiplication axioms for fields: Axioms 5.4.3 and 5.4.5.

Axiom 5.4.3 The addition axioms for fields are

- (A1) S is closed under addition: If $x, y \in S$, then $x + y \in S$.
- (A2) Addition is commutative: If $x, y \in S$, then $x + y = y + x$.
- (A3) Addition is associative: If $x, y, z \in S$, then $(x+y)+z = x+(y+z)$.
- (A4) There exists an additive identity element 0 in S: If $x \in S$, then $x + 0 = x$.
- (A5) Every $x \in S$ has an additive inverse: If $x \in S$, then there exists $-x \in S$ such that $x + (-x) = 0$.

Remark 5.4.4 The arithmetic axioms do not exhibit (A1), but they do exhibit (A2)-(A5).

Axiom 5.4.5 The multiplication axioms for fields are

- (M1) S is closed under multiplication: If $x, y \in S$, then $x \cdot y \in S$.
- (M2) Multiplication is commutative: If $x, y \in S$, then $x \cdot y = y \cdot x$.
- (M3) Multiplication is associative: If $x, y, z \in S$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- (M4) There exists a multiplicative identity element $1 \neq 0$ in S: If $x \in S$, then $x \cdot 1 = x$.
- (M5) If $x \in S$ and $x \neq 0$, then x has a multiplicative inverse: If $x \in S$, then there exists $x^{-1} \in S$ such that $x \cdot x^{-1} = 1$.

Remark 5.4.6 The arithmetic axioms preserve (M2)-(M4), but both of (M1) and (M5) are lost. The loss of (M5) was proven in Theorem 5.2.8.

Definition 5.4.7 An ordered field is a field F together with a relation $<$ which satisfies the field ordering axioms: Axiom 5.4.8.

Axiom 5.4.8 The field ordering axioms are

- (O1) Elements of F have trichotomy. If $x, y \in F$, then one and only one of the following is true: (i) $x < y$, (ii) $x = y$, or (iii) $x > y$.
- (O2) The $<$ relation is transitive. If $x, y, z \in F$, then $x < y$ and $y < z$ together imply $x < z$.
- (O3) If $x, y, z \in F$, then $x < y$ implies $x + z < y + z$.
- (O4) If $x, y, z \in F$, and if $z > 0$, then $x < y$ implies $x \cdot z < y \cdot z$.

It is understood that $x < y$ means $y > x$.

§5.5 Compliance of Cauchy Equivalence Classes with the Arithmetic Axioms

In this section, we give the usual definitions for arithmetic operations on Cauchy equivalence classes. We clarify the meanings for the extended case of $[x] \to [X + x] = [\aleph_{\mathcal{X}} + x]$ (Definition 4.2.13), and then we prove in a few cases that the arithmetic axioms are satisfied by the extended Cauchy equivalence classes $[X + x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \setminus C_{\mathbb{Q}} \implies [X + x] \notin \mathbb{R}_0$. The proofs in this section mostly follow References [11, 12].

Theorem 5.5.1 *Every convergent rational sequence of terms $a_n \in \mathbb{Q}$ is a Cauchy sequence.*

Proof. Per Definition 4.2.2, a sequence $\{a_n\}$ is a Cauchy sequence if and only if

$$\forall \delta \in \mathbb{Q} \quad \exists m, n, N \in \mathbb{N} \quad \text{s.t.} \quad m, n > N \quad \implies \quad \big|a_n - a_m\big| < \delta \ .$$

By the convergence of $\{a_n\}$, it is granted that there exists some $l \in \mathbb{R}$ such that

$$\lim_{n \to \infty} a_n = l \ .$$

Convergence guarantees that

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \implies \quad \big|a_n - l\big| < \frac{\delta}{2} \ .$$

Then, whenever $n, m > N$, we have

$$\big|a_n - a_m\big| = \big|\big(a_n - l\big) - \big(a_m - l\big)\big| \leq \big|a_n - l\big| + \big|a_m - l\big| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \ .$$

Therefore, every convergent rational sequence $\{a_n\}$ is a Cauchy sequence. ∎

Definition 5.5.2 If $x, y \in \mathbb{R}$ are such that there are two Cauchy equivalence classes $x = [(x_n)]$ and $y = [(y_n)]$, then $x + y = [(x_n + y_n)]$ and $x \cdot y = [(x_n \cdot y_n)]$.

Theorem 5.5.3 *The additive operation for equivalence classes given by Definition 5.5.2 is well defined.*

Proof. Define four Cauchy equivalence classes $[(a_n)], [(b_n)], [(c_n)]$, and $[(d_n)]$ having the properties

$$[a] = [b] \quad , \quad \text{and} \qquad [c] = [d] \quad ,$$

so that

$$\lim_{n\to\infty} \big(a_n - b_n\big) = 0 \quad , \quad \text{and} \qquad \lim_{n\to\infty} \big(c_n - d_n\big) = 0 \quad .$$

For addition to be proven well defined, we need to prove that $[(a_n + c_n)] = [(b_n + d_n)]$. This requires

$$[(a_n + c_n)] - [(b_n + d_n)] = 0 \quad .$$

The difference being equal to zero means that for sufficiently large n, and for any $\delta \in \mathbb{R}$, we must have

$$[(a_n + c_n)] - [(b_n + d_n)] = [(a_n - b_n)] + [(c_n - d_n)] < \delta \quad .$$

We will prove this by the method used in Theorem 5.5.1. The limits of $a_n - b_n$ and $c_n - d_n$ approaching zero tell us that

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \qquad \Longrightarrow \qquad \big|a_n - b_n\big| < \frac{\delta}{2} \quad , \quad \big|c_n - d_n\big| < \frac{\delta}{2} \quad .$$

Then, whenever $n, m > N$, we have

$$\big|\big(a_n - b_n\big) + \big(c_m - d_m\big)\big| \leq \big|a_n - b_n\big| + \big|c_m - d_m\big| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad .$$

This proves that $[a+c] = [b+d]$, and that, therefore, addition is a well-defined operation on Cauchy equivalence classes. ∎

Example 5.5.4 This example gives a specific case of Theorem 5.5.3 using numbers in the neighborhood of infinity. Suppose there are four subsets of $C_{\mathbb{Q}}^{\mathbf{AB}}$ with the properties

$$[\aleph_{[\mathcal{X}_1]} + x_1] = [\aleph_{[\mathcal{Y}_1]} + y_1] \quad , \quad \text{and} \qquad [\aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{Y}_2]} + y_2] \quad .$$

Since the big and little parts of equal numbers are equal, we have equality among all the matched pairs of $[x_1], [x_2], [y_1], [y_2], [\mathcal{X}_1], [\mathcal{X}_2], [\mathcal{Y}_1], [\mathcal{Y}_2] \subset C_{\mathbb{Q}}$. If addition is well-defined, then

$$[\aleph_{[\mathcal{X}_1]} + x_1] + [\aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{Y}_1]} + y_1] + [\aleph_{[\mathcal{Y}_2]} + y_2] \quad .$$

Evaluating the left and right sides independently yields

$$[\aleph_{[\mathcal{X}_1]} + x_1] + [\aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{X}_1]} + x_1 + \aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{X}_1 + \mathcal{X}_2]} + x_1 + x_2] \quad ,$$

and

$$[\aleph_{[\mathcal{Y}_1]} + y_1] + [\aleph_{[\mathcal{Y}_2]} + y_2] = [\aleph_{[\mathcal{Y}_1]} + y_1 + \aleph_{[\mathcal{Y}_2]} + y_2] = [\aleph_{[\mathcal{Y}_1+\mathcal{Y}_2]} + y_1 + y_2] \quad .$$

Considering first the small parts, Definition 5.5.2 gives $[x + y] = [x] + [y]$, so

$$[x_1 + x_2] = [y_1 + y_2] \quad \iff \quad [x_1] + [x_2] = [y_1] + [y_2] \quad .$$

This condition follows from Theorem 5.5.3. Considering the big parts yields

$$[\aleph_{[\mathcal{X}_1+\mathcal{X}_2]}] = [\aleph_{[\mathcal{Y}_1+\mathcal{Y}_2]}] \quad \iff \quad [\mathcal{X}_1] + [\mathcal{X}_2] = [\mathcal{Y}_1] + [\mathcal{Y}_2] \quad .$$

It follows as an obvious corollary of Theorem 5.5.3 that the additive operation is well defined for numbers in the neighborhood of infinity.

Remark 5.5.5 To prove that the multiplicative operation is well defined, we will rely on the boundedness of Cauchy sequences. First, we will give the proof of boundedness.

Theorem 5.5.6 *If $\{a_n\}$ is a Cauchy sequence of rationals, then there exists an $M \in \mathbb{R}$ such that $|a_n| < M$ for all $n \in \mathbb{N}$. In other words, every Cauchy sequence of rationals is bounded.*

Proof. Since $\{a_n\}$ is Cauchy, we know there is some sufficiently large $m, n \in \mathbb{N}$ such that

$$\big|a_n - a_m\big| < 1 \quad .$$

If follows for such n that

$$\big|a_{N+1} - a_n\big| < 1 \quad \implies \quad \big(a_{N+1} - 1\big) < a_n < \big(a_{N+1} + 1\big) \quad .$$

Define M as the greatest element of a set with a natural number of elements

$$M = \max\big\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\big\} \quad .$$

Every a_n with $n \leq N$ is in the set, and every a_n with $n > N$ is less than one of the last two elements of the set. Therefore, there exists a bound $M \in \mathbb{R}$ for every rational Cauchy sequence $\{a_n\}$. ∎

Theorem 5.5.7 *The multiplicative operation for equivalence classes given by Definition 5.5.2 is well defined.*

Proof. Define four Cauchy equivalence classes $[(a_n)], [(b_n)], [(c_n)]$, and $[(d_n)]$ having the properties

$$[a] = [b] \quad , \quad \text{and} \quad [c] = [d] \quad ,$$

so that

$$\lim_{n\to\infty} \big(a_n - b_n\big) = 0 \quad , \quad \text{and} \quad \lim_{n\to\infty} \big(c_n - d_n\big) = 0 \quad .$$

For multiplication to be proven well-defined, we need to prove that $[(a_n \cdot c_n)] = [(b_n \cdot d_n)]$, or, specifically, that for sufficiently large n

$$[(a_n \cdot c_n)] - [(b_n \cdot d_n)] < \delta \quad .$$

To that end, insert the additive identity as a difference of cross terms so that

$$\begin{aligned} a_n \cdot c_n - b_n \cdot d_n &= a_n \cdot c_n - b_n \cdot d_n + \big(c_n \cdot b_n - c_n \cdot b_n\big) \\ &= \big(a_n \cdot c_n - c_n \cdot b_n\big) + \big(c_n \cdot b_n - b_n \cdot d_n\big) \\ &= c_n \cdot \big(a_n - b_n\big) + b_n \cdot \big(c_n - d_n\big) \quad . \end{aligned}$$

It follows that

$$\big|a_n \cdot c_n - b_n \cdot d_n\big| \leq \big(|c_n| \cdot |a_n - b_n| + |b_n| \cdot |c_n - d_n|\big) \quad .$$

By Theorem 5.5.6, there exists bounds $|b_n| \leq B_0$ and $|c_n| \leq C_0$ for any $n \in \mathbb{N}$. Then let $M_0 = B_0 + C_0$ so that

$$\big|a_n \cdot c_n - b_n \cdot d_n\big| < M_0\big(|a_n - b_n| + |c_n - d_n|\big) \quad .$$

Since all four sequences are Cauchy, we have

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \Longrightarrow \quad \big|a_n - b_n\big| < \frac{\delta}{2M_0} \quad , \quad \big|c_n - d_n\big| < \frac{\delta}{2M_0} \quad .$$

We prove the theorem by writing

$$\big|a_n \cdot c_n - b_n \cdot d_n\big| < M_0 \left(\frac{\delta}{2M_0} + \frac{\delta}{2M_0}\right) = \delta \quad . \qquad \text{}$$

Remark 5.5.8 Theorem 5.5.6 proves the boundedness of Cauchy sequences of rationals in $C_{\mathbb{Q}}$ but not the boundedness of all sequences in $C_{\mathbb{Q}}^{\mathbf{AB}}$. Since numbers with non-zero big parts are represented as ordered pairs of elements of $C_{\mathbb{Q}}$, it is obvious that such numbers are bounded because each sequence in the pair is bounded. As a consequence of Theorem 5.5.7, which regards general Cauchy equivalence classes and does not restrict to the rationals, it follows that multiplication is well defined with numbers in the neighborhood of infinity. However, one must carefully note that the boundedness of such products will not always be such that the bound is in $\mathbb{R}$. By the identity $\aleph_{\mathcal{X}} \cdot \aleph_{\mathcal{Y}} = \aleph_{\aleph_{(\mathcal{XY})}}$, it can only be in $\mathbb{R}$ if $\mathcal{X} = 0$ and/or $\mathcal{Y} = 0$.

Remark 5.5.9 Assuming the field axioms, Definition 5.5.2 giving $x \cdot y = [(x_n \cdot y_n)]$ is good enough to allow us to prove the arithmetic operations are well defined. However, we have presently not defined division as multiplication by an inverse, so we need to give a definition for the quotient of two Cauchy equivalence classes.

Definition 5.5.10 If $x, y \in \mathbb{R}$ are such that there are two Cauchy equivalence classes $x = [(x_n)]$ and $y = [(y_n)]$, then $x \div y = [(x_n \div y_n)]$.

Theorem 5.5.11 *The quotient operation for equivalence classes of rationals given by Definition 5.5.10 is well defined.*

Proof. Define four Cauchy equivalence classes $[(a_n)], [(b_n)], [(c_n)]$, and $[(d_n)]$ having the properties

$$[a] = [b] \quad , \quad \text{and} \qquad [c] = [d] \quad ,$$

so that

$$\lim_{n\to\infty} \big(a_n - b_n\big) = 0 \quad , \quad \text{and} \qquad \lim_{n\to\infty} \big(c_n - d_n\big) = 0 \quad .$$

For division to be proven well defined, we need to prove that $[(a_n \div c_n)] = [(b_n \div d_n)]$. Specifically, for sufficiently large n, we must demonstrate

$$[(a_n \div c_n)] - [(b_n \div d_n)] < \delta \quad .$$

To that end, insert the additive identity as a difference of the cross terms so that

$$\begin{aligned} \frac{a_n}{c_n} - \frac{b_n}{d_n} &= \frac{a_n}{c_n} - \frac{b_n}{d_n} + \left(\frac{b_n}{c_n} - \frac{b_n}{c_n}\right) \\ &= \left(\frac{a_n}{c_n} - \frac{b_n}{c_n}\right) + \left(\frac{b_n}{c_n} - \frac{b_n}{d_n}\right) \\ &= \frac{a_n - b_n}{c_n} + \frac{b_n \cdot \big(d_n - c_n\big)}{c_n \cdot d_n} \quad . \end{aligned}$$

It follows that

$$\left|\frac{a_n}{c_n} - \frac{b_n}{d_n}\right| \leq \left(\frac{|a_n - b_n|}{|c_n|} + \frac{|b_n| \cdot |c_n - d_n|}{|c_n| \cdot |d_n|}\right) \quad .$$

By Theorem 5.5.6, there exist bounds $|b_n| \leq B_0$, $|c_n| \leq C_0$ and $|d_n| \leq D_0$ for any $n \in \mathbb{N}$. Since all four sequences are Cauchy, we have

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \Longrightarrow \quad \big|a_n - b_n\big| < \frac{C_0\delta}{2} \quad , \quad \big|c_n - d_n\big| < \frac{C_0 D_0 \delta}{2B_0} \quad .$$

We prove the theorem by writing

$$\left|\frac{a_n}{c_n} - \frac{b_n}{d_n}\right| < \left(\frac{\frac{C_0\delta}{2}}{C_0} + \frac{B_0 \frac{C_0 D_0 \delta}{2B_0}}{C_0 D_0}\right) = \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad .$$

Since we have assumed $[a], [b], [c], [d] \subset C_{\mathbb{Q}}$, we have proven the theorem with Axiom 5.2.12 granting associativity among division and multiplication. ∎

Main Theorem 5.5.12 *The quotient operation given by Definition 5.5.10 is well defined for equivalence classes in* $C_{\mathbb{Q}}^{\mathbf{AB}} \setminus C_{\mathbb{Q}}$.

Proof. Suppose there are four subsets of $C_{\mathbb{Q}}^{\mathbf{AB}}$ with the properties

$$[\aleph_{[\mathcal{A}]} + a] = [\aleph_{[\mathcal{B}]} + b] \quad , \quad \text{and} \qquad [\aleph_{[\mathcal{C}]} + c] = [\aleph_{[\mathcal{D}]} + d] \quad .$$

It follows from the equality of Cauchy sequences that

$$\begin{aligned} \lim_{n\to\infty} \left(\mathcal{A}_n - \mathcal{B}_n\right) &= 0 \\ \lim_{n\to\infty} \left(\mathcal{C}_n - \mathcal{D}_n\right) &= 0 \\ \lim_{n\to\infty} \left(a_n - b_n\right) &= 0 \\ \lim_{n\to\infty} \left(c_n - d_n\right) &= 0 \quad . \end{aligned}$$

For concision in notation, introduce the symbols

$$\begin{aligned} (A_n) &= (\aleph_{[(\mathcal{A}_n)]} + a_n) \\ (B_n) &= (\aleph_{[(\mathcal{B}_n)]} + b_n) \\ (C_n) &= (\aleph_{[(\mathcal{C}_n)]} + c_n) \\ (D_n) &= (\aleph_{[(\mathcal{D}_n)]} + d_n) \quad . \end{aligned}$$

For division to be proven well defined, we need to prove that $[(A_n \div C_n)] = [(B_n \div D_n)]$. Specifically, for sufficiently large n, we must demonstrate

$$[(A_n \div C_n)] - [(B_n \div D_n)] < \delta \quad .$$

Following the form of Theorem 5.5.11, we may insert the identity to obtain the inequality

$$\left| \frac{A_n}{C_n} - \frac{B_n}{D_n} \right| \leq \left(\frac{|A_n - B_n|}{|C_n|} + \frac{|B_n| \cdot |C_n - D_n|}{|C_n| \cdot |D_n|} \right) \quad .$$

Here, we make the major distinction with Theorem 5.5.11: the bounds on $(A_n), (B_n), (C_n), (D_n)$ are not in $\mathbb{R}_0$, and we must be careful not to allow associativity among multiplication and division when simplifying the expression. Since each of $(A_n), (B_n), (C_n), (D_n)$ are ordered pairs of Cauchy sequences of rationals (Axiom 4.2.17), we know the pairs of sequences are bounded. Let the bounds be defined by

$$\begin{aligned} [(A_n)] &= ([\mathcal{A}], [a]) \leq (A_0, a_0) \\ [(B_n)] &= ([\mathcal{B}], [b]) \leq (B_0, b_0) \\ [(C_n)] &= ([\mathcal{C}], [c]) \leq (C_0, c_0) \\ [(D_n)] &= ([\mathcal{D}], [d]) \leq (D_0, d_0) \quad , \end{aligned}$$

where the notation implies the ordering of each paired element respectively. It follows that

$$\begin{aligned}
\left|\frac{A_n}{C_n} - \frac{B_n}{D_n}\right| &\leq \frac{\left|\aleph_{A_0} + a_0 - \aleph_{B_0} - b_0\right|}{\left|\aleph_{C_0} + c_0\right|} + \frac{\left|\aleph_B + b_0\right| \cdot \left|\aleph_{C_0} + c_0 - \aleph_{D_0} - d_0\right|}{\left|\aleph_{C_0} + c_0\right| \cdot \left|\aleph_{D_0} + d_0\right|} \\
&\leq \frac{\left|\aleph_{(A_0 - B_0)} + a_0 - b_0\right|}{\left|\aleph_{C_0} + c_0\right|} + \frac{\left|\aleph_B + b_0\right| \cdot \left|\aleph_{(C_0 - D_0)} + c_0 - d_0\right|}{\left|\aleph_{C_0} + c_0\right| \cdot \left|\aleph_{D_0} + d_0\right|} \\
&\leq \frac{\left|A_0 - B_0\right|}{\left|C_0\right|} + \frac{\left|\aleph_{\left(\aleph_{(B_0C_0 - B_0D_0)} + B_0c_0 - B_0d_0 + b_0C_0 - b_0D_0\right)} + b_0c_0 - b_0d_0\right|}{\left|\aleph_{\left(\aleph_{(C_0D_0)} + D_0c_0 + d_0C_0\right)} + d_0c_0)\right|} \\
&\leq \frac{\left|A_0 - B_0\right|}{\left|C_0\right|} + \frac{\left|\aleph_{(B_0C_0 - B_0D_0)} + B_0c_0 - B_0d_0 + b_0C_0 - b_0D_0\right|}{\left|\aleph_{(C_0D_0)} + D_0c_0 + d_0C_0\right|} \\
&\leq \frac{\left|A_0 - B_0\right|}{\left|C_0\right|} + \frac{\left|B_0C_0 - B_0D_0\right|}{\left|C_0D_0\right|} \\
&\leq \frac{\left|A_0 - B_0\right|}{\left|C_0\right|} + \frac{\left|B_0\right| \cdot \left|C_0 - D_0\right|}{\left|C_0\right| \cdot \left|D_0\right|} \quad .
\end{aligned}$$

Since $A_0, B_0, C_0, D_0 \in \mathbb{R}_0$, this is the same form achieved in Theorem 5.5.11, and we will conclude the proof in the same way. Use the Cauchy property of the respective sequences to write

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \Longrightarrow \quad \left|A_n - B_n\right| < \frac{C_0\delta}{2} \quad , \quad \left|C_n - D_n\right| < \frac{C_0D_0\delta}{2B_0} \quad .$$

We prove the theorem writing

$$\left|\frac{A_n}{C_n} - \frac{B_n}{D_n}\right| < \frac{\frac{C_0\delta}{2}}{C_0} + \frac{B_0\frac{C_0D_0\delta}{2B_0}}{C_0D_0} = \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad . \qquad \text{leaf}$$

§6 Arithmetic Applications

§6.1 Properties of the Algebraic Fractional Distance Function Revisited

We have defined the algebraic FDF $\mathcal{D}^{\dagger}_{AB}$ to totally replicate the behavior of the geometric FDF $\mathcal{D}_{AB}$ with the added property that it should allow us to compute numerical quotients of the form $\frac{AX}{AB}$ in the absence of a supplemental constraint in the form $AX = cAB$, as might be obtained with a ruler. In verbose notation, we have

$$\mathcal{D}_{\mathbf{AB}} : \mathbf{AB} \to [0, 1] \quad , \quad \text{and} \qquad \mathcal{D}^{\dagger}_{\mathbf{AB}} : \{[0, \widehat{\infty}]; x\} \to [0, 1] \quad ,$$

so that the algebraic FDF provides more information by taking the line segment and the chart on the line segment whereas the geometric FDF doesn't know about the chart x.

In Section 3.1, we found that neither the algebraic FDF of the first kind nor the second has the analytic form of $\mathcal{D}^{\dagger}_{AB}$. The second kind was ruled out by Theorem 3.1.19 when we showed that $\mathcal{D}''_{AB}$ is not one-to-one. $\mathcal{D}'_{AB}$ was provisionally eliminated based on an unallowable discontinuity at infinity. Since $\mathcal{D}_{AB}$ is continuous on its domain, $\mathcal{D}^{\dagger}_{AB}$ is too. In Theorem 3.1.24 specifically, we showed that $\mathcal{D}'_{AB}$ cannot conform to the Cauchy criterion for continuity at infinity because that criterion always fails at infinity. The nature of the failure is the requirement

$$\left|x-\infty\right|<\delta \quad \iff \quad \delta>\infty \ .$$

There is no such $\delta\in\mathbb{R}$. What is the source of this discrepancy? The source is the additive absorptive property of infinity giving $\infty-x=\infty$. By now, however, we have shown that the absorptive properties of all infinite elements are not supported by the invariance of line segments under permutations of their endpoints, and we have otherwise given an artificial construction $\widehat{\infty}$ which does not have the problematic properties. With infinity hat in mind, now we will revisit the continuity and other properties of $\mathcal{D}'_{AB}$. We will show that ***the algebraic FDF of the first kind <u>does</u> satisfy the Cauchy criterion for a limit at infinity***, something which has been considered historically impossible. In the present section, we will also prove Conjecture 3.1.18 wherein it was postulated that $\mathcal{D}'_{AB}$ is injective. Having shown by the end of the present section that there are no obvious discrepancies between $\mathcal{D}^{\dagger}_{AB}$ and $\mathcal{D}'_{AB}$, we will assume that the algebraic FDF of the first kind is identically $\mathcal{D}^{\dagger}_{AB}$.

Main Theorem 6.1.1 *The algebraic fractional distance function of the first kind* $\mathcal{D}'_{\mathbf{AB}}(AX)$ *converges to a limit* $l=1$ *at* $B\in\mathbf{AB}$.

<u>Proof.</u> According to the Cauchy definition of the limit of $f(x)$ as x approaches $\widehat{\infty}$, we say that

$$\lim_{x\to\widehat{\infty}} f(x)=l \ ,$$

if and only if

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \text{s.t.} \quad \forall x\in D \ ,$$

we have

$$0<\left|x-\widehat{\infty}\right|<\delta \quad \implies \quad \left|f(x)-l\right|<\varepsilon \ .$$

In Theorem 3.1.24, we attempted to show this limit in the approach to geometric infinity $x\to\infty$. At that point, we had to stop because there is no $\delta\in\mathbb{R}$ such that $\infty-x<\delta$. Now we may choose $x\in\mathbb{R}$ with the given arithmetic axioms to obtain, for example,

$$\left|(\widehat{\infty}-b)-\widehat{\infty}\right|=b \ , \quad \text{or} \qquad \left|\aleph_{\mathcal{X}}-\widehat{\infty}\right|=\aleph_{(1-\mathcal{X})} \ .$$

Per the ordering axiom (Axiom 5.2.18), either of these can be less than some $\delta\in\mathbb{R}$. This remedies the blockage encountered in Theorem 3.1.24 where we

found that $\delta \in \mathbb{R}$ implies $\infty - x \not< \delta$. Now we may follow the usual prescription for the Cauchy definition of a limit, even at infinity! To that end, let $\delta = \aleph_{\left(\frac{\varepsilon}{2}\right)}$. Then the Cauchy definition requires that

$$0 < \left| x - \widehat{\infty} \right| < \aleph_{\left(\frac{\varepsilon}{2}\right)} \quad , \quad \text{and} \qquad \left| \mathcal{D}'_{\mathbf{AB}}(AX) - \mathcal{D}'_{\mathbf{AB}}(AB) \right| < \varepsilon \quad .$$

First, we will evaluate δ expression on the left as

$$\widehat{\infty} - x < \aleph_{\left(\frac{\varepsilon}{2}\right)} \qquad \Longleftrightarrow \qquad x > \aleph_{\left(1-\frac{\varepsilon}{2}\right)} \quad .$$

Definition 3.1.9 gives $\mathcal{D}'_{AB}$ as

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} & \text{for} \quad X \neq A,\ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} ,$$

where

$$\frac{\|AX\|}{\|AB\|} = \frac{\text{len}[a, x]}{\text{len}[a, b]} \quad .$$

Thus, evaluation of $|\mathcal{D}'_{\mathbf{AB}}(AX) - \mathcal{D}'_{\mathbf{AB}}(AB)|$ yields

$$\left| \frac{\text{len}[0, x]}{\text{len}[0, \widehat{\infty}]} - 1 \right| = \left| \frac{x}{\widehat{\infty}} - 1 \right| < \left| \frac{\aleph_{\left(1-\frac{\varepsilon}{2}\right)}}{\widehat{\infty}} - 1 \right| = \left| \left(1 - \frac{\varepsilon}{2}\right) - 1 \right| = \left| -\frac{\varepsilon}{2} \right| < \varepsilon \quad .$$

Therefore,

$$\lim_{x \to \widehat{\infty}} \mathcal{D}'_{\mathbf{AB}}(AX) = 1 \quad .$$

This limit demonstrates the continuity of $\mathcal{D}'_{\mathbf{AB}}$ at infinity.

Remark 6.1.2 When defining $\mathcal{D}'_{AB}$ and $\mathcal{D}''_{AB}$ in Section 3.1, we were able to show that $\mathcal{D}''_{AB}$ is not one-to-one, but we did not yet have the tools to prove that $\mathcal{D}'_{AB}$ is one-to-one on all real line segments. We conjectured it with Conjecture 3.1.18, and now we will use Lemma 6.1.3 to prove it in Theorem 6.1.4.

Lemma 6.1.3 *For any point $X \equiv \mathscr{X} = [x_1, x_2]$ in a real line segment AB, we have $x_1 \in \mathbb{R}_{\aleph}^{\mathcal{X}_0}$ if and only if $x_2 \in \mathbb{R}_{\aleph}^{\mathcal{X}_0}$.*

Proof. For proof by contradiction, suppose $x_1 \in \mathbb{R}_{\aleph}^{\mathcal{X}_1}$, $x_2 \in \mathbb{R}_{\aleph}^{\mathcal{X}_2}$, and $\mathcal{X}_1 \neq \mathcal{X}_2$. By Definition 4.1.14, there exist $b_1, b_2 \in \mathbb{R}_{\aleph}^{0}$ such that

$$x_1 = \aleph_{\mathcal{X}_1} + b_1 \quad , \quad \text{and} \qquad x_2 = \aleph_{\mathcal{X}_2} + b_2 \quad .$$

With the $a \leq b$ condition inherent to the $[a, b]$ interval notation, the algebraic FDF tells us that

$$\min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \infty]} = \frac{x_1}{\infty} = \mathcal{X}_1 \quad ,$$

and

$$\max[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{\text{len}[0, x_2]}{\text{len}[0, \infty]} = \frac{x_2}{\infty} = \mathcal{X}_2 \quad .$$

It follows from the identity $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX) = \mathcal{D}_{\mathbf{AB}}(AX)$ that

$$\min[\mathcal{D}_{\mathbf{AB}}(AX)] = \mathcal{X}_1 \quad , \quad \text{and} \qquad \max[\mathcal{D}_{\mathbf{AB}}(AX)] = \mathcal{X}_2 \quad .$$

By Definition 3.1.1, $\mathcal{D}_{\mathbf{AB}}(AX)$ is one-to-one which requires

$$\mathcal{X}_1 = \mathcal{X}_2 \quad .$$

This contradicts the assumed condition that $\mathcal{X}_1 \neq \mathcal{X}_2$.

Theorem 6.1.4 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ is injective (one-to-one) on all real line segments.*

Proof. (Proof of Conjecture 3.1.18.) Recall that $\mathcal{D}'_{AB} : AB \to [0, 1]$ is

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} = \dfrac{\text{len}[a, x]}{\text{len}[a, b]} & \text{for} \quad X \neq A, \ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Injectivity requires that

$$\mathcal{D}'_{AB}(AX_1) = \mathcal{D}'_{AB}(AX_2) \quad \Longleftrightarrow \quad AX_1 = AX_2 \quad \Longleftrightarrow \quad X_1 = X_2 \quad .$$

Even if there is an entire interval of numbers in the algebraic representations of each of X_1 and X_2, we have by Lemma 6.1.3:

$$\min[\mathcal{D}'_{AB}(AX_k)] = \max[\mathcal{D}'_{AB}(AX_k)] = \mathcal{X}_k \quad .$$

This tells us that choosing any $x \in \mathscr{X} \equiv X$ will yield the same $\mathcal{D}'_{AB}(AX)$. Therefore, the injectivity of $\mathcal{D}'_{AB}(AX)$ follows from the injectivity of $\mathcal{D}_{AB}(AX)$ through the constraint

$$\mathcal{D}'_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

Conjecture 6.1.5 The algebraic fractional distance function $\mathcal{D}^{\dagger}_{AB}$ is an algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$.

§6.2 Some Theorems for Real Numbers in the Neighborhood of Infinity

In Section 3.3, we listed four coarse bins of length as distinct modes in which a line segment might have a many-to-one or one-to-one relationship between its points and the numbers in their algebraic representations. The bins were

- $L \in \mathbb{R}_0$
- $L \in \mathbb{R}_{\aleph}^0 \setminus \mathbb{R}_0$
- $L \in \{\mathbb{R}_{\aleph}^{\mathcal{X}}\} \cup \mathbb{R}_{\aleph}^1$ (Recall that $0 < \mathcal{X} < 1$ is implicit in the absence of explicit statements to the contrary.)
- $L = \widehat{\infty}$

In Theorems 3.3.1 and 3.3.2, we were able to prove the cases $L \in \mathbb{R}_0$ and $L = \widehat{\infty}$, but we did not yet have sufficient tools to easily demonstrate the cases of $L \in \mathbb{R}_{\aleph}^0 \setminus \mathbb{R}_0$ and $L \in \{\mathbb{R}_{\aleph}^{\mathcal{X}}\} \cup \mathbb{R}_{\aleph}^1$. We still have not decided whether or not $\mathbb{R}_{\aleph}^0 \setminus \mathbb{R}_0 = \varnothing$ but, by this point, we have developed the tools needed to prove the many-to-one relationship between real numbers and points in a line segment with $L \in \{\mathbb{R}_0^{\mathcal{X}}\} \cup \mathbb{R}_0^1$. This is the third case above modified with a restriction to the natural neighborhoods of $\aleph_{\mathcal{X}}$ rather than the whole neighborhoods. This restriction guarantees $\text{Lit}(L) \in \mathbb{R}_0$. We will give this result in the present section which also contains various and sundry theorems and examples, the most exciting of which is left as a surprise!

Theorem 6.2.1 *If AB is a real line segment with finite length $L \in \{\mathbb{R}_0^{\mathcal{X}}\} \cup \mathbb{R}_0^1$, then no point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. From the statement of the theorem, we have $L = \text{len}(AB) = \aleph_{\mathcal{X}} + b$ with $0 < \mathcal{X} \leq 1$ and $b \in \mathbb{R}_0$. By Definition 2.3.14, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

It follows that

$$\min[\mathcal{D}_{AB}^{\dagger}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \aleph_{\mathcal{X}} + b]} = \frac{x_1}{\aleph_{\mathcal{X}} + b} \quad .$$

Now suppose $x_0 \in \mathbb{R}_0^+$ and $z = x_1 + x_0$, so $z > x_1$. Then

$$\frac{\text{len}[0, z]}{\text{len}[0, \aleph_{\mathcal{X}} + b]} = \frac{z}{\aleph_{\mathcal{X}} + b} = \frac{x_1 + x_0}{\aleph_{\mathcal{X}} + b} = \frac{x_1}{\aleph_{\mathcal{X}} + b} + \frac{x_0}{\aleph_{\mathcal{X}} + b} \quad .$$

By Axiom 5.2.13, the x_0 term vanishes, so we find

$$\frac{\text{len}[0, z]}{\text{len}[0, \aleph_{\mathcal{X}} + b]} = \frac{x_1}{\aleph_{\mathcal{X}} + b} = \min[\mathcal{D}_{AB}^{\dagger}(AX)] \quad .$$

Invoking the single-valuedness of bijective functions, we find that

$$\min[\mathcal{D}^{\dagger}_{AB}(AX)] = \max[\mathcal{D}^{\dagger}_{AB}(AX)] = \frac{x_2}{\aleph_{\mathcal{X}} + b} \quad \Longrightarrow \quad x_1 < z \leq x_2 \ .$$

Therefore $x_1 \neq x_2$, and the theorem is proven. ∎

Theorem 6.2.2 *The derivative of $f(x) = \aleph_x$ with respect to x is infinite.*

Proof. The definition of the derivative of $f(x)$ with respect to x is

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \ .$$

For $f(x) = \aleph_x$, we have

$$\begin{aligned} \frac{d}{dx} \aleph_x &= \lim_{\Delta x \to 0} \frac{\aleph_{(x+\Delta x)} - \aleph_x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\aleph_x + \aleph_{\Delta x} - \aleph_x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \aleph_{\Delta x} \\ &= \aleph_1 \ . \end{aligned}$$

∎

Definition 6.2.3 For $0 < \mathcal{X} < 1$, $\mathbb{N}_{\mathcal{X}}$ is a subset of real numbers such that

$$\mathbb{N}_{\mathcal{X}} = \left\{ \aleph_{\mathcal{X}} + w \;\middle|\; w \in \mathbb{W} \right\} \ ,$$

where the whole numbers are $\mathbb{W} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. The set $\{\mathbb{N}_{\mathcal{X}}\}$ is called the set of all $\mathbb{N}_{\mathcal{X}}$ such that $0 < \mathcal{X} < 1$. Complementing $\mathbb{N}$ in the neighborhood of the origin, define a set

$$\widehat{\mathbb{N}} = \left\{ \widehat{\infty} - n \;\middle|\; n \in \mathbb{N} \right\} \ ,$$

called natural numbers in the maximal neighborhood of infinity. The set of all extended natural numbers is

$$\mathbb{N}_{\infty} = \mathbb{N} \cup \{\mathbb{N}_{\mathcal{X}}\} \cup \widehat{\mathbb{N}} \ .$$

Definition 6.2.4 The function E^x is defined as

$$E^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \ ,$$

where the sum is taken to mean all $k \in \mathbb{N}_{\infty} \cup \{0\}$. This function is called the big exponential function.

Theorem 6.2.5 *For any $x \in \mathbb{R}_0$, the big exponential function is equal to the usual exponential function:*

$$x \in \mathbb{R}_0 \quad \Longrightarrow \quad E^x = e^x \quad .$$

Proof. The usual exponential function is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad ,$$

with the upper bound on k meaning "as the integer k increases without bound." This notation also gives an implicit restriction $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. To prove the theorem, it will suffice to show that all terms vanish when $k \neq 0$ and $k \notin \mathbb{N}$. We have

$$\begin{aligned} E^x &= \sum_{k \in \mathbb{N}_0} \frac{x^k}{k!} + \sum_{k \in \mathbb{N}_{\mathcal{X}_1}} \frac{x^k}{k!} + \sum_{k \in \mathbb{N}_{\mathcal{X}_2}} \frac{x^k}{k!} + \dots \\ &= e^x + \sum_{\substack{k=0 \\ k \in \mathbb{N}_0}}^{\infty} \frac{x^{(\aleph_{\mathcal{X}_1} + k)}}{\big(\aleph_{\mathcal{X}_1} + k\big)!} + \sum_{\substack{k=1 \\ k \in \mathbb{N}}}^{\infty} \frac{x^{(\aleph_{\mathcal{X}_1} - k)}}{\big(\aleph_{\mathcal{X}_1} - k\big)!} + \sum_{\substack{k=0 \\ k \in \mathbb{N}_0}}^{\infty} \frac{x^{(\aleph_{\mathcal{X}_2} + k)}}{\big(\aleph_{\mathcal{X}_2} + k\big)!} + \dots \quad . \end{aligned}$$

Now it will suffice to show that the sum over $k \in \mathbb{N}_{\mathcal{X}}$ vanishes for any $\mathcal{X} > 0$. This follows if the summed quotients vanish for all such k. Although multiplication and division are not associative in general, Axiom 5.2.12 allows us to move factors of $x \in \mathbb{R}_0$ into or out of quotients, as per usual. Therefore, let $z = x^{-1}$ so that

$$\frac{x^{(\aleph_{\mathcal{X}} \pm k)}}{\big(\aleph_{\mathcal{X}} \pm k\big)!} = \frac{1}{z} \frac{x^{(\aleph_{\mathcal{X}} \pm k - 1)}}{\big(\aleph_{\mathcal{X}} \pm k\big)\big(\aleph_{\mathcal{X}} \pm k - 1\big)!} = \frac{x^{(\aleph_{\mathcal{X}} \pm k - 1)}}{\big(\aleph_{(z\mathcal{X})} \pm kz\big)\big(\aleph_{\mathcal{X}} \pm k - 1\big)!} \quad .$$

The number of parenthetical terms in the factorial is equal to the number of factors of x in the numerator, so, after bringing down every factor of x, we obtain

$$\frac{x^{(\aleph_{\mathcal{X}} \pm k)}}{\big(\aleph_{\mathcal{X}} \pm k\big)!} = \frac{1}{\big(\aleph_{(z\mathcal{X})} \pm kz\big)\big(\aleph_{(z\mathcal{X})} \pm kz - z\big)\dots} = 0 \quad .$$

This quotient always vanishes if $\mathcal{X} > 0$ and $x \in \mathbb{R}_0$. The theorem is proven. ∎

Example 6.2.6 This example gives a good thinking device for understanding limits in the form $n \to \infty$ when n steps in integer increments. Usually, $n \to \infty$ is taken to mean "as the iterator n increases without bound," but, in this example, we will suggest that $n \to \infty$ might be better read as "the sum over every $n \in \mathbb{N}_\infty$," at least in certain contexts.

Definition 2.2.2 gives two definitions for the ∞ symbol, one of which is

$$\lim_{n\to\infty}\sum_{k=1}^{n} k = \infty \quad .$$

Axiom 5.2.1 grants the closure of $\mathbb{R}_0$ under its operations, so the partial sums will always be another $\mathbb{R}_0$ number for any $n \in \mathbb{N}$. In other words, it will always be less than $\aleph_{\mathcal{X}}$ for any $\mathcal{X} > 0$. However, the statement "as n increases without bound" induces the notion of the non-convergence of the partial sums. In turn, this allows us to think of the sum as exceeding certain $\aleph_{\mathcal{X}}$, so it may more plainly demonstrate the notion of non-convergence when we take $n \to \infty$ to mean the sum over all $n \in \mathbb{N}_\infty$. In that case, the partial sums will eventually have individual terms greater than $\aleph_{\mathcal{X}}$ for any $0 < \mathcal{X} < 1$, and it is obvious that $\aleph_{\mathcal{X}}$ cannot be an upper bound on the partial sums over $n \in \mathbb{N}_\infty$. Taking $m \in \mathbb{N}$, observe that the $n \in \mathbb{N}_\infty$ convention gives

$$\lim_{n\to\infty}\sum_{k=1}^{n} k > \lim_{n\to m}\sum_{k=1}^{n}\left(\aleph_{(m^{-1})} + k\right) > m\aleph_{(m^{-1})} = \aleph_1 \quad .$$

It is obvious that the limit of the partial sums diverges in $\mathbb{R}$ as the big part of the partial sums will easily exceed $\aleph_1 = \widehat{\infty}$. Certainly, it is obvious that the partial sums diverge if $n \in \mathbb{N}$ or $n \in \mathbb{N}_\infty$, but it may be *more* obvious when $n \in \mathbb{N}_\infty$. When n is said to increase without bound and is also taken as $n \in \mathbb{N}$, then there is an intuitive hiccup seeing that the sequence of the sums should diverge when every element in the sequence of partial sums is less than any $\aleph_{\mathcal{X}} \in \{\mathbb{R}^{\mathcal{X}}\}$. So, instead, it may be better to think of the $n \to \infty$ notation as meaning the sum over all $n \in \mathbb{N}_\infty$.

This example has demonstrated the utility of $\mathbb{N}_\infty$ as a thinking device, and it makes a distinction between the two formulae

$$\lim_{x\to 0^+}\frac{1}{x} = \left|\widehat{\infty}\right| \quad , \quad \text{and} \qquad \lim_{n\to\infty}\sum_{k=1}^{n} k = \infty \quad . \tag{6.1}$$

In the partial sums definition, and under the $\mathbb{N}_\infty$ convention, the distinction between geometric infinity and algebraic infinity is suggested as

$$\widehat{\infty} = \aleph_1 \quad , \quad \text{and} \qquad \infty = \aleph_{\aleph_{\aleph_{\cdots}}} \quad .$$

This would require a significant revision of the entire text to accommodate $|\aleph_1| \neq \infty$, but we point out the possibility of the alternative convention with a nod toward future inquiry. Until then, we still have $|\widehat{\infty}| = \infty$. Per Definition 2.2.1, both limits in Equation (6.1) diverge in $\mathbb{R}$, and we cannot differentiate $|\aleph_1|$ from $|\aleph_{\aleph_{\cdots}}|$ without first making a transfinite continuation. This continuation is surely something to be explored because it is the longitudinal continuation of $\mathbb{R}$ beyond its endpoints perfectly dual to the famous transverse continuation of $\mathbb{R}$ onto $\mathbb{C}$. Where the latter has yielded so much fruit in the history of mathematics, the former ought to bear some fruit as well.

Main Theorem 6.2.7 *If ABC is a right triangle such that (i) $\angle ABC = \frac{\pi}{2}$, (ii) $\|AB\|, \|BC\| \in \{\mathbb{R}_0^{\mathcal{X}}\}$, and (iii) $\|AB\| \neq c_0\|BC\|$, and if the Pythagorean theorem is phrased as*

$$\|AC\| = \sqrt{\|AB\|^2 + \|BC\|^2} \quad , \quad \textit{with} \qquad \|AC\| = \textit{len}(AC) \quad ,$$

then

$$\textit{len}(AC) \notin \mathbb{R}_{\aleph}^0 \cup \{\mathbb{R}_{\aleph}^{\mathcal{X}}\} \cup \mathbb{R}_{\aleph}^1 \quad .$$

Proof. From $\|AB\|, \|BC\| \in \{\mathbb{R}_0^{\mathcal{X}}\}$, the squared lengths of the legs are

$$\|AB\|^2 = \left(\aleph_{\mathcal{X}} + x\right)^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2)} + 2x\mathcal{X}\right)} + x^2 \quad ,$$

and

$$\|BC\|^2 = \left(\aleph_{\mathcal{Y}} + y\right)^2 = \aleph_{\left(\aleph_{(\mathcal{Y}^2)} + 2y\mathcal{Y}\right)} + y^2 \quad .$$

If we directly state the Pythagorean theorem in terms of the lengths (per the statement of the theorem), then we find

$$\|AC\|^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2 + \mathcal{Y}^2)} + 2(x\mathcal{X} + y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Assuming $\|AC\| = \aleph_{\mathcal{A}} + a$, we find

$$\aleph_{\left(\aleph_{(\mathcal{A}^2)} + 2a\mathcal{A}\right)} + a^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2 + \mathcal{Y}^2)} + 2(x\mathcal{X} + y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Setting the big parts equal yields

$$\aleph_{(\mathcal{A}^2)} + 2a\mathcal{A} = \aleph_{(\mathcal{X}^2 + \mathcal{Y}^2)} + 2(x\mathcal{X} + y\mathcal{Y}) \quad ,$$

which still has separable big and little parts. Doing the maximum possible separation of all the big and little parts yields

$$\begin{aligned} \mathcal{A}^2 &= \mathcal{X}^2 + \mathcal{Y}^2 \\ a\mathcal{A} &= x\mathcal{X} + y\mathcal{Y} \\ a^2 &= x^2 + y^2 \quad . \end{aligned}$$

Since $\|AB\| \neq c_0\|BC\|$, meaning that there does not exist a c_0 such that $x = c_0 y$ and $\mathcal{X} = c_0\mathcal{Y}$, we have three inconsistent equations in two variables: a and $\mathcal{A}$. No real-valued length $\|AC\|$ squared will satisfy the Pythagorean theorem as stated. The theorem is proven. ∎

Definition 6.2.8 A number is a complex number $z \in \mathbb{C}$ if and only if

$$z = x + iy \quad , \quad \text{and} \qquad x, y \in \mathbb{R} \quad .$$

Theorem 6.2.9 *If we assign an algebraic representation to the hypotenuse $AC \equiv z \in \mathbb{C}$ rather than the $AC \equiv \|AC\| \in \mathbb{R}$ disallowed by Main Theorem 6.2.7, then the Pythagorean identity is satisfied by $AC^2 \equiv \bar{z}z$.*

Proof. Given two legs, we want to find the hypotenuse through the Pythagorean theorem. We assume that the legs are real line segments so that

$$AB^2 \equiv \|AB\|^2 \quad , \quad \text{and} \qquad BC^2 \equiv \|BC\|^2 \quad .$$

If we take $\|AB\|, \|BC\| \in \mathbb{C}$ such that $\mathrm{Im}(\|AB\|) = \mathrm{Im}(\|BC\|) = 0$, then each is its own complex conjugate, and the quantity squared retains its usual meaning. Still, the geometric identity

$$AC^2 = AB^2 + BC^2 \quad ,$$

needs an algebraic interpretation if we are to do trigonometry. The present theorem concerns the "squared," exponent 2 operation being identified as multiplication by the complex conjugate in the sense that the inner product of a 1D vector $\vec{z} \in \mathbb{C}^1$ with itself is $\vec{z}^2 = \langle z|z\rangle = \bar{z}z$. The vector space axioms are known to be satisfied in $\mathbb{C} = \mathbb{C}^1$, so it is only an irrelevant matter of notation whether we specify a complex number z or a 1D complex vector $\vec{z}$. As in the previous theorem, we find

$$\big\langle AB\big|AB\big\rangle = \big(\aleph_{\mathcal{X}} + x\big)^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2)} + 2x\mathcal{X}\right)} + x^2 \quad ,$$

and

$$\big\langle BC\big|BC\big\rangle = \big(\aleph_{\mathcal{Y}} + y\big)^2 = \aleph_{\left(\aleph_{(\mathcal{Y}^2)} + 2y\mathcal{Y}\right)} + y^2 \quad .$$

The present statement of the Pythagorean theorem is

$$\big\langle AC\big|AC\big\rangle = \aleph_{\left(\aleph_{(\mathcal{X}^2+\mathcal{Y}^2)} + 2(x\mathcal{X}+y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Let $z \in \mathbb{C}$ be such that (i) it conforms to Definition 6.2.8, (ii) $AC \equiv z = |AC\rangle$, and (iii)

$$z = \aleph_{(\mathcal{X}\pm i\mathcal{Y})} + x \pm iy = \big(\aleph_{\mathcal{X}} + x\big) + i\big(\aleph_{\mathcal{Y}} + y\big) \quad .$$

We have

$$\begin{aligned}
\bar{z}z = \big\langle AC\big|AC\big\rangle &= \big[\aleph_{(\mathcal{X}\pm i\mathcal{Y})} + (x \pm iy)\big]^*\big[\aleph_{(\mathcal{X}\pm i\mathcal{Y})} + (x \pm iy)\big] \\
&= \big[\aleph_{(\mathcal{X}\mp i\mathcal{Y})} + (x \mp iy)\big]\big[\aleph_{(\mathcal{X}\pm i\mathcal{Y})} + (x \pm iy)\big] \\
&= \aleph_{(\mathcal{X}\mp i\mathcal{Y})\aleph_{(\mathcal{X}\pm i\mathcal{Y})}} + \aleph_{(\mathcal{X}\mp i\mathcal{Y})}(x \pm iy) \\
&\qquad + \aleph_{(\mathcal{X}\pm i\mathcal{Y})}(x \mp iy) + (x \mp iy)(x \pm iy) \\
&= \aleph_{\left(\aleph_{(\mathcal{X}^2+\mathcal{Y}^2)}\right)} + \aleph_{(x\mathcal{X}\pm iy\mathcal{X}\mp ix\mathcal{Y}+y\mathcal{Y})} \\
&\qquad + \aleph_{(x\mathcal{X}\mp iy\mathcal{X}\pm ix\mathcal{Y}+y\mathcal{Y})} + x^2 + y^2 \\
&= \aleph_{\left(\aleph_{(\mathcal{X}^2+\mathcal{Y}^2)} + 2(x\mathcal{X}+y\mathcal{Y})\right)} + x^2 + y^2 \quad .
\end{aligned}$$

Therefore,

$$\big\langle AC\big|AC\big\rangle = \big\langle AB\big|AB\big\rangle + \big\langle BC\big|BC\big\rangle \quad ,$$

and the Pythagorean theorem is satisfied as stated. This proves the theorem.

Corollary 6.2.10 *If we assign an algebraic representation to the hypotenuse $AC \equiv \vec{x} \in \mathbb{R}^2$ rather than the $AC \equiv \|AC\| \in \mathbb{R}$ disallowed by Main Theorem 6.2.7, then the Pythagorean identity is satisfied by $AC^2 \equiv \vec{x} \cdot \vec{x}$.*

Proof. This corollary follows from Theorem 6.2.9 in the way that everything in $\mathbb{C}$ has two equivalent vector space representations in $\mathbb{C}^1$ and $\mathbb{R}^2$. Let $\vec{x} \in \mathbb{R}^2$ be a vector in the Cartesian plane equipped as a vector space. We have three real vectors defining ABC in $\mathbb{R}^2$:

$$\vec{AB} = (\aleph_{\mathcal{X}} + x, 0) \ , \ \vec{BC} = (0, \aleph_{\mathcal{Y}} + y) \ , \quad \text{and} \qquad \vec{AC} = (\aleph_{\mathcal{X}} + x, \aleph_{\mathcal{Y}} + y) \ .$$

The Pythagorean theorem yields

$$\vec{AC} \cdot \vec{AC} = \vec{AB} \cdot \vec{AB} + \vec{BC} \cdot \vec{BC} \ .$$

Again, we find

$$\vec{AB} \cdot \vec{AB} = \big(\aleph_{\mathcal{X}} + x\big)^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2)} + 2x\mathcal{X}\right)} + x^2 \ ,$$

and

$$\vec{BC} \cdot \vec{BC} = \big(\aleph_{\mathcal{Y}} + y\big)^2 = \aleph_{\left(\aleph_{(\mathcal{Y}^2)} + 2y\mathcal{Y}\right)} + y^2 \ .$$

Checking the given form of $\vec{AC} \in \mathbb{R}^2$, we find

$$\begin{aligned} \vec{AC} \cdot \vec{AC} &= (\aleph_{\mathcal{X}} + x, \aleph_{\mathcal{Y}} + y) \cdot (\aleph_{\mathcal{X}} + x, \aleph_{\mathcal{Y}} + y) \\ &= \vec{AB} \cdot \vec{AB} + \vec{BC} \cdot \vec{BC} \ . \end{aligned}$$

The Pythagorean identity is satisfied with an algebraic representation of the hypotenuse AC such that $AC \equiv \vec{AC} \in \mathbb{R}^2$. The theorem is proven. ∎

Example 6.2.11 If a right triangle has two equal legs $\|AB\| = \|BC\|$, then the hypotenuse AC should be such that $\|AC\| = \sqrt{2}\|AB\|$. Since this is a case of $\|AB\| = c_0\|BC\|$ not considered in Main Theorem 6.2.7, we will include it now for completeness. We have two equal legs AB and BC with length $\aleph_{\mathcal{X}} + x$ such that

$$\sqrt{2}\|AB\| = \sqrt{2}\|BC\| = \aleph_{\sqrt{2}\mathcal{X}} + \sqrt{2}x \ .$$

We square it to check the Pythagorean theorem and find

$$\begin{aligned} \big(\aleph_{\sqrt{2}\mathcal{X}} + \sqrt{2}x\big)^2 &= \aleph_{\left(\sqrt{2}\mathcal{X}\aleph_{\sqrt{2}\mathcal{X}}\right)} + 2\aleph_{\sqrt{2}\mathcal{X}}\sqrt{2}x + 2x^2 \\ &= \aleph_{\aleph_{(2\mathcal{X}^2)}} + \aleph_{4x\mathcal{X}} + 2x^2 \\ &= \aleph_{\left(\aleph_{2\mathcal{X}^2} + 4x\mathcal{X}\right)} + 2x^2 \ . \end{aligned}$$

Indeed, the hypotenuse is real-valued and scaled by $\sqrt{2}$ when the legs are equal:

$$\aleph_{\left(\aleph_{2\mathcal{X}^2} + 4x\mathcal{X}\right)} + 2x^2 = 2\big(\aleph_{\mathcal{X}} + x\big)^2 \quad \Longrightarrow \quad \|AC\| = \sqrt{2}\|AB\| \ .$$

It is good that the geometry of a small triangle remains intact even when it is uniformly resized to have a characteristic scale in the neighborhood of infinity. However, what would happen if we rescaled one leg and not the other? Would it suddenly gain an imaginary part? In the complex algebraic representation of the hypotenuse $AC \equiv z = \aleph_{(\mathcal{X} \pm i\mathcal{Y})} + x \pm iy$, as in Theorem 6.2.9, we have randomly chosen $BC \equiv ([\mathcal{Y}], [y])$ as governing the imaginary part. However, if the two legs are real, then why should $([\mathcal{Y}], [y])$ govern the imaginary part of z and not $AB \equiv ([\mathcal{X}], [x])$? Having raised these issues, we leave them to future work on 2D planar Euclidean geometry. After one more related example, we will return to the present considerations regarding the 1D geometry of a straight line (which is not as simple as might be assumed.)

Example 6.2.12 This example demonstrates a ramification of Main Theorem 6.2.7 for the ordinary notions of trigonometry. Consider a right triangle ABC such that $\angle ABC = \frac{\pi}{2}$. Suppose $\|AB\| = \aleph_{\mathcal{X}} + x$ and $\|BC\| = \aleph_{\mathcal{Y}} + y$. Let $\alpha = \angle CAB$ such that $0 < \alpha < \frac{\pi}{2}$. Ordinary notions of trigonometry suggest

$$\|AC\| \sin(\alpha) = \aleph_{\mathcal{Y}} + y \quad , \quad \text{and} \qquad \|AC\| \cos(\alpha) = \aleph_{\mathcal{X}} + x \quad . \tag{6.2}$$

It follows that

$$\|AC\| = \aleph_{\left(\frac{\mathcal{Y}}{\sin(\alpha)}\right)} + \frac{y}{\sin(\alpha)} \quad , \quad \text{and} \qquad \|AC\| = \aleph_{\left(\frac{\mathcal{X}}{\cos(\alpha)}\right)} + \frac{x}{\cos(\alpha)}$$

Equating the big and little parts yields

$$\tan(\alpha) = \frac{\mathcal{Y}}{\mathcal{X}} \quad , \quad \text{and} \qquad \tan(\alpha) = \frac{y}{x} \quad .$$

Although this is a contradiction for every case in which $\frac{\mathcal{Y}}{\mathcal{X}} \neq \frac{y}{x}$—which is equally stated as $\|AB\| \neq c_0 \|BC\|$—the result is self-consistent in the framework of fractional distance analysis. If the trigonometry functions in Equation (6.2) are real-valued, then the equality cannot hold when $\mathrm{Im}(\|AC\|) \neq 0$ because each RHS is real-valued. We have shown in Example 6.2.11 that the hypotenuse is real-valued for the case of $\|AB\| = c_0 \|BC\|$ with $c_0 = 1$, and the trigonometry functions should work as usual for any $c_0 \in \mathbb{R}$. That will enforce the equal relative scale $\frac{\mathcal{Y}}{\mathcal{X}} = \frac{y}{x}$ for the ratios of the big and little parts of the lengths of the legs. In other cases, we might expect the angle α to acquire an imaginary part.

Remark 6.2.13 In leaving the real line and going onto the plane, we have exceeded the scope of this analysis. Other than a $\mathbb{C}$ application in Section 8 to demonstrate the negation of the Riemann hypothesis, we will not continue to exceed the confines of $\mathbb{R}$. Even given the solution to that famous problem in Section 8, however, it is the opinion of this writer that the principle demonstrated in Main Theorem 6.2.7 is certainly the most important result given herein. It cleanly demonstrates that the extension $L \in \mathbb{R}_0 \to L \in \mathbb{R}_\infty$

is not the trivial exercise that might be intuitively assumed. Among the two valid interpretations given for the Pythagorean identity (Theorem 6.2.9 and Corollary 6.2.10) the $z \in \mathbb{C}$ representation of the length of the hypotenuse is more relevant than $\vec{x} \in \mathbb{R}^2$ because z is a 1D scalar number whose real part is a cut in the real number line. In other words, z is a cut in the real number line added to a cut in the imaginary number line. Since cuts in the real number line are known to have both zero and non-zero imaginary parts, meaning some real numbers are the real parts of complex numbers with non-zero imaginary parts, $z \in \mathbb{C}$ is more germane to the standard analysis of $\mathbb{R}$ than is $\vec{x} \in \mathbb{R}^2$. Vector structure in vector analysis requires an entire axiomatic framework for vector arithmetic, but all of the arithmetic for $z \in \mathbb{C}$ can be derived immediately if i is added to the arithmetic axioms.

Theorem 6.2.14 *A real number $x \in \{\mathbb{R}_{\aleph}^{\mathcal{X}}\} \cup \mathbb{R}_{\aleph}^{1}$ does not have a real-valued square root.*

Proof. Let there be two real numbers

$$z = \aleph_{\mathcal{Z}} + a \quad , \quad \text{and} \qquad x = \aleph_{\mathcal{X}} + b \quad .$$

Assume $z^2 = x$ so that

$$\left(\aleph_{\mathcal{Z}} + a\right)^2 = \aleph_{\mathcal{X}} + b \quad . \tag{6.3}$$

We have

$$\left(\aleph_{\mathcal{Z}} + a\right)^2 = \aleph_{\left(\aleph_{(\mathcal{Z}^2)} + 2a\mathcal{Z}\right)} + z^2 \quad ,$$

so we should set the big and little parts of the left and right sides of Equation (6.3) equal to each other. This gives

$$\aleph_{(\mathcal{Z}^2)} + 2a\mathcal{Z} = \mathcal{X} \quad , \quad \text{and} \qquad z^2 = b \quad .$$

The former constraint equation gives $\mathcal{Z} = 0$ because the RHS has zero big part. It follows that $\mathcal{X} = 0$. This is a contradiction because we have already selected $\aleph_{\mathcal{X}}$ as the non-zero big part of $x \in \{\mathbb{R}_{\aleph}^{\mathcal{X}}\} \cup \mathbb{R}_{\aleph}^{1}$. ∎

Example 6.2.15 Consider the limit

$$\lim_{b \to \infty} \aleph_{\mathcal{X}} - b = l \quad .$$

It remains to be clarified precisely what is meant by the notation $b \to \infty$ because we should have options for at least two distinct behaviors. For example, one might wish to define

$$\lim_{b \to \infty} \aleph_{\mathcal{X}} - b = -\infty \quad , \quad \text{and} \qquad \lim_{b \to \widehat{\infty}} \aleph_{\mathcal{X}} - b = \aleph_{\mathcal{X}} - \widehat{\infty} = -\aleph_{(1-\mathcal{X})} \quad ,$$

where $b \to \widehat{\infty}$ means that b approaches $\widehat{\infty}$, while $b \to \infty$ would mean that b increases without bound (even including transfinite numbers larger than

$\widehat{\infty}$) such that b approaches some geometric infinity whose absolute value is in some sense greater than that of algebraic infinity. We will not make such definitions here because the requisite formal definitions for $x > \widehat{\infty}$ are out of scope. However, simply based on the absorption or non-absorption of ∞ and $\widehat{\infty}$ respectively, the limits given in this example should be presumed correct.

§6.3 The Archimedes Property of Real Numbers

While there are many ways to state the Archimedes property of real numbers with symbolic logic, the modern establishment has adopted a standard statement

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

For this statement to accurately characterize the property as it appeared in the first edition Greek language copy of Euclid's *Elements*, it must depend on an unstated axiom that every real number is less than some natural number. Without that axiom, the statement is wrong, and there is no other word than "wrong" by which it should be described. In this section, we will consult the original text in *The Elements* [1]. We will use the original text to prove absolutely that the above symbolic statement is not the Archimedes property of real numbers given so famously by Euclid in antiquity.

For the above statement to agree with that which was given by Euclid in Greek, one must first take the axiom that every real number is less than some natural number. Without a statement or implicit acknowledgment of such an axiom, the above Latin symbolic statement is *wrongly* called the Archimedes property of real numbers.

Definition 6.3.1 The statement of the Archimedes property which appears in Euclid's tome *The Elements*, which was attributed by Archimedes to his predecessor Eudoxus, *and* which must be taken as ***the definitive statement of the Archimedes property of real numbers***, appears as Definition 4 in Book 5 of *The Elements*. The original Greek is translated as follows [1].

> "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another."

Remark 6.3.2 As it appears in *The Elements*, a straightforward mathematical statement of the property would be

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \quad .$$

There is no mention of multiplication by a positive integer $n \in \mathbb{N}$. To prove that the Archimedes property of real numbers does not implicitly restrict the multiplier to $n \in \mathbb{N}$, we will examine the context in the original text.

Definition 6.3.3 In Reference [1], Fitzpatrick translates Book 5, Definitions 1 through 5 as follows.

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.
2. And the greater is a multiple of the lesser whenever it is measured by the lesser.
3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.
4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.
5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and third both exceed, are both equal to, or are both less than, equal multiples of the second and fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.

Remark 6.3.4 Though we may prove directly from Euclid's own words that the multiplier in the Archimedes property is not defined as a natural number, Fitzpatrick gives footnotes qualifying his translations of Euclid's original Greek. These footnotes support the wrongness of the supposition that Euclid meant to imply that the multiplier must always be a natural number. We will list the footnotes here for thoroughness, though we will not rely on them in Theorem 6.3.5. The footnotes are as follows.

1. In other words, α is said to be a part of β if $\beta = m\alpha$.
2. (*No footnote given.*)
3. In modern notation, the ratio of two magnitudes, α and β, is denoted $\alpha : \beta$.
4. In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n.
5. In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\alpha > n\beta$ whenever $m\gamma > n\delta$, $m\alpha = n\beta$ whenever $m\gamma = n\delta$, and $m\alpha < n\beta$ whenever $m\gamma < n\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, *etc.*, are irrational.

Footnote 5 makes it exceedingly obvious that the multipliers are "all m and n" in $\mathbb{R}$. As Definition 5 is the "kernel of Eudoxus' theory of proportion," and to the extent that Eudoxus is the originator of the Archimedes property, we will argue that the multiplicative criterion in that property regards "any kind of multiplication whatever" rather than strict natural number multiplication.

Theorem 6.3.5 *The statement*

$$\forall x, y \in \mathbb{R} \quad s.t. \quad x < y \quad \exists n \in \mathbb{N} \quad s.t. \quad nx > y \quad .$$

is not a proper statement of the Archimedes property of real numbers as given in antiquity.

Proof. It follows from Book 5, Definition 5, of Euclid's original text that if $y \in \mathbb{R}$ is a multiple of $z \in \mathbb{R}$, then there exists some "multiplier" $x \in \mathbb{R}$ such that $xy = z$. To prove the present theorem by contradiction, assume that Euclid meant to restrict the multiplier in his definitions as $n \in \mathbb{N}$, and then consider Definition 2:

> "And the greater is a multiple of the lesser whenever it is measured by the lesser."

Suppose $y = 2$ and $z = 3$ so that, among the two numbers, z is the greater. If the multiplier by which z is to be measured by y is restricted to $n \in \mathbb{N}$ rather than $x \in \mathbb{R}$, then z cannot be measured by y. This is ***an affront to reason***, firstly, and it directly contradicts Definition 1:

> "A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater."

It is self-evidently true that $3 > 2$, so, for 2 to be a part of 3, it must measure the greater. "Measure" is defined by Definition 2 in terms of multiples, which are thence defined in terms of multiplication. For 2 be a part of 3 in the sense of Definition 1, we must do multiplication with a multiplier $x = 1.5 \notin \mathbb{N}$. This proves the theorem. 🍃

Remark 6.3.6 In Book 7, Definition 2, Euclid defines "numbers" as natural numbers, but what are today called real numbers are instead the "magnitudes" described in Book 5. Euclid in no way implied that the multiplier in Definition 4 should be taken strictly as $n \in \mathbb{N}$, and, so, neither was Euclid of the opinion that Archimedes meant to do so in his own earlier paraphrasing of Eudoxus.

Example 6.3.7 This example demonstrates that if one presupposes the non-existence of real numbers greater than any natural number, taking it purely as an unproven axiomatic definition, one which violates the contrary proof of the existence of such numbers given in Main Theorem 3.2.6, then the statement

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad ,$$

does adequately encapsulate the Archimedes property of antiquity. Proof of this statement is given by Rudin as follows [11].

> "Let A be the set of all nx, where n runs through the positive integers. If [*the symbolic statement given above in the present example*]

> were false, then y would be an upper bound of A. But then A has a least upper bound in $\mathbb{R}$. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < (m+1)x \in A$, which is impossible since α is an upper bound of A."

Here, Rudin follows the reasoning of Proposition 7.1.14 in which it will be claimed again that $\mathbb{R}_0$ cannot have a supremum in $\mathbb{R}$. After we discuss the existence of this supremum in Section 7.1, we will revisit the issue most specifically in Section 7.5.

Remark 6.3.8 If we adopt

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \quad .$$

as the definitive statement of the Archimedes property, as in Remark 6.3.2, then we will have taken away the Archimedes property of real numbers from the maximal whole neighborhood of infinity $\mathbb{R}^1_{\aleph}$. For instance, if

$$\widehat{\infty} - a < \widehat{\infty} - b \quad ,$$

then we cannot multiply the LHS by a number greater than one and have a real-valued product due to the identity $\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty}$. If we multiply by a positive number less than one, call it δ, then

$$\aleph_\delta - \delta a < \widehat{\infty} - a < \widehat{\infty} - b \quad ,$$

does *not* conform to the usual Archimedean requirement that $\delta(\widehat{\infty} - a) > \widehat{\infty} - b$. If this were to force the ejection of $\mathbb{R}^1_{\aleph}$ from $\mathbb{R}$ because such numbers were found not to exhibit the Archimedes property, then that would cause a breakdown in Axiom 2.1.7 giving $\mathbb{R} = (-\infty, \infty)$. If we suppose, correctly, that all real numbers obey the Archimedes property, then we might write

$$\mathbb{R}^+ = (0, \infty) \setminus \mathbb{R}^1_{\aleph} \quad , \tag{6.4}$$

but this is highly disfavorable because we lose the perfect infinite geometric line construction that we have sought to preserve. In terms of the topology, Equation (6.4) breaks the usual topology of $\mathbb{R}$ such that its basis is no longer all open subsets $(a, b) \subset (-\infty, \infty)$.

In what manner shall the maximal neighborhood of infinity exhibit the Archimedes property of real numbers? How might we solve this problem? The answer lies in Euclid's original Greek:

> "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another."

In Remark 6.3.2, we have adopted the convention that the multiplier must attach to the lesser part of $x < y$, but no such requirement is given by Euclid's

totally symmetric statement. For one to exceed the other upon multiplication allows us to state the property in terms of multiplication of either the greater or the lesser among x and y, *i.e.*: ***according to any kind of multiplication whatsoever***. In Euclid's own parlance, for one to exceed the other only requires that each is a "part" or "multiple" of the other without specifying a requirement for which is which. Taking careful note of the non-specificity of the ordering relation in Euclid's Definition 4, we will preserve the highly favorable definition $\mathbb{R} = (-\infty, \infty)$ by giving a symbolic statement of the Archimedes property obeyed by $x \in \mathbb{R}^1_{\aleph}$, as well as the rest of $\mathbb{R}$ (Definition 6.3.9). At the end of this section, we will give a new, modern statement of the Archimedes property such that its application is greatly simplified, but first we will show that the fractional distance model of $\mathbb{R}$ obeys a symbolically appropriate Latin restatement of Euclid's small handful of original Greek words. Once we show that the ancient definition is satisfied, we will make a simplifying axiom such that demonstrating the property is simplified.

Definition 6.3.9 The most general statement of the ancient Archimedes property of real numbers is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad .$$

Main Theorem 6.3.10 *The $C^{\mathbf{AB}}_{\mathbb{Q}}$ construction of $\mathbb{R}$ is such that every $x, y \in \mathbb{R}^0_{\aleph} \cup \{\mathbb{R}^{\mathcal{X}}_{\aleph}\} \cup \mathbb{R}^1_{\aleph}$ exhibit the ancient Archimedes property of real numbers.*

Proof. By Definition 6.3.9, it suffices to demonstrate

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad .$$

We will consider the general forms

$$x = \aleph_{\mathcal{X}} + b \quad , \quad \text{and} \qquad y = \aleph_{\mathcal{Y}} + a \quad ,$$

such that $x \in \mathbb{R}^{\mathcal{X}}_{\aleph}$ and $y \in \mathbb{R}^{\mathcal{Y}}_{\aleph}$, and we will assume constraints $x < y$ and $0 \leq \mathcal{X} \leq \mathcal{Y} \leq 1$. Further assume that a and b are constrained appropriately for $\mathcal{X}$ and/or $\mathcal{Y}$ equal to one or zero so that x and y are always in $\mathbb{R}^+$. The starting point for demonstrating the Archimedes property is $x < y$, which we write as

$$\aleph_{\mathcal{X}} + b < \aleph_{\mathcal{Y}} + a \quad .$$

To prove the theorem, we will consider the distinct cases. In each equality listed below, we will put $z_1 x$ on the left and $z_2 y$ on the right.

• ($x \in \mathbb{R}^0_{\aleph}$ and $y \in \mathbb{R}^0_{\aleph}$) Here, both x and y have vanishing big parts, so $x < y$ defines the ordering of the little parts. Choose $z_1 = \aleph_{\mathcal{Z}} + z$ such that $0 < \mathcal{Z}b < 1$ and choose $z_2 = 1$. Then

$$\big(\aleph_{\mathcal{Z}} + z\big)b = \aleph_{(\mathcal{Z}b)} + zb \qquad > \qquad a \quad .$$

• ($x \in \mathbb{R}^0_{\aleph}$ and $y \in \mathbb{R}^{\mathcal{Y}}_{\aleph}$) Here, x has a vanishing big part, and y has a non-vanishing big part. Choose $z_1 = \aleph_{\left(\frac{1+\mathcal{Y}}{2b}\right)} + z$ and $z_2 = 1$. Then

$$\left(\aleph_{\left(\frac{1+\mathcal{Y}}{2b}\right)} + z\right) b = \aleph_{\left(\frac{1+\mathcal{Y}}{2}\right)} + zb \qquad > \qquad \aleph_{\mathcal{Y}} + a \quad .$$

Since $\frac{1+\mathcal{Y}}{2}$ is the average of $\mathcal{Y}$ and 1, it is guaranteed to be a number in the open interval $(\mathcal{Y}, 1)$.

• ($x \in \mathbb{R}^0_{\aleph}$ and $y \in \mathbb{R}^1_{\aleph}$) Here, x has a vanishing big part, and y has big part $\aleph_1$. Choose $z_1 = \aleph_{\mathcal{Z}} + z$ and z_2 such that $0 < z_2 < \mathcal{Z}b < 1$. Then

$$\left(\aleph_{\mathcal{Z}} + z\right) b = \aleph_{(\mathcal{Z}b)} + zb \qquad > \qquad z_2\big(\aleph_1 + a\big) = \aleph_{(z_2)} + z_2 a \quad .$$

• ($x \in \mathbb{R}^{\mathcal{X}}_{\aleph}$ and $y \in \mathbb{R}^{\mathcal{Y}}_{\aleph}$ such that $\mathcal{X} < \mathcal{Y}$) Here, neither x nor y has a vanishing big part, and the big part of x is less than big part of y. Choose $z_1 = \frac{1+\mathcal{Y}}{2\mathcal{X}}$ and $z_2 = 1$. Then

$$\frac{1+\mathcal{Y}}{2\mathcal{X}}\left(\aleph_{\mathcal{X}} + b\right) = \aleph_{\left(\frac{1+\mathcal{Y}}{2}\right)} + b\frac{1+\mathcal{Y}}{2\mathcal{X}} \qquad > \qquad \aleph_{\mathcal{Y}} + a \quad .$$

• ($x, y \in \mathbb{R}^{\mathcal{X}}_{\aleph}$ such that $\mathcal{X} = \mathcal{Y}$) Here, x and y have equal big parts, so it follows from $x < y$ that the little parts are ordered accordingly. Choose $z_1 = z$ such that $\mathcal{X} < z\mathcal{X} < 1$ and $z_2 = 1$. Then

$$z\left(\aleph_{\mathcal{X}} + b\right) = \aleph_{(z\mathcal{X})} + zb \qquad > \qquad \aleph_{\mathcal{X}} + a \quad .$$

• ($x \in \mathbb{R}^{\mathcal{X}}_{\aleph}$ and $y \in \mathbb{R}^1_{\aleph}$) Here, x and y have unequal, non-zero big parts with the big part of y being the greater. Choose $z_1 = 1$ and $z_2 = \frac{\mathcal{X}}{2}$. Then

$$\aleph_{\mathcal{X}} + b \qquad > \qquad \frac{\mathcal{X}}{2}\big(\aleph_1 - a\big) = \aleph_{\left(\frac{\mathcal{X}}{2}\right)} - \frac{a\mathcal{X}}{2} \quad .$$

• ($x, y \in \mathbb{R}^1_{\aleph}$) Here, x and y have equal big parts $\aleph_1$, and $x < y$ defines the ordering of the little parts. Choose $z_1 = 1$ and $z_2 = \frac{1}{2}$. Then

$$\aleph_1 + b \qquad > \qquad \frac{1}{2}\big(\aleph_1 - a\big) = \aleph_{\left(\frac{1}{2}\right)} - \frac{a}{2} \quad .$$

We have considered every combination of $x < y$ among the various neighborhoods and shown that they comply uniformly with Definition 6.3.9. The theorem is proven.

Remark 6.3.11 If it were not for the extremal case of $x \in \mathbb{R}^0_{\aleph}$ and $y \in \mathbb{R}^1_{\aleph}$, we might have formulated the symbolic statement of the Archimedes property as

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R}^+ \quad \text{s.t.} \quad zx > y \quad \text{or} \quad x > zy \quad .$$

This form is nice because it uses only a single multiplication operation and exactly reflects Fitzpatrick's footnote:

> "In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n."

However, it is not possible to phrase the symbolic statement of the property with only a single multiplier due to the extremal case in which (i) x is in the neighborhood of the origin, and (ii) y is in the maximal neighborhood of infinity. Even then, however, Euclid does not precisely require a condition of the form, "multiplication of just one can exceed the other." As it is written:

> "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another."

This statement absolutely allows the two multiplier form given in Definition 6.3.9. This statement is equally well clarified with a similar but slightly different footnote than that given by Fitzpatrick. An alternative footnote explaining the meaning of the property would be the following.

> In other words, α has a ratio with respect to β if $m_1\alpha > n_1\beta$ and $m_2\beta > n_2\alpha$, for some m_1, m_2, n_1, and n_2.

This is exactly what is given in Definition 6.3.9. It is well consistent with the ratio-of-ratios language seen in Book 5, Definition 5, as well as the statement about "any kind of multiplication."

In general, we have made a rather large statement

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad , \tag{6.5}$$

of Euclid's few original words. The reasoning behind including the Archimedes property of real numbers as a supplemental constraint on the behavior of cuts in the real number line is that it is supposed to be an elegantly simple statement of a simple behavior, but Equation (6.5) is only somewhat elegant. Therefore, having already independently demonstrated rigorous compliance with Euclid in the absence of a simplifying modern axiom, now we will give such an axiom.

Axiom 6.3.12 The Archimedes property of 1D transfinitely continued real numbers is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad \text{and/or} \quad \aleph_n x > y \quad .$$

This axiom defines the implicit transfinite ordering required for $\leq$ to be a relation among real numbers $\mathbb{R}$ and 1D transfinitely continued real numbers $\mathbb{T}$

whose big parts $\aleph_{\mathcal{X}}$ are such that $\mathcal{X} > 1$. As a subset of the 1D transfinitely continued real numbers, the real numbers themselves automatically inherit compliance with the Archimedes property.

Remark 6.3.13 If the real number line ends at infinity, that indicates an endpoint there, and endpoints are associated with $\widehat{\infty}$ when we take the convention that the notion of infinite geometric extent precludes the existence of endpoints at ∞. Therefore, the lack of a terminating point for the line at infinity automatically implies the 1D transfinite continuation of $\mathbb{R} = (-\aleph_1, \aleph_1)$ onto $\mathbb{T} = (-\aleph_\infty, \aleph_\infty)$. If $\mathbb{R}$ didn't continue onto $\mathbb{T}$, then it would end at ∞: a contradiction if ∞ is not allowed to be an endpoint. Since it is only required that $\widehat{\infty} = \aleph_1$ is the supremum of the real numbers, there is no requirement that it is the largest number. Axiom 6.3.12 generates the requisite definition of transfinite ordering such that given $x, y \in \mathbb{R}$ and $x < y$, zx can be greater than y without zx itself being $zx \in \mathbb{R}$. Here, we imply an intuitive ordering for $zx \in \mathbb{T}$, and then we use this ordering to satisfy the $zx > y$ condition irreducibly cited in *The Elements*. In the scheme of Axiom 6.3.12, all the bulleted cases of $x < y$ statements in Main Theorem 6.3.10 are replaced with elegantly simple formulae.

§7 The Topology of the Real Number Line

§7.1 Basic Set Properties

In this section, we give some elementary set properties of the natural neighborhoods and begin to approach the logical connection to the whole neighborhoods. Recall that the natural neighborhoods $\mathbb{R}_0^{\mathcal{X}}$ are defined with little part $|b| \in \mathbb{R}_0^0$, and the whole neighborhoods are defined with $|b| \in \mathbb{R}_{\aleph}^0$.

Lemma 7.1.1 *Every natural neighborhood in* $\{\mathbb{R}_0^{\mathcal{X}}\}$ *is an open set.*

Proof. By Definition 4.1.4, the set of all intermediate natural neighborhoods of infinity is

$$\left\{\mathbb{R}_0^{\mathcal{X}}\right\} = \left\{\aleph_{\mathcal{X}} + b \;\middle|\; b \in \mathbb{R}_0,\ 0 < \mathcal{X} < 1\right\} \quad .$$

A given $\mathbb{R}_0^{\mathcal{X}}$ defined with a particular $\mathcal{X}$ is open if and only if there is a δ-neighborhood of each of its elements such that every element of that neighborhood is also an element of $\mathbb{R}_0^{\mathcal{X}}$. We will use the ball function δ-neighborhood as in Definition 4.1.11 rather than Definition 4.1.12 because the elements of $\mathbb{R}_0^{\mathcal{X}}$ are numbers, not points. This theorem is proven with a δ-neighborhood of an arbitrary $x \in \mathbb{R}_0^{\mathcal{X}}$. Defining $b^{\pm} = b \pm \delta$, we have

$$\text{Ball}(x \in \mathbb{R}_0^{\mathcal{X}}, \delta) = (\aleph_{\mathcal{X}} + b - \delta, \aleph_{\mathcal{X}} + b + \delta) = (\aleph_{\mathcal{X}} + b^-, \aleph_{\mathcal{X}} + b^+) \quad .$$

Axiom 5.2.1 requires that $\mathbb{R}_0$ is closed under the $\pm$ operations so $b, \delta \in \mathbb{R}_0$ implies $b^{\pm} \in \mathbb{R}_0$. The set $\mathbb{R}_0^{\mathcal{X}}$ is open because

$$(\aleph_{\mathcal{X}} + b^-, \aleph_{\mathcal{X}} + b^+) \subset \mathbb{R}_0^{\mathcal{X}} = \left\{\aleph_{\mathcal{X}} + b \;\middle|\; b \in \mathbb{R}_0\right\} \quad .$$

Alternatively, no $\mathbb{R}_0^{\mathcal{X}}$ set in $\{\mathbb{R}_0^{\mathcal{X}}\}$ contains its boundary points, so each such set is open. ∎

Theorem 7.1.2 *Given two natural neighborhoods $\mathbb{R}_0^{\mathcal{X}}$ and $\mathbb{R}_0^{\mathcal{Y}}$ with $0 \leq \mathcal{X} < \mathcal{Y} \leq 1$, there exists another natural neighborhood $\mathbb{R}_0^{\mathcal{Z}}$ such that $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$.*

Proof. Consider the interval

$$(\aleph_{\mathcal{X}}, \aleph_{\mathcal{Y}}) \subset \mathbb{R} \quad .$$

By Definition 3.2.1, the number at the center of this interval is

$$\frac{\aleph_{\mathcal{Y}} + \aleph_{\mathcal{X}}}{2} = \aleph_{\left(\frac{\mathcal{Y}+\mathcal{X}}{2}\right)} \quad .$$

We have

$$\mathcal{X} < \frac{\mathcal{Y} + \mathcal{X}}{2} < \mathcal{Y} \quad ,$$

so let $\mathcal{Z} = \frac{\mathcal{Y}+\mathcal{X}}{2}$. Any number $z \in Z \in \mathbf{AB}$ of the form

$$z = \aleph_{\mathcal{Z}} + z_0 \quad , \quad \text{for} \qquad z_0 \in \mathbb{R}_0 \quad ,$$

will be such that

$$\mathcal{D}_{\mathbf{AB}}(AZ) = \mathcal{Z} \quad .$$

Since $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$, the theorem is proven. ∎

Corollary 7.1.3 *Given two whole neighborhoods $\mathbb{R}_{\aleph}^{\mathcal{X}}$ and $\mathbb{R}_{\aleph}^{\mathcal{Y}}$ with $0 \leq \mathcal{X} < \mathcal{Y} \leq 1$, there exists another whole neighborhood $\mathbb{R}_{\aleph}^{\mathcal{Z}}$ such that $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$.*

Proof. Following the proof of Theorem 7.1.2, we arrive at a number $z \in Z \in \mathbf{AB}$ of the form

$$z = \aleph_{\mathcal{Z}} + z_0 \quad , \quad \text{for} \qquad z_0 \in \mathbb{R}_{\aleph}^0 \quad ,$$

Even in the whole neighborhood exceeding the natural neighborhood, z_0 has no fractional magnitude with respect to **AB**. Therefore, the total fractional distance is completely determined by the big part as

$$\text{Big}(z) = \aleph_{\mathcal{Z}} \qquad \Longleftrightarrow \qquad \mathcal{D}_{\mathbf{AB}}(AZ) = \mathcal{Z} \quad .$$

Proof follows from $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$, as in Theorem 7.1.2. ∎

Definition 7.1.4 An open set S is disconnected if and only if there exist two open, non-empty sets U and V such that

$$S = U \cup V \quad , \quad \text{and} \qquad U \cap V = \varnothing \quad .$$

If a set is not disconnected, it is connected.

Corollary 7.1.5 *$\mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_0^{\mathcal{Y}}$ is a disconnected set for any $0 \leq \mathcal{X} < \mathcal{Y} \leq 1$.*

Proof. An open set is disconnected if it is the union of two disjoint, non-empty open sets. By Lemma 7.1.1, $\mathbb{R}_0^{\mathcal{X}}$ is open, and it is obvious that such sets are non-empty. It follows from Theorem 7.1.2 that they are disjoint, *i.e.*:

$$\mathbb{R}_0^{\mathcal{X}} \cap \mathbb{R}_0^{\mathcal{Y}} = \varnothing \quad .$$

The union $\mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_0^{\mathcal{Y}}$ satisfies the definition of a disconnected set. ✐

Remark 7.1.6 During the development of the intermediate neighborhoods of infinity, we found it useful to separate the $\mathcal{X} = 0$ and $\mathcal{X} = 1$ cases from the intermediate neighborhoods $\{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$. For efficacy in notation, now we will combine all the different neighborhoods into a streamlined, unified notation. The following definitions supplement Definitions 4.1.3 and 4.1.4 to include the cases of $\mathcal{X} = 0$ and $\mathcal{X} = 1$.

Definition 7.1.7 To streamline notation, define

$$\mathbb{R}_{\aleph}^{\cup} = \bigcup_{0 \leq \mathcal{X} \leq 1} \mathbb{R}_{\aleph}^{\mathcal{X}} = \mathbb{R}_{\aleph}^{0} \cup \left\{\mathbb{R}_{\aleph}^{\mathcal{X}}\right\} \cup \mathbb{R}_{\aleph}^{1}$$
$$\mathbb{R}_{0}^{\cup} = \bigcup_{0 \leq \mathcal{X} \leq 1} \mathbb{R}_{0}^{\mathcal{X}} = \mathbb{R}_{0}^{0} \cup \left\{\mathbb{R}_{0}^{\mathcal{X}}\right\} \cup \mathbb{R}_{0}^{1} \quad .$$

Definition 7.1.8 The complement of $\mathbb{R}_0^{\mathcal{X}}$ in $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is $\mathbb{R}_C^{\mathcal{X}}$:

$$\mathbb{R}_C^{\mathcal{X}} = \mathbb{R}_{\aleph}^{\mathcal{X}} \setminus \mathbb{R}_0^{\mathcal{X}} \quad .$$

Theorem 7.1.9 *There exist more positive real numbers than are in $\mathbb{R}_0^{\cup}$. In other words,*

$$\mathbb{R}^+ \setminus \mathbb{R}_0^{\cup} \neq \varnothing \quad .$$

Proof. By the definition of an interval, and through Axiom 2.1.7 explicitly granting the connectedness of $\mathbb{R} = (-\infty, \infty)$, the interval $\mathbb{R}^+ = (0, \infty)$ is connected. To prove the present theorem, it will suffice to show that $\mathbb{R}_0^{\cup}$ is disconnected. Disconnection follows from Corollary 7.1.5. ✐

Remark 7.1.10 It was already expected that there may be real numbers not contained in the natural neighborhoods. It was for this reason that we defined distinct whole neighborhoods $\mathbb{R}_{\aleph}^{\mathcal{X}} \supseteq \mathbb{R}_0^{\mathcal{X}}$. In Section 7.4, we will conjecture $\mathbb{R}_C^{\mathcal{X}} = \varnothing$, but first we will prove another result: one far more interesting.

Main Theorem 7.1.11 *There exist more positive real numbers than are in $\mathbb{R}_{\aleph}^{\cup}$. In other words,*

$$\mathbb{R}^+ \setminus \mathbb{R}_{\aleph}^{\cup} \neq \varnothing \quad .$$

Proof. $\mathbb{R}^+$ is a connected interval. $\mathbb{R}^{\cup}_{\aleph} \setminus \{0\}$ is a disjoint union of open subsets of $\mathbb{R}^+$. A connected interval cannot be covered with a disconnected subset. The theorem is proven. ✐

Theorem 7.1.12 *For any $\mathcal{X} > 0$, $\aleph_{\mathcal{X}}$ is an upper bound of $\mathbb{R}_0$.*

Proof. An upper bound of a set is greater than or equal to every element of that set. Suppose

$$X, Y \in \mathbf{AB} \quad , \quad x \in \mathbb{R}_0 \quad , \quad x \in X \quad , \quad \text{and} \qquad \aleph_{\mathcal{X}} \in Y \quad .$$

It follows that

$$\mathcal{D}_{\mathbf{AB}}(AX) = 0 \quad , \quad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AY) = \mathcal{X} \quad .$$

By the ordering of $\mathbb{R}$ (Axioms 3.1.13 and 5.2.18), $\aleph_{\mathcal{X}}$ is an upper bound of $\mathbb{R}_0$ whenever $\mathcal{X} > 0$. ✐

Corollary 7.1.13 $\mathbb{N}$ *is bounded from above.*

Proof. If $n \in \mathbb{N}$, then $n \in \mathbb{R}_0$. By Theorem 7.1.12, all $x \in \mathbb{R}_0$ are bounded from above. $\mathbb{N}$ is bounded from above. ✐

Proposition 7.1.14 $\mathbb{R}_0 \subset \mathbb{R}$ does not have a least upper bound $\sup(\mathbb{R}_0) \in \mathbb{R}$. In other words, $\mathbb{R}$ does not have the least upper bound property.

Justification. To invoke a contradiction, suppose $s \in \mathbb{R}$ is the least upper bound of $\mathbb{R}_0$. If $s - 1$ was an upper bound of $\mathbb{R}_0$, then s could not be the least upper bound because $s - 1 < s$. Therefore, $s = \sup(\mathbb{R}_0)$ implies $(s - 1) \in \mathbb{R}_0$. By Axiom 5.2.1, $\mathbb{R}_0$ is closed under addition. It follows that $(s - 1 + 2) \in \mathbb{R}_0$ because $2 \in \mathbb{R}_0$. Since $s + 1 > s$, we obtain a contradiction having shown that there must exist elements of $\mathbb{R}_0$ greater than the assumed supremum s. ✐

Remark 7.1.15 The issue described in the justification of Proposition 7.1.14 shall be referred to as "the least upper bound problem." This proposition is usually presented as a theorem, and it brings us to one of the most finely nuanced issues in the present treatment of $\mathbb{R}$. This proposition makes a convincing case that $\mathbb{R}_0$ cannot have a supremum in $\mathbb{R}$, but, if $\mathbb{R}_0$ is a subset of the connected interval $(-\widehat{\infty}, \widehat{\infty})$, then ***it most certainly must have a least upper bound*** in $\mathbb{R}$. Otherwise $(-\widehat{\infty}, \widehat{\infty})$ is not connected. In Section 7.5, we will return to the topic of algebraic contradictions related to the suprema required for the connectedness of the interval containing the various disconnected neighborhoods. If $\mathbb{R}$ is to have the usual topology overall, then it *must* have the least upper bound property.

§7.2 Cantor-like Sets of Real Numbers

In this section, we will continue to develop the properties of $\mathbb{R}$ by comparing the properties of $\mathbb{R}^+ \setminus \mathbb{R}_{\aleph}^{\cup}$ and $\mathbb{R}^+ \setminus \mathbb{R}_0^{\cup}$ to the well-known properties of Cantor sets.

Definition 7.2.1 Munkres constructs a Cantor set C as follows [13].

> "Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its 'middle third' $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its 'middle thirds' $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation
>
> $$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \quad .$$
>
> The intersection
>
> $$C = \bigcap_{n \in \mathbb{Z}^+} A_n \quad ,$$
>
> is called the [*ternary*] Cantor set; it is a subspace of $[0, 1]$."

Remark 7.2.2 The interval $[0, 1]$ is the image of **AB** under the fractional distance map. This likeness will serve as the basis for the analytical direction of the present section.

Definition 7.2.3 Define two Cantor-like sets

$$\mathbb{F}_0 = [0, \infty] \setminus \mathbb{R}_0^{\cup} \quad , \quad \text{and} \quad \mathbb{F}_{\aleph} = [0, \infty] \setminus \mathbb{R}_{\aleph}^{\cup} \quad .$$

Corollary 7.2.4 *Neither* $\mathbb{F}_0$ *nor* $\mathbb{F}_{\aleph}$ *is the empty set.*

Proof. Proof follows from Theorem 7.1.9 and Main Theorem 7.1.11: there exist more positive real numbers than are in either variant of $\mathbb{R}^{\cup}$. The interval $[0, \infty]$ is connected by Axiom 2.1.7, and a connected set cannot be covered by a disjoint union of its open subsets. (We say $\mathbb{R}_0^0$ and $\mathbb{R}_{\aleph}^0$ are open sets in the subspace topology of $[0, \infty]$ even though they are not strictly open in $\mathbb{R}$.) The corollary is proven. ∎

Remark 7.2.5 To construct $\mathbb{F}_0$ and $\mathbb{F}_{\aleph}$, we have subtracted from $[0, \infty]$ the relevant neighborhood of $\aleph_{\mathcal{X}}$ for every infinite decimal number $0 \leq \mathcal{X} \leq 1$. Whatever remains is a "dust" of some sort. For this reason, we call $\mathbb{F}_0$ and $\mathbb{F}_{\aleph}$ Cantor-like sets.

Lemma 7.2.6 $\mathbb{F}_{\aleph}$ *is a strict subset of* $\mathbb{F}_0$*, or it is exactly equal to* $\mathbb{F}_0$*, i.e.:* $\mathbb{F}_{\aleph} \subseteq \mathbb{F}_0$.

Proof. Proof follows from Definition 7.2.3. $\mathbb{F}_\aleph$ is constructed by deleting open intervals whose lengths are at least as great as those deleted in the construction of $\mathbb{F}_0$. Each variant of deleted interval is centered about $\aleph_\mathcal{X}$. $\mathbb{F}_\aleph \subseteq \mathbb{F}_0$ because $\mathbb{R}_0^\mathcal{X} \subseteq \mathbb{R}_\aleph^\mathcal{X}$. If $\mathbb{R}_C^\mathcal{X} = \varnothing$, then $\mathbb{F}_\aleph = \mathbb{F}_0$ (which is what we will choose in Section 7.4.)

Theorem 7.2.7 *$\mathbb{F}_0$ and $\mathbb{F}_\aleph$ are closed subsets of $[0, \infty]$.*

Proof. A subset $S \subset T$ is closed in T if and only if its complement in T is open. The complements of $\mathbb{F}_0$ and $\mathbb{F}_\aleph$ in $[0, \infty]$ are $\mathbb{R}_0^\cup$ and $\mathbb{R}_\aleph^\cup$ respectively, both of which are disjoint unions of open sets. $\mathbb{F}_0$ and $\mathbb{F}_\aleph$ are closed in $[0, \infty]$.

Remark 7.2.8 When constructing the ternary Cantor set (Definition 7.2.1), the least element of the final result of iterative deletions is zero. By construction, the endpoints of the intervals left after each deletion of a middle third will remain forever, so it is already given at the A_1 step that the least number in the parent interval $[0, 1]$, which is zero, will be the least element of the resultant Cantor set. When defining $\mathbb{F}$ in either variant, it is not immediately apparent what will be the least element because $0 \in \mathbb{R}_\aleph^0 \supseteq \mathbb{R}_0^0$ is deleted at the first step. Since $\mathbb{F}$ is closed, however, we know it does have a least element.

Definition 7.2.9 The connected elements of $\mathbb{F}_0$ are provisionally labeled $\mathbb{F}_0(n)$, and the connected elements of $\mathbb{F}_\aleph$ are provisionally labeled $\mathbb{F}_\aleph(n)$. The labeling convention in either variant is such that

$$\forall x \in \mathbb{F}(n) \quad \forall y \in \mathbb{F}(m) \quad \text{s.t.} \quad n > m \quad \implies \quad x > y \ .$$

Each $\mathbb{F}(n)$ is connected, and every two $\mathbb{F}(n), \mathbb{F}(m)$ are disconnected whenever $n \neq m$.

Remark 7.2.10 We have deleted an uncountable infinity of $\mathbb{R}_\aleph^\mathcal{X}$ neighborhoods to construct $\mathbb{F}_\aleph$. The elements of $\mathbb{F}_\aleph$ separate these neighborhoods, so the number of disconnected elements $\mathbb{F}_\aleph$ must be uncountably infinite. Such elements cannot be enumerated with $n \in \mathbb{N}$. To the contrary, the set $\mathbb{N}_\infty$ (Definition 6.2.3) has a countably infinite number of elements $\aleph_\mathcal{X} - n$ and a similar number of $\aleph_\mathcal{X} + n$ for each of an uncountably infinite number of $\mathcal{X}$. It is guaranteed that $n \in \mathbb{N}_\infty$ will provide a sufficient labeling scheme for $\mathbb{F}(n)$.

Definition 7.2.11 Among a collection of $\mathbb{R}_0^\mathcal{X}$ or $\mathbb{R}_\aleph^\mathcal{X}$ sets, the set with the greatest value of $\mathcal{X}$ is said to be the rightmost set, and that with the least value of $\mathcal{X}$ is said to be the leftmost set.

Proposition 7.2.12 For every $\mathbb{F}_0(n)$ or $\mathbb{F}_\aleph(n)$, the respective subset of $\mathbb{R}_0^\cup$ or $\mathbb{R}_\aleph^\cup$ whose elements are less than any x in $\mathbb{F}_0(n)$ or $\mathbb{F}_\aleph(n)$ has a rightmost set,

and the subset of $\mathbb{R}_0^{\cup}$ or $\mathbb{R}_{\aleph}^{\cup}$ whose elements are greater than x has a leftmost set.

Justification. We will neglect the 0 and $\aleph$ subscripts in this justification. Let $\mathbb{R}_-^{\cup}$ be the set of all $\mathbb{R}^{\mathcal{X}}$ whose elements are less than any $x \in \mathbb{F}(n)$, and let $\mathbb{R}_+^{\cup}$ be the set of all $\mathbb{R}^{\mathcal{X}}$ whose elements are greater. By Definition 7.2.9, $\mathbb{F}(n)$ is a connected interval, and every two $\mathbb{F}(j), \mathbb{F}(k)$ are disconnected whenever $j \neq k$. Furthermore, Corollary 7.1.5 proves that every two $\mathbb{R}^{\mathcal{X}} \neq \mathbb{R}^{\mathcal{Y}}$ are disconnected. Since $[0, \infty]$ is a connected union of $\mathbb{F}(n)$ and $\mathbb{R}^{\cup}$, with the former being closed intervals and the latter being open, it follows that the structure of $\mathbb{R}^+$ is an ordered union

$$\mathbb{R}^+ = \ldots \mathbb{F}(n) \cup \mathbb{R}^{\mathcal{X}} \cup \mathbb{F}(n+1) \cup \mathbb{R}^{\mathcal{Y}} \cup \mathbb{F}(n+2) \ldots \quad .$$

This contradicts Theorem 7.1.2, however. If there was an $\mathbb{R}^{\mathcal{Z}}$ between $\mathbb{R}^{\mathcal{X}}$ and $\mathbb{R}^{\mathcal{Y}}$, then it would necessarily be $\mathbb{R}^{\mathcal{Z}} \subset \mathbb{F}(n+1)$ contradicting the definition of $\mathbb{F}$. The connected property of $\mathbb{R}$ requires, therefore, that we introduce an alternative labeling scheme before continuing with this justification.

Definition 7.2.13 For $n \in \mathbb{N}_\infty$, the connected elements of $\mathbb{R}_0^{\cup}$ are labeled $\mathbb{R}_0(n)$, and the connected elements of $\mathbb{R}_{\aleph}^{\cup}$ are labeled $\mathbb{R}_{\aleph}(n)$. The labeling convention in either variant is such that

$$\forall x \in \mathbb{R}(n) \quad \forall y \in \mathbb{R}(m) \quad \text{s.t.} \quad n > m \quad \Longrightarrow \quad x > y \quad .$$

It follows that $\mathbb{R}_0^0 = \mathbb{R}_0(1)$ and $\mathbb{R}_{\aleph}^0 = \mathbb{R}_{\aleph}(1)$. We say that $\mathbb{R}_0(n)$ is the natural neighborhood of $\aleph(n)$ and $\mathbb{R}_{\aleph}(n)$ is its whole neighborhood. Specifically, $\aleph(1) = \aleph_0 = 0$.

Continuing with the justification of Proposition 7.2.12, we may now infer that $[0, \infty)$ is constructed as an ordered union in the form

$$[0, \infty) = \mathbb{R}(1) \cup \mathbb{F}(1) \ldots \mathbb{R}(n) \cup \mathbb{F}(n) \cup \mathbb{R}(n+1) \cup \mathbb{F}(n+1) \ldots \quad .$$

Since the connectedness of $\mathbb{R}$ requires the sequential alternation of the disconnected $\mathbb{R}(n)$ and $\mathbb{F}(n)$ in the total ordered union, it follows that $\mathbb{R}(k)$ is the rightmost set in the $\mathbb{R}_-^{\cup}$ whose elements are less than any $x \in \mathbb{F}(k)$, and $\mathbb{R}(k+1)$ is the leftmost set in the $\mathbb{R}_+^{\cup}$ whose elements are greater than any $x \in \mathbb{F}(k)$. This concludes the justification of Proposition 7.2.12. ∎

Remark 7.2.14 In Example 6.2.15, we considered

$$\lim_{b \to \infty} \big(\aleph_{\mathcal{X}} - b\big) = -\infty \quad , \quad \text{and} \qquad \lim_{b \to \widehat{\infty}} \big(\aleph_{\mathcal{X}} - b\big) = \aleph_{\mathcal{X}} - \widehat{\infty} = -\aleph_{(1-\mathcal{X})} \quad ,$$

as two desirable modes of limit behavior. Now the $\mathbb{F}(n)$ notation suggests a third desirable behavior such that

$$\lim_{b \to \aleph(2)} \aleph(n) + b = \aleph(n+1) \quad .$$

(Recall $\aleph(1) = \aleph_0 = 0$.) It may or may not be possible to accommodate this limiting mode. It might be that any sequence which does not converge its own local neighborhood of fractional distance must diverge all the way to infinity in one variety or another. Indeed, the property $\frac{d}{dx}\aleph_x = \widehat{\infty}$ (Theorem 6.2.2) suggests in some sense that once a sequence fails to converge in its local $\aleph_{\mathcal{X}}$-neighborhood, it has to keep diverging to some maximal value.

The likely issue with such a limiting mode as $b \to \aleph(2)$, something which may even amount to a flaw in the justification of Proposition 7.2.12, is that $\aleph(2) = \aleph_{\mathcal{X}_{\min}}$ is such that $\mathcal{X}_{\min}$ is the smallest positive real number. It is generally understood that no such number exists. We have developed a requirement for such a number in the course of supporting Proposition 7.2.12, but the lack of a smallest positive real number is so well established that we might suppose no such number exists. Contrary to our conjuring of $\aleph_{\mathcal{X}}$ by requirement when $\aleph_{\mathcal{X}}$ has only been supposed not to exist, there exists a large body of demonstrations that $\mathcal{X}_{\min}$ cannot exist. If such a number as $\mathcal{X}_{\min}$ can be derived from the ordered union given in Proposition 7.2.12, then that will be very exciting. However, there are many problems associated with such a line of reasoning. We will present a few of them in Section 7.3, and then we will not use the (n) notations in a problematic way moving forward.

Definition 7.2.15 This definition regards both $0, \aleph$ subscript variants of the relevant objects. To avoid the problematic (n) notation, label each connected element of $\mathbb{F}$ as $\mathbb{F}^{\mathcal{X}}$. For every $\mathbb{R}^{\mathcal{X}}$, there exists a unique $\mathbb{F}^{\mathcal{X}}$ such that

$$x \in \mathbb{R}^{\mathcal{X}} \quad , \quad z \in \mathbb{F}^{\mathcal{X}} \quad \implies \quad x < z \quad ,$$

and

$$y \in \mathbb{R}^{\mathcal{Y}} \quad , \quad z \in \mathbb{F}^{\mathcal{X}} \quad , \quad \mathcal{Y} > \mathcal{X} \quad \implies \quad y > z \quad .$$

In other words, there is a closed interval $\mathbb{F}^{\mathcal{X}}$ right-adjacent to every $\mathbb{R}^{\mathcal{X}}$ whenever $0 \leq \mathcal{X} < 1$. With this definition, we move the elements of $\mathbb{F}$ into the non-problematic superscript $\mathcal{X}$ labeling scheme as opposed to moving the $\mathbb{R}^{\mathcal{X}}$ into the (n) scheme, as in Definition 7.2.13.

§7.3 Paradoxes Related to Infinitesimals

In this section, we demonstrate a few paradoxes, or contradictions, invoked by the $\mathbb{R}_0(n)$ enumeration scheme and its corollary concept of a least positive real number. We solve some of the paradoxes in this section with the superscript $\mathcal{X}$ label (Definition 7.2.15), and the other paradoxes are resolved in Section 7.5.

Definition 7.3.1 $\mathcal{F}(n) \in \mathbb{R}$ is the unique real number in the center of $\mathbb{F}_0(n)$ and $\mathbb{F}_{\aleph}(n) \subseteq \mathbb{F}_0(n)$. If $\mathbb{F}_0(n) = [a, b]$ and $\mathbb{F}_{\aleph}(n) = [c, d]$, then

$$\text{len}[a, \mathcal{F}(n)] = \text{len}[\mathcal{F}(n), b] \quad , \quad \text{and} \qquad \text{len}[a, \mathcal{F}(n)] = \text{len}[\mathcal{F}(n), b] \quad ,$$

as in Definition 3.2.1. In the alternative labeling scheme, $\mathcal{F}_{\mathcal{X}}$ is the number in the center of $\mathbb{F}_0^{\mathcal{X}}$ and $\mathbb{F}_{\aleph}^{\mathcal{X}}$. In either label, the number has the properties

$$\text{Big}(\mathcal{F}_{\mathcal{X}}) = \mathcal{F}_{\mathcal{X}} \quad , \quad \text{and} \qquad \text{Lit}(\mathcal{F}_{\mathcal{X}}) = 0 \quad .$$

Theorem 7.3.2 *The number $\mathcal{F}(1) = \mathcal{F}_0$ has infinitesimal fractional magnitude with respect to* **AB**.

Proof. We will use Robinson's standard non-standard definition of a hyperreal infinitesimal [14,15]. A number ε is a positive infinitesimal number if and only if

$$\forall x \in \mathbb{R}^+ \quad \exists \varepsilon \notin \mathbb{R} \quad \text{s.t.} \quad 0 < \varepsilon < x \quad .$$

By construction, $\mathcal{F}(1)$ is the number in the center of the gap between $\mathbb{R}_{\aleph}(1) = \mathbb{R}_{\aleph}^0$ and $\mathbb{R}_{\aleph}(2) \in \{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$. Since $\mathcal{F}(1)$ is not in $\mathbb{R}_{\aleph}(1) = \mathbb{R}_{\aleph}^0$, it cannot have zero fractional magnitude; $\mathbb{R}_{\aleph}^0$ is the set of all numbers having zero fractional magnitude along **AB**. If $\mathcal{F}(1)$ had non-zero real fractional magnitude, then it would be $\mathcal{F}(1) \in \{\mathbb{R}_0^{\mathcal{X}}\}$, an obvious contradiction because $\mathcal{F}(1)$ has less fractional magnitude than any nested element in that set of sets. If we denote the fractional magnitude of $\mathcal{F}(1)$ with the symbol ε, the properties of this magnitude are exactly those given above in the definition of an infinitesimal. The theorem is proven. ∎

Definition 7.3.3 A number is said to be a measurable number if it can exist within the algebraic representation of some $X \in$ **AB**. If a number is not measurable, then it is immeasurable.

Theorem 7.3.4 *Every $x \in \mathbb{F}$ is an immeasurable number.*

Proof. The FDFs are bijective between their domain AB and range $[0,1] \subset \mathbb{R}$. The range is a real interval containing no numbers with infinitesimal parts (Axiom 2.1.6), so this tells us that $\mathcal{F}(n)$ is not in the algebraic representation of any geometric $X \in$ **AB**. In the $\mathcal{X}$ notation, the numbers in each $\mathbb{F}^{\mathcal{X}}$ have infinitesimally more fractional distance along **AB** than the numbers in each $\mathbb{R}^{\mathcal{X}}$. ∎

Remark 7.3.5 We have granted that every geometric point X has an algebraic representation (Axiom 2.3.12), but we have not required the opposite. Therefore, there is no problem with an infinitesimal fractional magnitude for $\mathcal{F}(1)$ because there is no corresponding point $X \in$ **AB** that is required to have infinitesimal fractional distance. Although $\mathcal{F}(1) = \mathcal{F}_0$ has infinitesimal fractional magnitude, the number itself is very large. It is greater than any natural number.

Paradox 7.3.6 As the number in the center of the set separating successive neighborhoods of fractional distance, every $\mathcal{F}(n)$ has the property

$$\mathcal{F}(n) = \frac{\aleph(n) + \aleph(n+1)}{2} \quad . \tag{7.1}$$

Every $\mathbb{R}_0^{\mathcal{X}}$ can be obtained by a translation operation on another element of $\{\mathbb{R}_0^{\mathcal{X}}\}$. Doing set-wise arithmetic, we may write, for instance

$$\hat{T}(\aleph_\delta)\mathbb{R}_0^{(\mathcal{X}-\delta)} = \aleph_\delta + \mathbb{R}_0^{(\mathcal{X}-\delta)} = \mathbb{R}_0^{\mathcal{X}} \quad .$$

Letting $AB \equiv [\aleph(n), \aleph(n+1)]$ for some $n \geq 2$, define

$$S_n = \mathbb{R}_0(n) \cap AB \quad , \quad \text{and} \qquad S_{n+1} = \mathbb{R}_0(n+1) \cap AB \quad ,$$

and note that translational symmetry requires

$$\text{len}\big(S_n\big) = \text{len}\big(S_{n+1}\big) \quad .$$

Since

$$AB = S_n \cup \mathbb{F}_0(n) \cup S_{n+1} \quad ,$$

and since $\mathcal{F}(k)$ is the number in the center of the closed interval $\mathbb{F}_0(k)$, it is obvious that $\mathcal{F}(n)$ is the unique number in the center of the line segment AB. In the Euclidean metric, this number is always the average of the least and greatest numbers in the algebraic representations of A and B respectively. However, if the $\mathbb{R}_0(n)$ notation is a label for $\mathbb{R}_0^{\mathcal{X}}$ where $\mathcal{X}$ is strictly a real number, then, using the original labeling scheme without (n), we find

$$\mathcal{F}(n) = \frac{\aleph_{\mathcal{X}} + \aleph_{\mathcal{Y}}}{2} = \aleph_{\left(\frac{\mathcal{X}+\mathcal{Y}}{2}\right)} \quad .$$

This number is most obviously an element of $\mathbb{R}_0^{\left(\frac{\mathcal{X}+\mathcal{Y}}{2}\right)}$. This contradicts the definition $\mathcal{F}(n) \in \mathbb{F}_0(n)$. It is paradoxical that $\aleph(n+1)$ as in Equation (7.1) cannot have any corresponding $\aleph_{\mathcal{Y}}$.

Paradox Resolution 7.3.7 Paradox 7.3.6 does not exist in the $\mathcal{F}_{\mathcal{X}}$ notation. If we never suppose the existence of $\aleph(n+1)$, then there is no starting point in Equation (7.1), and the paradox cannot be demonstrated, *i.e.*:

$$\mathcal{F}_{\mathcal{X}} \neq \frac{\aleph_{\mathcal{X}} + \aleph_{\mathcal{Y}}}{2} \quad .$$

Neither can we write in that case $[\aleph(n), \aleph(n+1)] = [\aleph_{\mathcal{X}}, \aleph_{\mathcal{Y}}]$ because the continuous spectrum of fractional distance does not allow us to select consecutive $\mathcal{X}, \mathcal{Y}$.

Paradox 7.3.8 The neighborhood of the origin $\mathbb{R}_0^0 = \mathbb{R}_0(1)$ contains numbers of the form $\aleph_{\mathcal{X}} + b$ for b strictly non-negative (and $\mathcal{X} = 0$), but every intermediate neighborhood allows both signs for b. It follows that

$$\text{len}\big(\mathbb{R}_0(1)\big) = \frac{1}{2}\,\text{len}\big(\mathbb{R}_0(2)\big) \quad , \quad \text{and} \qquad \mathcal{F}(1) = \frac{1}{3}\,\mathcal{F}(2) \quad .$$

Every element of $\mathbb{R}_0(2)$ has positive real fractional magnitude because $\mathbb{R}_0(2) \subset \{\mathbb{R}_0^{\mathcal{X}}\}$, but, if $\mathcal{F}(1)$ has infinitesimal fractional magnitude ε, then 3ε is also less than any real number (according to Robinson's arithmetic for hyperreal numbers [14, 15].) If 3ε is infinitesimal, then that contradicts the ordering

$$x \in \mathbb{R}_0(n) \qquad \Longrightarrow \qquad x < \mathcal{F}(n) \quad .$$

Paradox Resolution 7.3.9 Paradox 7.3.8 is resolved in the $\mathcal{F}_{\mathcal{X}}$ formalism. We can uniquely associate $\mathcal{F}(1) = \mathcal{F}_0$ but there is no $\mathcal{F}_{\mathcal{X}_{\min}}$ that we might associate with $\mathcal{F}(2)$.

Paradox 7.3.10 Each $\mathbb{F}_0(n) \subset \mathbb{F}_0$ is a closed, connected interval. It is required, then, that

$$\mathbb{F}_0(1) = [a, b] \quad , \quad \text{and} \qquad \mathcal{F}(1) = \frac{b-a}{2} \quad .$$

Assuming the normal arithmetic for $x \in \mathbb{F}$ (which has not yet been axiomatized), it follows that

$$a = \sup(\mathbb{R}_0) = b - 2\mathcal{F}(1) \quad .$$

This is paradoxical for the reasons presented in Proposition 7.1.14: $\mathbb{R}_0$ ought not have a supremum.

Paradox 7.3.11 If $\mathcal{F}(1)$ is a real number centered in the closed interval $\mathbb{F}(1)$, then, assuming the normal arithmetic for $x \in \mathbb{F}$, we find that $2\mathcal{F}(1) = \aleph_{\mathcal{X}_{\min}}$ with $\mathcal{X}_{\min}$ being the least positive real number. This number is thought not to exist, with good reason. Therefore, the implied identity

$$\mathcal{F}(1) = \frac{\aleph_{\mathcal{X}_{\min}}}{2} \quad ,$$

has the appearance of being inadmissibly paradoxical.

§7.4 Complements of Natural Neighborhoods

In this section, we take many of the facts established in the previous sections and begin to put them together to form a coherent picture of $\mathbb{F}^{\mathcal{X}}$, $\mathbb{R}_0^{\mathcal{X}}$, $\mathbb{R}_{\aleph}^{\mathcal{X}}$, $\mathbb{R}_C^{\mathcal{X}}$, and the rest. This is what we know so far:

- We have defined $\mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_C^{\mathcal{X}} = \mathbb{R}_{\aleph}^{\mathcal{X}}$.

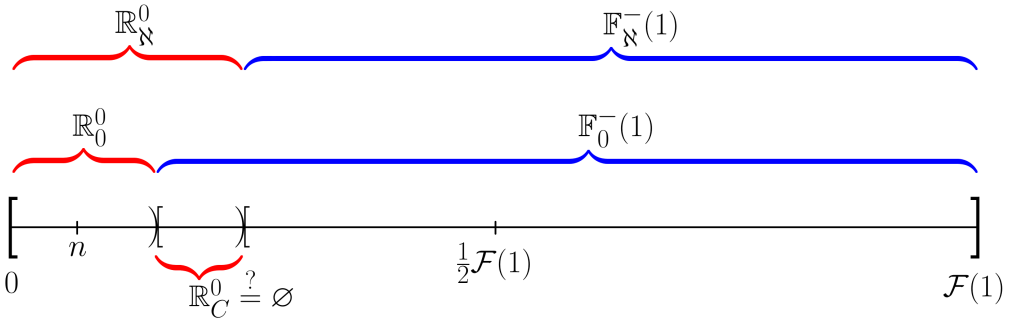

Figure 2: This figure (*not to scale!*) shows the neighborhood of the origin $\mathbb{R}^0_{\aleph}$, the substructure of that neighborhood, and the associated structure from the Cantor-like sets. $\mathbb{F}^-(1)$ refers to the subset of $\mathbb{F}(1)$ which is less than or equal to $\mathcal{F}(1) = \mathcal{F}_0$.

- We do not know whether $\mathbb{R}^{\mathcal{X}}_C = \varnothing$. This will be the main issue decided in the present section.
- We have not yet defined any arithmetic operations for $x \in \mathbb{F}^{\mathcal{X}} \cup \mathbb{R}^{\mathcal{X}}_C$.
- We have not yet given an algebraic construction for $x \in \mathbb{F}^{\mathcal{X}} \cup \mathbb{R}^{\mathcal{X}}_C$.

Remark 7.4.1 In this section, we will use $\mathcal{F}(1) = \mathcal{F}_0$ to refer to a real number which is an upper bound on $\mathbb{R}_0$ without assuming an attendant problematic (n) enumeration scheme.

Theorem 7.4.2 *If we assume the usual arithmetic for $\mathcal{F}_{\mathcal{X}}$, then the set $\mathbb{R}^0_{\aleph}$ lies within the left endpoint A of the line segment $AB \equiv [0, \mathcal{F}_0]$. In other words, every element of $\mathbb{R}^0_{\aleph}$ has zero fractional magnitude even with respect to len$[0, \mathcal{F}_0] \lll$ len* **AB**.

Proof. Every interval has a unique number at its center. For $AB \equiv [0, \mathcal{F}_0]$, this number is $c = \frac{1}{2}\mathcal{F}_0$, as in Figure 2. If $c \in \mathbb{R}^0_{\aleph}$, meaning that the fractional magnitude with respect to **AB** was zero, then $2c = \mathcal{F}_0$ would also have $2 \times 0 = 0$ fractional magnitude with respect to **AB**. This is contradictory because it would require $\mathcal{F}_0 \in \mathbb{R}^0_{\aleph}$. Continuing the argument, we find that for any $n \in \mathbb{N}$, the number $\frac{1}{n}\mathcal{F}_0$ must not be an element of $\mathbb{R}^0_{\aleph}$. Now assume $\frac{1}{n}\mathcal{F}_0 \in X \in AB \equiv [0, \mathcal{F}_0]$ and $X \neq A$. Since the quotient of two line segments is defined as a real number (Definition 3.1.1), and since the difference of two real numbers is always greater than some inverse natural number (Axiom 2.1.6), we may write for some $m \in \mathbb{N}$

$$\frac{AX}{AB} - \frac{AA}{AB} > \frac{1}{m} \quad .$$

This is satisfied for any $X \neq A$, so $\frac{1}{n}\mathcal{F}_0$ can be a number in the algebraic representation of any $X \neq A$. Since $\frac{1}{n}\mathcal{F}_0 \notin \mathbb{R}^0_{\aleph}$, $\mathbb{R}^0_{\aleph}$ must be a subset of, or equal to, the algebraic representation of the left endpoint A of $AB \equiv [0, \mathcal{F}_0]$.

Corollary 7.4.3 *If we assume the usual arithmetic for $\mathcal{F}_{\mathcal{X}}$, then for any $x \in \mathbb{R}_0$ such that $x \in X$, and for $X \in AB$ such that $AB \equiv [0, \mathcal{F}_0]$, we have*

$$\mathcal{D}_{AB}(AX) = 0 \quad .$$

Proof. By the property $\mathbb{R}_0^0 \subseteq \mathbb{R}_{\aleph}^0$, proof follows from Theorem 7.4.2.

Alternatively, $\mathcal{D}_{AB}$ is such that

$$\mathcal{D}_{AB}(AX) = \mathcal{D}_{AB}^{\dagger}(AX) = \frac{x}{\mathcal{F}_0} \quad .$$

The case of $x = 0$ is trivial. To prove the other cases by contradiction, suppose $z > 0$ and that

$$\frac{x}{\mathcal{F}_0} = z \quad .$$

Since $x < \mathcal{F}_0$ and $x, \mathcal{F}_0 \in \mathbb{R}^+$, it follows that $0 < z < 1$. All such $z \in \mathbb{R}_0$ have a multiplicative inverse $z^{-1} \in \mathbb{R}_0$ so

$$\frac{x}{z\mathcal{F}_0} = 1 \quad \iff \quad z^{-1}x = \mathcal{F}_0 \quad .$$

This is a contradiction because $z^{-1}x \in \mathbb{R}_0$ but $\mathcal{F}_0$ is greater than any element of $\mathbb{R}_0$. ∎

Remark 7.4.4 Suppose we define $\digamma_{\mathcal{X}} = \mathcal{X} \cdot \mathcal{F}_0$ so that it mirrors the structure of $\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty}$. Since $\mathbb{R}_{\aleph}^0$ has zero fractional distance even along $AB \equiv [0, \mathcal{F}_0]$, we could define a set of whole neighborhoods along AB

$$\mathbb{R}_{\mathcal{F}}^{\mathcal{X}} = \left\{ \digamma_{\mathcal{X}} + b \;\middle|\; b \in \mathbb{R}_{\aleph}^0 \right\} \quad ,$$

exactly dual to the elements of $\{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$ spaced along $\mathbf{AB} \equiv [0, \infty]$. By subtracting every $\mathbb{R}_{\mathcal{F}}^{\mathcal{X}}$ from the interval $[0, \mathcal{F}_0]$, we would create another Cantor-like set. Following the prescription given in Section 7.2, we would invoke the connectedness of the interval to label the disconnected elements of the new Cantor-like set, and we would label the numbers in the centers of each of those disconnected intervals. Call those numbers $\mathcal{G}(n)$ labeled with $n \in \mathbb{N}_{\infty}$ so that they are dual to the $\mathcal{F}(n)$ in the duality transformation $[0, \infty] \to [0, \mathcal{F}_0]$. The $\mathcal{G}(n)$ will have a non-problematic labeling scheme $\mathcal{G}_{\mathcal{X}}$ with $\mathcal{X}$ measuring fractional distance along $AB \equiv [0, \mathcal{F}_0] \not\equiv \mathbf{AB}$. By replicating the present course of analysis, we could show that no element of $\mathbb{R}_{\aleph}^0$ has non-zero fractional magnitude even with respect to $\text{len}[0, \mathcal{G}_0] \lll \text{len}[0, \mathcal{F}_0] \lll \text{len}(\mathbf{AB})$. We could do this forever, defining more and more, tinier and tinier Cantor-like sets, and $\mathbb{R}_{\aleph}^0$ would never leave the left endpoint A of the line segment whose algebraic representation is $[0, \Gamma_0]$ with Γ_0 being the number in the center of the leftmost connected element of the umpteenth Cantor-like set. This overarching condition of scale invariance among $[0, \widehat{\infty}]$, $[0, \mathcal{F}_0]$, $[0, \mathcal{G}_0]$, and so on reflects what is called fractal structure.

Remark 7.4.5 To accommodate the interpretation of the positive branch of $\mathbb{R}$ as a Euclidean line segment, we were forced to introduce numbers in the form $\aleph_{\mathcal{X}}$. As a consequence, we were forced to introduce numbers of the form $\mathcal{F}(n) = \mathcal{F}_{\mathcal{X}}$ to describe the numbers in the Cantor-like sets whose elements are $\mathbb{F}^{\mathcal{X}}$. If we tried to define $\mathbb{F}^{\mathcal{X}}$ as a neighborhood in the form

$$\mathbb{F}^{\mathcal{X}} \stackrel{?}{=} \left\{ \mathcal{F}_{\mathcal{X}} + b \;\middle|\; |b| \in \mathbb{R}_{\aleph}^{0} \right\} \quad ,$$

then we would immediately encounter a problem. This set is clearly open, while we have already proven that it must be closed (Theorem 7.2.7). Therefore, we are left with a mystery set $\mathbb{F}_0$ whose elements are not easily decided. Since this is the second such set we have—$\mathbb{R}_C^{\mathcal{X}}$ being the first—we ought to combine them into a single mystery set. To do so, we will conjecture $\mathbb{R}_{\aleph}^{\mathcal{X}} \setminus \mathbb{R}_0^{\mathcal{X}} = \varnothing$, which will be tantamount to axiomatizing it in advance of the reliance on this conjecture. With this conjecture (Conjecture 7.4.6), we will establish that every number which is (i) not in $\mathbb{R}_0^{\mathcal{X}}$, (ii) greater than any $x \in \mathbb{R}_0^{\mathcal{X}}$, and yet (iii) not greater than $y \in \mathbb{R}_0^{\mathcal{Y}}$ for any $\mathcal{Y} > \mathcal{X}$, must be in $\mathbb{F}^{\mathcal{X}}$ right-adjacent to $\mathbb{R}_0^{\mathcal{X}}$.

We have not proven that $\mathbb{R}_{\aleph}^{\mathcal{X}} \setminus \mathbb{R}_0^{\mathcal{X}} = \varnothing$, but neither have we proven the existence of such numbers ($\mathbb{R}_{\aleph}^{\mathcal{X}} \setminus \mathbb{R}_0^{\mathcal{X}} \neq \varnothing$) as we have with $\aleph_{\mathcal{X}}$ and $\mathcal{F}_{\mathcal{X}}$. Rather, we have required with Axioms 4.2.6 and 4.2.17 that every $x \in \mathbb{R}$ may be constructed algebraically as an element of $C_{\mathbb{Q}}^{\mathbf{AB}} \subset C_{\mathbb{Q}} \times C_{\mathbb{Q}}$ or as a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$. So far, we have only given such constructions for the numbers in the natural neighborhoods of $\aleph_{\mathcal{X}}$. To avoid needless complication, therefore, we will conjecture that $\mathbb{R}_C^{\mathcal{X}}$ is the empty set and that, consequently, $\mathbb{F}_0 = \mathbb{F}_{\aleph}$. Then we will have all of the $\mathbb{R}_0^{\mathcal{X}} = \mathbb{R}_{\aleph}^{\mathcal{X}} = \mathbb{R}^{\mathcal{X}}$ neighborhoods cleanly defined with elements of $C_{\mathbb{Q}}^{\mathbf{AB}}$, and we will move everything else into the Cantor-like set $\mathbb{F}$, whose numbers we will define as partitions of $C_{\mathbb{Q}}^{\mathbf{AB}}$. By the following conjecture, we will have given algebraic constructions and arithmetic axioms for every number in $\mathbb{R}_0^{\cup} = \mathbb{R}_{\aleph}^{\cup}$. Everything which remains to be completed is transferred by Conjecture 7.4.6 into $\mathbb{F}$.

Conjecture 7.4.6 Every number having zero fractional magnitude with respect to **AB** is an element of $\mathbb{R}_0$. Most generally,

$$\mathbb{R}_{\aleph}^{\mathcal{X}} = \mathbb{R}_0^{\mathcal{X}} \quad , \quad \text{and} \qquad \mathbb{R}_C^{\mathcal{X}} = \varnothing \quad .$$

Remark 7.4.7 One would also want to conjecture the countercase to Conjecture 7.4.6 and examine the requirements for establishing naturally numbered tiers of increasingly large numbers, larger than any natural number but still having zero fractional distance with respect to **AB**. However, we will take the opposite tack here. Now that we have conjectured that the whole and natural neighborhoods are the same, we will drop the 0 and $\aleph$ subscripts from their respective objects.

§7.5 Immeasurable Numbers and The Least Upper Bound Problem

Axiom 4.2.6 gave every $x \in \mathbb{R}$ as the value of a function $f : C_{\mathbb{Q}} \times C_{\mathbb{Q}} \to C_{\mathbb{Q}}^{\mathbf{AB}}$ (Definition 4.2.13), or as a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$. Later, Axiom 4.2.17 granted that every element of $C_{\mathbb{Q}}^{\mathbf{AB}}$ may be represented as an ordered pair $([\mathcal{X}], [b])$ specifying the big and little parts of some $x \in \mathbb{R}^{\cup}$. In this section, we will invoke the partition clause to define the immeasurable $x \in \mathbb{F}$ as Dedekind partitions of $C_{\mathbb{Q}}^{\mathbf{AB}}$. We have already defined the arithmetic of the equivalence classes in Section 5.2, but we have not proven that partitions obey the arithmetic axioms in the way that we have for the direct equivalence classes $[x] \in C_{\mathbb{Q}}^{\mathbf{AB}} \subset C_{\mathbb{Q}} \times C_{\mathbb{Q}}$. In this section, we will also prove that the immeasurables do not, and cannot, obey the arithmetic axioms. This will be due mainly to the least upper bound problem (Proposition 7.1.14) which we will solve and avoid. We will prove that every $x \in \mathbb{F}$ is some $\mathcal{F}_{\mathcal{X}}$, or that, equivalently, $\mathbb{F} = \{\mathcal{F}_{\mathcal{X}}\}$, meaning the subsets of $\mathbb{F}$ are unions of isolated points. $\mathbb{F}$ will be a discrete set of numbers. This will require that each successive pair of $\mathbb{R}^{\mathcal{X}}$ share an extremum; the supremum of one is the infimum of the next. In dealing with $\mathbb{F}$, we will suggest that the $x \in \mathbb{F}$ can be used as a set of analogue natural numbers for the construction of a chart related to the original Euclidean chart by a scale factor $\mathcal{F}_0$, where $\mathcal{F}_0 = \mathcal{F}(1)$ is the least immeasurable number. Motivated by the regular spacing of the $\mathcal{F}_{\mathcal{X}} = \mathcal{F}(n)$, we would define (hypothetically) a transfinite, rescaled version of $\mathbb{N}$, call it $\mathbb{N}_{\mathbb{F}}$, whose unit increment is such that natural numbers $n \in \mathbb{N}$ have vanishing fractional distance with respect to $1 \in \mathbb{N}_{\mathbb{F}}$. Then we would define zero as the least number in the algebraic representation of the left endpoint of some AB to infer an analogue $\mathbb{Q}_{\mathbb{F}}$ supporting the construction of $\mathbb{F}$-labeled analogues of $C_{\mathbb{Q}}$ and $C_{\mathbb{Q}}^{\mathbf{AB}}$. In this way, the formal algebraic construction of a transfinite number system follows as direct consequence of $\mathbb{R}$ without any further, extraneous inputs beyond the initial supposition that $\mathbb{N}$ does exist. We will conclude this section showing that the immeasurables conform to the requirements of the Archimedes property of real numbers.

Definition 7.5.1 A Dedekind cut is a partition of the rationals $\mathbb{Q}$ into two sets L and R such that every real number is equal to some partition $x = (L, R)$ with the following properties.

- $L \neq \mathbb{Q}$ is non-empty.
- If (i) $x, y \in \mathbb{Q}$, (ii) $x < y$, and (iii) $y \in L$, then $x \in L$.
- If $x \in L$, then there exists $y \in L$ such that $y > x$.

Definition 7.5.2 An extended Dedekind cut is a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$ into two sets L and R such that every real number is equal to some partition $x = (L, R)$ with the following properties.

- $L \neq C_{\mathbb{Q}}^{\mathbf{AB}}$ is non-empty.

- If (i) $x, y \in C_{\mathbb{Q}}^{\mathbf{AB}}$, (ii) $x < y$, and (iii) $y \in L$, then $x \in L$.
- If $x \in L$, then there exists $y \in L$ such that $y > x$.

Theorem 7.5.3 *The number $\mathcal{F}_0 \in \mathbb{R}$ is an extended Dedekind partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$.*

Proof. Let

$$L_0 = \left\{ [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\middle|\; \mathrm{Big}(x) = 0 \right\} \;, \quad \text{and} \qquad R_0 = \left\{ [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\middle|\; \mathrm{Big}(x) > 0 \right\} \;,$$

and consider that Dedekind himself wrote the following [3].

> "In every case in which a cut (A_1, A_2) is given that is not produced by a rational number, we create a new number, an irrational number a, which we consider to be completely defined by this cut; we will say that the number corresponds to this cut or that it produces the cut."

In that vein, the cut (L_0, R_0) is not produced by a measurable number $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$, so we create the new number $\mathcal{F}_0$: an immeasurable number. We say that the cut "produces" this number, or vice versa. Therefore, $\mathcal{F}_0 = (L_0, R_0)$, and the theorem is proven. ∎

Definition 7.5.4 The extended Dedekind form of $\mathcal{F}_{\mathcal{X}} = (L_{\mathcal{X}}, R_{\mathcal{X}}) \in \mathbb{R}$ is such that

$$L_{\mathcal{X}} = \left\{ [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\middle|\; \mathrm{Big}(x) \leq \aleph_{\mathcal{X}} \right\}$$
$$R_{\mathcal{X}} = \left\{ [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\middle|\; \mathrm{Big}(x) > \aleph_{\mathcal{X}} \right\} \;.$$

Main Theorem 7.5.5 *$\mathcal{F}_{\mathcal{X}}$ is the only number in $\mathbb{F}^{\mathcal{X}} \subset \mathbb{R}$. In other words, $\mathbb{F}^{\mathcal{X}} = \mathcal{F}_{\mathcal{X}}$ or, equivalently, $\mathbb{F} = \{\mathcal{F}_{\mathcal{X}}\}$.*

Proof. It will suffice to prove this theorem if we show that $\mathbb{F}^{\mathcal{X}}$ is a one-point set. Definition 7.2.15 gives

$$x \in \mathbb{R}^{\mathcal{X}} \;\;,\;\; z \in \mathbb{F}^{\mathcal{X}} \quad \Longrightarrow \quad x < z \;\;,$$

and

$$y \in \mathbb{R}^{\mathcal{Y}} \;\;,\;\; z \in \mathbb{F}^{\mathcal{X}} \;\;,\;\; \mathcal{Y} > \mathcal{X} \quad \Longrightarrow \quad y > z \;\;.$$

This definition establishes that $z \in \mathbb{F}^{\mathcal{X}}$ is an upper bound on $\mathbb{R}^{\mathcal{X}}$, and a lower bound on $\mathbb{R}^{\mathcal{Y}}$ for any $\mathcal{Y} > \mathcal{X}$. If $z \in \mathbb{F}^{\mathcal{X}}$ is simultaneously (i) the least upper bound of $\mathbb{R}^{\mathcal{X}}$, and (ii) the greatest lower bound of some $\mathbb{R}^{\mathcal{Y}}$, then $z = \mathcal{F}_{\mathcal{X}}$ is the unique $z \in \mathbb{F}^{\mathcal{X}}$, and the proof will be completed. For proof by contradiction, assume $u \in \mathbb{R}$ is an upper bound on $\mathbb{R}^{\mathcal{X}}$ with the property

$$u < \mathcal{F}_{\mathcal{X}} \;\;.$$

By Axioms 4.2.6 and 4.2.17, u must be a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$ or an equivalence class therein. Therefore, we will divide the proof into two parts.

• (Equivalence class) If u was $[u] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$, then $u \in \mathbb{R}^{\mathcal{U}} \subset \mathbb{R}^{\cup}$. If $\mathcal{U} > \mathcal{X}$, then $u > \mathcal{F}_{\mathcal{X}}$, a contradiction. If $\mathcal{U} \leq \mathcal{X}$, then u is not an upper bound on $\mathbb{R}^{\mathcal{X}}$, another contradiction. Now it is proven that $u \neq [u] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$.

• (Partition) The partition definition is

$$u = (L_u, R_u) \quad .$$

If $u < \mathcal{F}_{\mathcal{X}}$, and $\mathcal{F}_{\mathcal{X}} = (L, R)$, then there exists a non-empty $\Sigma \subset L$ such that

$$L_u = L \setminus \Sigma \quad , \quad \text{and} \qquad R_u = R \cup \Sigma \quad . \tag{7.2}$$

Specifically, $\Sigma = [u, \mathcal{F}_{\mathcal{X}})$. From Definition 7.5.4, we have

$$L = \left\{ [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\middle|\; \mathrm{Big}(x) \leq \aleph_{\mathcal{X}} \right\} \quad .$$

Since u is an upper bound on $\mathbb{R}^{\mathcal{X}}$, there is no non-empty $\Sigma \subset L$ that can satisfy Equation (7.2). We contradict the supposition that such a Σ does exist. Since there is no upper bound on $\mathbb{R}^{\mathcal{X}}$ less than $\mathcal{F}_{\mathcal{X}}$, $\mathcal{F}_{\mathcal{X}}$ must be the least upper bound of $\mathbb{R}^{\mathcal{X}}$.

A similar demonstration proves that $\mathcal{F}_{\mathcal{X}}$ must be the greatest lower bound of some $\mathbb{R}^{\mathcal{Y}}$ with $\mathcal{Y} > \mathcal{X}$. It follows that $\mathbb{F}^{\mathcal{X}}$ is a one-point set. The theorem is proven.

Remark 7.5.6 The least upper bound problem rears its comely head. With Main Theorem 7.5.5, we have given $\mathcal{F}_{\mathcal{X}} = \sup(\mathbb{R}_{\mathcal{X}})$, but we have already made a strong case that no such supremum can exist (Proposition 7.1.14). In the remainder of this section, we will conclude the development of the fractional distance approach to $\mathbb{R}$ such that the reasoning behind the least upper bound problem is carefully sidestepped.

Before we continue, we will outline exactly what it means "to conclude the development." With Main Theorem 7.5.5, we have now given a construction by Cauchy equivalence classes for every $x \in \mathbb{R}$. Every measurable $x \in \mathbb{R}^{\cup}$ is directly a subset of $C_{\mathbb{Q}}^{\mathbf{AB}}$. The arithmetic of such numbers is given in Section 5. Every immeasurable $x \in \mathbb{F}$ is now formally constructed as an extended Dedekind partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$. Since $\mathbb{R} = \mathbb{R}^{\cup} \cup \mathbb{F}$, all real numbers now have a direct algebraic construction. We assume $\mathbb{N}$, define 0, construct $\mathbb{Q}$, $C_{\mathbb{Q}}$ and $C_{\mathbb{Q}}^{\mathbf{AB}}$ thence, and then we take $x \in \mathbb{R}$ as the elements and partitions of $C_{\mathbb{Q}}^{\mathbf{AB}}$. In Section 5.5, however, it was only proven that the equivalence classes themselves obey the arithmetic axioms. We cannot simply throw $\mathcal{F}_{\mathcal{X}}$ in there—not in any rigorous fashion—because there is not a Cauchy equivalence class $[\mathcal{F}_{\mathcal{X}}] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ for any $[\mathcal{X}] \subset C_{\mathbb{Q}}$. Even if we forced the arithmetic axioms onto $\mathcal{F}_{\mathcal{X}}$ with more

axioms, we would immediately hit the least upper bound problem developed in Proposition 7.1.14. Main Theorem 7.5.5 shows that $\mathcal{F}_{\mathcal{X}}$ is the supremum of $\mathbb{R}^{\mathcal{X}}$, but $\mathbb{R}^{\mathcal{X}}$ is not supposed to have a supremum. Now we will solve the least upper bound problem and begin moving toward the Archimedes property of $\mathbb{F} \subset \mathbb{R}$.

Definition 7.5.7 If a real number is $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$, then it is called an arithmatic number (pronounced arith·matic.) All measurable numbers $\mathbb{R}^{\cup} = C_{\mathbb{Q}}^{\mathbf{AB}}$ are arithmatic. If a real number is a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$ not given by any element therein, then it is called a non-arithmatic number. All immeasurable numbers are non-arithmatic.

Axiom 7.5.8 Arithmetic operations are not defined among arithmatic and non-arithmatic numbers.

Remark 7.5.9 What we have done with the separation of the reals into arithmatic and non-arithmatic numbers mirrors the usual separation between canonically algebraic numbers, which are the roots of non-zero polynomials with rational coefficients, and canonically non-algebraic numbers which are not the roots of any such polynomials. Canonically non-algebraic numbers are supposed to exist because they are needed to fill in the gaps in $\mathbb{Q}$ which are not allowed if $\mathbb{R}$ is to satisfy the definition of a simply connected 1D interval. Now we have gone one step further and shown that ***non-arithmatic numbers are needed to fill in the connectedness of the many neighborhoods***.

Definition 7.5.10 If $\mathbb{R}$ has the least upper bound property, then every non-empty subset $S \subset \mathbb{R}$ that has an upper bound must have a least upper bound $u \in \mathbb{R}$ such that $u = \sup(S)$.

Proposition 7.5.11 (Restatement of Proposition 7.1.14: the least upper bound problem.) $\mathbb{R}$ does not have the least upper bound property because $\mathbb{R}_0$ cannot have a supremum if arithmetic is defined for $x \in \mathbb{F}$ in the usual way. If $\mathcal{F}_0 = \sup(\mathbb{R}_0)$, and if $\mathbb{F}$ were to obey the arithmetic axioms in compositions with $x \in \mathbb{R}^{\cup}$, then $\mathcal{F}_0 - 1$ would be an element of $\mathbb{R}_0$. By the closure of $\mathbb{R}_0$, $\mathcal{F}_0 - 1 + 2$ would be another element of $\mathbb{R}_0$. This contradicts the identity $\mathcal{F}_0 = \sup(\mathbb{R}_0)$ because $\mathcal{F}_0 < \mathcal{F}_0 + 1$.

Refutation. Let x be an arithmatic real number. Axiom 7.5.8 is such that

$$\mathcal{F}_0 = \sup(\mathbb{R}_0) \quad \Longrightarrow \quad \sup(\mathbb{R}_0) \pm x = \text{undefined} \ .$$

The arithmetic axioms cannot be used to demonstrate the condition in the justification of this proposition. $\mathbb{R}_0$ ***most certainly can have a supremum***. The supremum of each $\mathbb{R}^{\mathcal{X}}$ (the open set of numbers that are $100 \times \mathcal{X}\%$ of the way down the real number line) is $\mathcal{F}_{\mathcal{X}}$: a non-arithmatic number.

Theorem 7.5.12 $\mathbb{R}$ *has the least upper bound property, which is also called the Dedekind property or Dedekind completeness.*

Proof. If L and R are two non-empty subsets of $\mathbb{R}$ such that $\mathbb{R} = L \cup R$, meaning that (L, R) is a partition of $\mathbb{R}$, and

$$x \in L \ , \ y \in R \quad \Longrightarrow \quad x < y \ ,$$

then the Dedekind property requires that either L has a greatest member or R has a least member. This property is implicit in the connectedness of the algebraic interval. With Main Theorem 7.5.5, we have established the connectedness of the successive disconnected intervals in $\mathbb{R}^{\cup}$. They are connected by the elements of $\mathbb{F}$. All subsets of $\mathbb{R}$ with an upper bound also have a least upper bound because $\{\mathcal{F}_{\mathcal{X}}\}$ guarantees the simple connectedness. The connectedness proves the theorem. ∎

Remark 7.5.13 The non-arithmatic, immeasurable numbers inherit their ordering with respect to the $\leq$ relation from the total ordering of $\mathbb{R}$. The supremum of one measurable neighborhood is the infimum of the next. The maximal neighborhood $\mathbb{R}^1$ does not have a supremum in $\mathbb{R}$, but it is exempted from the Dedekind property because $\mathbb{R}^1$ does not have a real upper bound at all. The upper bound of the maximal neighborhood of infinity diverges. Since no upper bound $u \in \mathbb{R}$ exists for $\mathbb{R}^1$ at all, a real least upper bound on $\mathbb{R}^1$ cannot exist.

Definition 7.5.14 A set S is totally ordered if it obeys the following order axioms.

- (O1a) Elements of S have trichotomy. If $x, y \in S$, then one and only one of the following is true: (i) $x < y$, (ii) $x = y$, or (iii) $x > y$.
- (O2a) The $<$ relation is transitive. If $x, y, z \in S$, then $x < y$ and $y < z$ together imply $x < z$.
- (O3a) If $x, y, z \in S$, then $x < y$ implies $x + z < y + z$, or at least one sum is undefined.
- (O4a) If $x, y, z \in S$, and if $z > 0$, then $x < y$ implies $xz < yz$, or at least one product is undefined.

Remark 7.5.15 Axioms (O1a)-(O4a) are almost exactly the (O1)-(O4) ordering axioms of a complete ordered number field (Axiom 5.4.8). We have changed the last two axioms involving the arithmetic operations $+$ and $\times$. The changes make allowances for the immeasurable $x \in \mathbb{F}$ which are truncated from existence when one adopts a contra-Euclidean, field-related axiom that every real number is less than some natural number.

Theorem 7.5.16 $\mathbb{R}$ *is a totally ordered set.*

Proof. We will prove each of (O1a) to (O4a).

• (O1a) Trichotomy is trivially inherent to the order established for $\mathbb{R}^{\cup}$ (Axiom 5.2.18). Trichotomy is satisfied in $\mathbb{R} = \mathbb{R}^{\cup} \cup \mathbb{F}$ by the result $\mathcal{F}_{\mathcal{X}} = \sup(\mathbb{R}^{\mathcal{X}})$.

• (O2a) Transitivity is satisfied by Axiom 5.2.18 and the corollary results for the extrema $\{\mathcal{F}_{\mathcal{X}}\}$.

• (O3a) Since we have restricted the inequalities in this part of Definition 7.5.14 to the arithmatic numbers, the arithmetic axioms give compliance as stated.

• (O4a) Satisfaction follows in the manner of (O3a).

The ordering relation $\leq$ for $\mathbb{R}$ is such that $\mathbb{R}$ is totally ordered. ✐

Remark 7.5.17 Since we are not using a number field approach to $\mathbb{R}$, we need not state the definition of total order in the exact form of the axioms of a totally ordered, complete number field having unified laws of arithmetic. In Definition 7.5.14, we have minimally modified the usual definition of total order (Axiom 5.4.8) so that (O3a) and (O4a) distinguish arithmatic and non-arithmatic numbers. We will justify this exception to the usual definition of total order as follows.

The lack of arithmetic definitions for immeasurable numbers doesn't affect their well ordering with respect to the measurable ones. The lack of defined operations has no bearing on the concept of the ordering of the set. Indeed, regarding the *geometric notions* of addition and multiplication to which we have referred throughout this analysis of fractional distance, there should not exist geometrically identical arithmetic operations for numbers with and without geometrically measurable fractional distance. Shared geometry-based arithmetic operations would necessarily suggest a shared character of measurablity or immeasurablity, but not both characteristics shared simultaneously.

A supposed requirement for this shared character is the root of the discrepancy in the least upper bound problem (Proposition 7.5.11). In our initial development of the least upper bound problem (Proposition 7.1.14), it was implicitly assumed that $\sup(\mathbb{R}_0)$ must be an arithmatic number in the way that ***all real numbers were supposed to be algebraic until the connectedness of the interval demanded non-algebraic numbers to fill in the gaps***. A number is said to be canonically algebraic if it is the root of a certain polynomial, but this is not what is meant when we use the adjective "algebraic" to describe the immeasurables. Presently, $x \in \mathbb{F}$ may be called *algebraic* due to the lack of a geometric picture of fractional distance for $x \in \mathbb{F}$. The geometric picture of $x \in \mathbb{F}$ comes from an algebraic ordering with the $\leq$ relation being defined over all of $\mathbb{R}$. It is clear that the $\mathcal{F}_{\mathcal{X}}$ are

geometric Euclidean magnitudes, or cuts, measured relative to the origin $\hat{0}$ of an infinitely long line, but it is not clear how this works in the metric space picture of $\mathbb{R}$ where

$$d(0, \mathcal{F}_{\mathcal{X}}) = \left|\mathcal{F}_{\mathcal{X}} - 0\right| = \text{undefined} \quad .$$

Toward the metric space picture, it is interesting that we have shown in Main Theorem 6.1.1 that arithmetic in the neighborhood of infinity allows us to take all-important limits at infinity within the realm of standard analysis. There is no need to invoke the metric space definition of $\mathbb{R}$ to take these limits. The metric space is the canonical workaround for a supposed failure of the Cauchy limit criterion at infinity, but here we take the Cauchy limit at infinity with the modernized definition of $\mathbb{R}$ (Main Theorem 6.1.1). Why, then, should it be a problem that the metric function is undefined? In Definition 2.1.2, we defined a number line as a line equipped with a chart x and the Euclidean metric $d(x, y) = |y - x|$, but, now that we have closely examined all the details, we can otherwise define a number line as a line equipped with the totally ordered fractional distance chart $|x| \in \mathbb{R}^{\cup} \cup \mathbb{F}$ (which is just the Euclidean chart.) With this definition taken *a priori* as an axiom, it is possible to reproduce the entire fractional distance analysis without any dependence on a metric function whose domain only holds the measurables $\mathbb{R}^{\cup}$. If desired, it could be assumed that the metric definition of a number line (Definition 2.1.2) is overwritten as needed for consistency. By the end of this section, however, we will have restored the metric functionality.

Paradox Resolution 7.5.18 (Resolution of Paradox 7.3.10.) The paradox depends on assumed usual arithmetic for non-arithmatic immeasurable numbers. The paradox is remedied by the non-arithmatic property of $\mathcal{F}_{\mathcal{X}}$.

Paradox Resolution 7.5.19 (Resolution of Paradox 7.3.11.) The paradox depends on assumed usual arithmetic for non-arithmatic immeasurable numbers. The paradox is remedied by the non-arithmatic property of $\mathcal{F}_{\mathcal{X}}$.

Remark 7.5.20 Throughout this analysis, we have referred many times to the geometric notions of addition and multiplication. If $\mathcal{F}_{\mathcal{X}}$ is an immeasurable number $x \notin \mathbb{R}^{\cup}$ such that ordinary notions of geometry cannot be applied to it universally, by what means might we axiomatize the arithmetic of non-arithmatic numbers? We have shown in Section 7.3 that the straightforward introduction of infinitesimals is not the correct way forward, and we have shown it for all the reasons that infinitesimals are usually not allowed into standard analysis. So, to pierce the reader's *probable* veil of *likely* skepticism regarding some perceived absurdity about ordered but non-arithmatic real numbers $\mathcal{F}_{\mathcal{X}}$, which are not one iota more specious a construction than the non-algebraic numbers which were brought into standard analysis for the *exact* same reason, we now point out that the $\{\mathcal{F}_{\mathcal{X}}\}$ are the only real numbers not

forced into the arithmetic axioms by some identification as an element of $C_{\mathbb{Q}}^{\mathbf{AB}}$. Immeasurable real numbers $x \in \mathbb{F}$ are partitions of the big parts of $C_{\mathbb{Q}}^{\mathbf{AB}}$ only. They are not uniquely identified with any $[x] \in C_{\mathbb{Q}}^{\mathbf{AB}}$ whose corresponding Dedekind partition would specify a big part *and* a little part. Measurable numbers differ from immeasurable numbers because the Dedekind partition corresponding to any $x \in \mathbb{R}^{\cup}$ must specify a little part. Thus, having entirely different algebraic constructions, measurable and immeasurable numbers are so markedly different in their qualia, it is *certainly reasonable* to suppose that they obey different arithmetic axioms.

Remark 7.5.21 Now, in advance of the material presented in the remainder of this section, we will laboriously belabor the point that ***there is no preferred scale for standard analysis***. As mentioned in the preamble to this section, we will suggest taking the immeasurables $\mathcal{F}(n)$ as the extended naturals on a number line whose unit of distance is the spacing between successive immeasurables $\mathcal{F}_k, \mathcal{F}_{k+1} \in \mathbb{R}$. We will use the label $\mathbb{R}_{\mathbb{F}}$ to distinguish the new number line, and we will identify the cut $\mathcal{F}_0 \in \mathbb{R}$ with $1 \in \mathbb{R}_{\mathbb{F}}$. In Theorem 7.5.29, we will show that this identification requires that the origin of $\mathbb{R}_{\mathbb{F}}$ is shifted with respect to that in $\mathbb{R}$.

Geometric infinity ∞ is not affected in any way, ever, by any conformal rescaling parameter. From a geometric perspective, then, we should expect $\mathbb{R}_{\mathbb{F}}$ to be $\mathbb{R}$ itself identically. For instance, consider a convention such that

$$\overline{\mathbb{R}} = [-\aleph_1, \aleph_1] \quad , \quad \text{and} \qquad \overline{\mathbb{T}} = [-\aleph_{\widehat{\infty}}, \aleph_{\widehat{\infty}}] \quad ,$$

where the existence of some $\mathbb{T}$ separate from $\mathbb{R}$ is implied by the notion of geometric infinity as proffered in Example 4.4.10: a number which cannot be written as an endpoint without contradicting the notion of the infinite geometric extent of a number line. In the hypothetical formalization of such a scheme, and in the context of this remark only, one might introduce a new symbol or a term like "ultimate transfinite infinity" to avoid ambiguity with ∞ (geometric infinity) in our working identity $|\aleph_1| = \infty$. If $\overline{\mathbb{R}}$ had an endpoint at ultimate transfinite infinity, then that would contradict the notion of infinite geometric extent forbidding it being an endpoint, so the existence of a continued superset such as $\mathbb{T}$ is implied. Even then, $\aleph_{\widehat{\infty}}$ must be less than ultimate transfinite infinity because transfinite infinity can no more be an endpoint for $\overline{\mathbb{T}}$ than it could for $\overline{\mathbb{R}}$. Thus, $\mathbb{R}$, $\mathbb{R}_{\mathbb{F}}$, and $\mathbb{T}$ are three rescaled copies of the same object, and we might rightfully call any of them $\mathbb{R}$. All that is required is a chart to normalize the interval as $(-\aleph_1, \aleph_1)$.

The limit definition of algebraic infinity

$$\left|\aleph_1\right| = \lim_{x \to 0^{\pm}} \frac{1}{x} \quad ,$$

is such that the scale of the unit $1 \in \mathbb{N}$ in the numerator does not affect our ability to construct $(-\aleph_1, \aleph_1) \subseteq (-\aleph_{\widehat{\infty}}, \aleph_{\widehat{\infty}})$. However large or small we take

the scale of $1 \in \mathbb{N}$, the resultant extended real line $[-\widehat{\infty}, \widehat{\infty}]$ will always be a subset of $\mathbb{T}$, which cannot itself extend to ultimate transfinite infinity because, in part, $\aleph_{\widehat{\infty}} < \aleph_{\widehat{\infty}+1}$. So, regardless of the scale of $1 \in \mathbb{N}$, the interval $(-\widehat{\infty}, \widehat{\infty})$ is always going to be the same set $\mathbb{R}$, even if we add supplemental labeling to denote a relative scale among different copies of it, and even if the relative scale is on the order of $\mathcal{F}_0$ or greater. There is no scale factor by which we might stretch algebraic infinity as in $\mathbb{R} = (-\widehat{\infty}, \widehat{\infty})$ to be more than a drop in the bucket with respect to an interval extending to ultimate transfinite infinity in both directions. Even $\mathbb{T}$, which necessarily falls short of ultimate transfinite infinity due to its $\pm\aleph_{\widehat{\infty}}$ endpoints, will always be infinitely longer than $\mathbb{R}$ on a given scale. For every instance of $\overline{\mathbb{R}}$ charted with x, there is a rescaled chart over $\mathbb{R}'$ containing x' to the right of $x = \widehat{\infty}$ such that those x' are still real numbers to the left of the $\widehat{\infty}$ that bounds $\mathbb{R}'$. As this is true on any scale, those transfinite points will always belong to an interval infinitely longer than any given instance of $\mathbb{R}$: one extending to ultimate transfinite infinity in both directions. Thus, we are well enough motivated to use the immeasurables as the natural numbers in some $\mathbb{R}_{\mathbb{F}}$ such that $\text{len}(\mathbb{R}_{\mathbb{F}}) \ggg \text{len}(\mathbb{R})$.

Now we have *laboriously* belabored the point about there being no preferred scale for $\mathbb{R}$. In the remainder of this remark, we will continue to do so and infer that the immeasurables behave like the naturals on a copy of $\mathbb{R}$ which is quite large relative to another copy: one scaled by an immeasurable quantity. We will define $\mathcal{F}(n) = n \in \mathbb{N}_{\mathbb{F}}$ and then claim that $\mathbb{N}_{\mathbb{F}} = \mathbb{N}$ because there is no unique scale for $\mathbb{R}$ (or $\mathbb{N}$.) Having supposed the existence of natural numbers as a prerequisite to $\mathbb{Q}$ in our construction of $\mathbb{R}$ by Cauchy sequences of rationals (Section 4.2), we will find that the curious remainder set $\mathbb{F}$ left over at the end is exactly what we have put in to begin with: $\mathbb{N}$. Indeed, we will find that the natural numbers which served as the starting point for $C_{\mathbb{Q}}^{\mathbf{AB}}$ were simply the $\mathcal{F}(n)$ on a smaller copy of $\mathbb{R}$ whose existence can be inferred by the self-similarity of $\mathbb{R}$ on any scale. It's tortoises all the way down, basically.

By the least upper bound problem (Proposition 7.5.11), we have demonstrated a requirement that standard analysis must not contain any binary operations mapping to

$$\left.\begin{array}{l} \left\{[x] + \mathcal{F}_{[\mathcal{X}]} \;\middle|\; [x], [\mathcal{X}] \subset C_{\mathbb{Q}}\right\} \\ \left\{[x] \cdot \mathcal{F}_{[\mathcal{X}]} \;\middle|\; [x], [\mathcal{X}] \subset C_{\mathbb{Q}}\right\} \\ \left\{[x] \div \mathcal{F}_{[\mathcal{X}]} \;\middle|\; [x], [\mathcal{X}] \subset C_{\mathbb{Q}}\right\} \end{array}\right\} \subset \mathbb{R}^{\cup} \times \mathbb{F} \quad .$$

Accordingly, the fractional distance axioms forbid such products. What, then, shall we do with $\mathbb{F}$? Undefined definitions beg for definitions. En route to some definitions for the arithmetic of $\{\mathcal{F}_{\mathcal{X}}\}$, let us examine the process of $\mathbb{R}$'s algebraic construction.

To construct $\mathbb{R}$ by Cauchy equivalence classes of rationals in Section 4.2, we have implicitly assumed $\mathbb{N}$ as a discrete set without first requiring that the natural numbers are regularly spaced cuts along the real line. Rather, we have assumed $\mathbb{N}$ as an abstract (algebraic) set of counting of numbers and

then appended it to the least number $x = 0$ in the algebraic representation of the geometric left endpoint of **AB**. We have also defined $x = 0$ to be the additive identity for $n \in \mathbb{N}$ so that $n = 0 + n$. By doing so, we associate the Cauchy equivalence classes of rationals with cuts in $[0, \widehat{\infty})$. Due to **AB** being equipped with a metric (Definition 2.1.2), the notion of $x \in \mathbb{Q}$ as a cut some distance away from $x = 0$ is implied, and then we may infer the irrationals as cuts from the connectedness of the interval. This stands as a parallel construction of $\mathbb{R}$ in which we have not presupposed that numbers are cuts in a line. We have only supposed that lines (or line segments) exist, that $\mathbb{N}$ exists, that $n \in \mathbb{N}$ has an additive identity, and that the ratios of $n, m \in \mathbb{N}$ exist as elements of $\mathbb{Q}$. Strictly speaking, the idea of one apple or 52 apples does not require the presupposition of the existence of a 1D Hausdorff space extending infinitely far in both directions. It suffices to grant that if whole apples are had, we can have a natural number of them: one, a few, or a lot. With the introduction of a metric function, however, it is granted that units of distance can be counted like apples, and that increments of distance separate unequal numbers: numbers are cuts in a line.

We have assumed $\mathbb{N}$ at the outset of our algebraic constructive process to write

$$\mathbb{N} \longrightarrow \mathbb{N} \cup \{0\} \longrightarrow \mathbb{Q} \longrightarrow C_{\mathbb{Q}} \longrightarrow C_{\mathbb{Q}}^{\mathbf{AB}} \longrightarrow \mathbb{R} \longrightarrow \mathbb{F} \quad .$$

If we complement the undefined definitions for the arithmetic of immeasurable numbers such that ***they obey the arithmetic of the natural numbers in a rescaled chart*** whose $1 \in \mathbb{N}_{\mathbb{F}}$ is on the order of $\mathcal{F}_0 \in \mathbb{R}$ in a "smaller" chart, then the process leading to the mystery set $\mathbb{F}$ becomes

$$\mathbb{N} \longrightarrow \mathbb{N} \cup \{0\} \longrightarrow \mathbb{Q} \longrightarrow C_{\mathbb{Q}} \longrightarrow C_{\mathbb{Q}}^{\mathbf{AB}} \longrightarrow \mathbb{R} \longrightarrow \mathbb{F} \longrightarrow \mathbb{N}_{\mathbb{F}} \quad ,$$

Having constructed a bigger version of $\mathbb{N}$, we can use it to construct a transfinite chart without having to assume a second transfinite copy of the naturals with which to begin another constructive process for $\mathbb{R}_{\mathbb{F}}$. We can continue to construct "bigger and bigger" copies of $\mathbb{R}$ forever with scale factors greater than or equal to any $n \in \mathbb{N}$, and we will never require any intuitive suppositions beyond the existence of the $n \in \mathbb{N}$ that we have already supposed. We might rescale bigger and bigger forever, and we will never fail to have a resultant set which is infinitely small with respect to a given instance of $\mathbb{T}$. Even if we chose the scale factor $\aleph_{\widehat{\infty}}$, we could make use of the notation being such that $\aleph_{(\aleph_{\mathcal{X}})} = \mathcal{X} \cdot \widehat{\infty}^2$ to preserve the infinitely-larger-ness of $\mathbb{T}$ as

$$\mathbb{T} = (-\aleph_{\aleph_{\aleph_{\aleph_{\cdots}}}}, \aleph_{\aleph_{\aleph_{\aleph_{\cdots}}}}) = (-\widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\cdots}}}}, \widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\cdots}}}}) \quad .$$

Since the previous $(-\aleph_{\widehat{\infty}}, \aleph_{\widehat{\infty}})$ construction for $\mathbb{T}$ makes it identical to $\mathbb{R}$ on another scale, a better definition for $\mathbb{T}$, one which would distinguish it from $\mathbb{R}$, might be to set $\mathbb{T}$ itself as the interval extending to ultimate transfinite infinity in both directions. However, there is no need for such distinctions and

clarifications in real analysis. Real analysis ends at $\aleph_1$, while $\mathbb{T}$ lives only in transfinite analysis where "beyond infinity" has meaning.

Due to the absorptive properties of geometric infinity (in the usual sense of the absolute value of algebraic infinity $|\widehat{\infty}|$ being equal to ∞), a conformal rescaling parameter such as $\mathcal{F}_0$ or $\aleph_{\mathcal{X}}$ will never change the Euclidean conceptual component underlying everything: $\mathbb{R} = (-\infty, \infty)$. For this reason, the scale that we assign to any particular copy of $\mathbb{R}$ will never disrupt the configuration of an infinite line along which the interval $(-\widehat{\infty}, \widehat{\infty})$ can be shown to exist as a conformal chart on some finite subset. It follows that $\mathbb{N}_{\mathbb{F}}$ is just another copy of $\mathbb{N}$ with a rescaled unit increment, so it must be that $\mathbb{N}_{\mathbb{F}} = \mathbb{N}$. However, since the scale of the two sets is different, we cannot allow them to interact by arithmetic. That would require $\mathbb{N}_{\mathbb{F}} \neq \mathbb{N}$. Thus, we take Axiom 7.5.8 to disallow arithmetic among measurable and immeasurable numbers. If we took $\mathbb{N}_{\mathbb{F}} \neq \mathbb{N}$, then there would be an implied preferred scale for $\mathbb{R}$ when no such scale exists. Thus, we have that which was assumed at the outset of the constructive process as the output at the end of the same:

$$\mathbb{N} \longrightarrow \mathbb{N} \cup \{0\} \longrightarrow \mathbb{Q} \longrightarrow C_{\mathbb{Q}} \longrightarrow C_{\mathbb{Q}}^{\mathbf{AB}} \longrightarrow \mathbb{R} \longrightarrow \mathbb{F} \longrightarrow \mathbb{N} \quad .$$

Since arithmetic between the measurables and immeasurables is forbidden, there is little reason to retain the subscript label $\mathbb{F}$ which reminds us that the scale of one instance of $\mathbb{N}$ differs from another. When that matters, we may call the larger numbers $\mathcal{F}(n)$ noting that the problems associated with the (n) notation have been done away with by Axiom 7.5.8. Citing one such problem, there are irreparable contradictions with the idea that $\frac{1}{2}\mathcal{F}_0 = \frac{1}{2}\mathcal{F}(1)$ should be a cut $x \in \mathbb{R}_0$, but there is no problem when $\frac{1}{2}\mathcal{F}(1)$ is cut in the separate big real line $\mathbb{R}_{\mathbb{F}}$ where the quantity is expressed as $\frac{1}{2}\mathcal{F}(1) \to \frac{1}{2} \cdot 1 = \frac{1}{2}$ with $\frac{1}{2}, 1 \in \mathbb{Q}_{\mathbb{F}}$. Now the label (n) tells us which (extended) natural number $n \in \mathbb{N}_{\mathbb{F}}$ is each $\mathcal{F}(n) \in \mathbb{R}$, and we have a ready-made set of arithmetic axioms for $\{\mathcal{F}_{\mathcal{X}}\}$: they should obey the arithmetic laws of natural numbers with themselves. We will formalize this with Axiom 7.5.24.

Going back to Definition 2.1.3, we see that the real line is only a line with an appended label "real," so the requirement $\mathbb{R} = \mathbb{R}_{\mathbb{F}}$ simply augments the label we put on the line. In this case, we have $\mathbb{R}$ as "the real line" and $\mathbb{R}_{\mathbb{F}}$ as "the big real line" with a specification of relative scale between them. Among the two number lines, however, it necessarily follows that *the* real line is identically *the big real line* in another convention where we attach the label *real* to what would be "the small real line," call it $\mathbb{R}_{\mathbb{F}^{-1}}$. Having supposed the preexistence of $\mathbb{N}$ as something separate from $\mathbb{R}$, now we have a nice explanation for the abstract n which show up as regularly spaced cuts in $\mathbb{R}$ once we have it built from geometry: $\mathbb{N} \subset \mathbb{R}$ are the $\mathcal{F}(n)$ from the small real line. In this way, we might construct an infinite tier of increasingly large or small copies of $\mathbb{R}$, call them $\mathbb{R}_{\mathbb{F}^k}$, without any further supplemental abstractions. Since all copies of $\mathbb{R}$ are the same, meaning that they differ by nothing more than a label to specify the scale, there is no strict requirement to define bigger or smaller lines at all.

It will suffice for an extended transfinite analysis to define separate charts on a single line: $x_{\mathbb{F}^k} \in \mathbb{R}_{\mathbb{F}^k} \subset \mathbb{T}$. Since we have taken $(-\widehat{\infty}, \widehat{\infty})$ as embeddable on the finite subintervals of $(-\infty, \infty)$, there is no problem assuming other charts over $\mathbb{R}$ whose linear extent exceeds or falls short of $(-\widehat{\infty}, \widehat{\infty})$ as measured according to another $x_{\mathbb{F}^j}$.

Example 7.5.22 As an example of a possible application for a framework of analysis in $\mathbb{R}_{\mathbb{F}^k}$, consider the widely accepted formula for the sum of the positive integers

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \quad .$$

Perhaps taking this result to mean that the sum of all $n \in \mathbb{N}_{\mathbb{F}^j}$ is equal to $-\frac{1}{12} \in \mathbb{Q}_{\mathbb{F}^{j+1}}$ could serve as a Rosetta stone of sorts for translating between $\mathbb{R}$ and $\mathbb{R}_{\mathbb{F}}$ in the new transfinite analysis proposed here.

Definition 7.5.23 Given a chart x on $\mathbb{R} \subset \mathbb{T}$, there exists another chart $x_{\mathbb{F}}$ on $\mathbb{R}_{\mathbb{F}} \subset \mathbb{T}$ such that $x = \mathcal{F}(n)$ and $x_{\mathbb{F}} = n$ are the same cut in $\mathbb{T}$. Successive immeasurables in $\mathbb{R}$ are identified with successive naturals in $\mathbb{R}_{\mathbb{T}}$. The identification of $\mathcal{F}(n) \in \mathbb{R}$ with $n \in \mathbb{R}_{\mathbb{F}}$ means that the two charts are pinned as follows. If $\mathcal{X} > 0$, and line segment AB has simultaneous algebraic representations

$$AB \equiv [0, \aleph_{\mathcal{X}}] \subset \mathbb{R} \quad , \quad \text{and} \qquad AB \equiv [x_{\mathbb{F}}, y_{\mathbb{F}}] \subset \mathbb{R}_{\mathbb{F}} \quad ,$$

then splitting AB into AX and XB such that

$$AX \equiv [0, \mathcal{F}(n)) \subset \mathbb{R} \quad , \quad \text{and} \qquad XB = [\mathcal{F}(n), \aleph_{\mathcal{X}}] \subset \mathbb{R} \quad ,$$

requires that

$$AX \equiv [x_{\mathbb{F}}, n) \subset \mathbb{R}_{\mathbb{F}} \quad , \quad \text{and} \qquad XB = [n, y_{\mathbb{F}}] \subset \mathbb{R}_{\mathbb{F}} \quad .$$

Axiom 7.5.24 The arithmetic operations of immeasurable real numbers with themselves are the arithmetic operations of the extended natural numbers $\mathbb{N}_{\infty}$ (Definition 6.2.3) such that there exists a one-to-one correspondence

$$\mathbb{R} \ni \mathcal{F}(n) \quad \longleftrightarrow \quad n \in \mathbb{R}_{\mathbb{F}} \quad .$$

These operations take two immeasurable $\mathcal{F}_{\mathcal{X}} \in \mathbb{R}$ and return a cut in another number line which is an identical but distinctly labeled copy of $\mathbb{R}$. In particular, we have

$$\begin{aligned} \mathcal{F}(n) + \mathcal{F}(m) = \mathcal{F}(n+m) \quad &\longrightarrow \quad n + m \in \mathbb{R}_{\mathbb{F}} \\ \mathcal{F}(n) \cdot \mathcal{F}(m) = \mathcal{F}(n \cdot m) \quad &\longrightarrow \quad n \cdot m \in \mathbb{R}_{\mathbb{F}} \\ \mathcal{F}(n) \div \mathcal{F}(m) = \mathcal{F}(n \div m) \quad &\longrightarrow \quad n \div m \in \mathbb{R}_{\mathbb{F}} \quad . \end{aligned}$$

Remark 7.5.25 With Axiom 7.5.8, we have avoided the least upper bound problem. Namely, expressions such as $\mathcal{F}_{\mathcal{X}} \pm b$ are undefined when $b \in \mathbb{R}_0$. However, it remains to show that $\mathcal{F}_{\mathcal{X}} \in \mathbb{R}$ satisfies the Archimedes property of real numbers. There is most likely no concise symbolic representation of the statement, "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another," that does not rely on arithmetic, so, therefore, we should define arithmetic operations for non-arithmatic numbers with themselves. This is achieved with Axiom 7.5.24. Axiom 7.5.8 only forbids arithmetic between measurables and immeasurables, so we have retained the freedom to define arithmetic operations among immeasurable numbers.

Main Theorem 7.5.26 *Immeasurable numbers $\mathcal{F}_{\mathcal{X}} \in \mathbb{R}$ obey the Archimedes property of real numbers.*

Proof. Definition 6.3.9 gives the Archimedes property of real numbers as

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad .$$

As in the proof of Main Theorem 6.3.10, we will divide this proof into cases of the Archimedean statement $x < y$.

• ($x \in \mathbb{R}^{\mathcal{X}}$, $y = \mathcal{F}_{\mathcal{Y}}$, $\mathcal{Y} \geq \mathcal{X}$) We have $x < y$ so choose two multipliers $z_1 = \frac{\mathcal{Z}}{\mathcal{X}}$ such that $\mathcal{Y} < \mathcal{Z} < 1$, and $z_2 = \mathcal{F}_0 = \mathcal{F}(1)$ where $\mathcal{F}_0 \in \mathbb{R}$ is the multiplicative identity $1 \in \mathbb{R}_{\mathbb{F}}$ for the arithmetic of the immeasurables granted by Axiom 7.5.24. Then

$$\frac{\mathcal{Z}}{\mathcal{X}}\big(\aleph_{\mathcal{X}} + b\big) = \aleph_{\mathcal{Z}} + \frac{b\mathcal{Z}}{\mathcal{X}} \quad > \quad \mathcal{F}_{\mathcal{Y}} \quad .$$

• ($x = \mathcal{F}_{\mathcal{X}}$, $y \in \mathbb{R}^{\mathcal{Y}}$, $\mathcal{Y} \geq \mathcal{X}$) Choose $z_1 = \mathcal{F}_0$ and $z_2 = \frac{\mathcal{Z}}{\mathcal{Y}}$ such that $\mathcal{Z} \leq \mathcal{X}$. Then

$$\mathcal{F}_{\mathcal{X}} \quad > \quad \frac{\mathcal{Z}}{\mathcal{Y}}\big(\aleph_{\mathcal{Y}} + b\big) = \aleph_{\mathcal{Z}} + \frac{b\mathcal{Z}}{\mathcal{Y}} \quad .$$

• ($x = \mathcal{F}_{\mathcal{X}}$, $y = \mathcal{F}_{\mathcal{Y}}$, $\mathcal{Y} \geq \mathcal{X}$) This case will be remarked upon, and then proven below.

Remark 7.5.27 For the purposes of the arithmetic of the immeasurables, there is no problem using the $\mathcal{F}_{\mathcal{X}} = \mathcal{F}(n)$ notation. For $x = \mathcal{F}_{\mathcal{X}}$ and $y = \mathcal{F}_{\mathcal{Y}}$ with $\mathcal{Y} \geq \mathcal{X}$, we take the $\mathcal{F}(n) = n \in \mathbb{N}_{\mathbb{F}}$ as a discrete exterior set upon which the cuts in $\mathbb{R}_{\mathbb{F}}$ are constructed. All of the problems demonstrated in and around Section 7.3 relied upon assumptions of the form

$$\begin{aligned} x + y = f_+(x, y) &: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ x \cdot y = f_\times(x, y) &: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ x \div y = f_\div(x, y) &: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad , \end{aligned}$$

which we find are now better written in the respective forms of

$$f(x,y) : \mathbb{R}^{\cup} \times \mathbb{R}^{\cup} \to \mathbb{R} \quad ,$$

because the arithmetic of $x \in \mathbb{R}^{\cup}$ and $x \in \mathbb{F}$ is differentiated. Noting that Axiom 7.5.8 forbids the domain $\mathbb{R}^{\cup} \times \mathbb{F}$ for any of f_+, $f_\times$, or $f_\div$, and recalling that Euclid wrote in Book 5 of *The Elements*, Definition 3, "A ratio is a certain type of condition with respect to size of two magnitudes of the same kind," Axiom 7.5.24 now grants forms following

$$f(\mathcal{F}_\mathcal{X}, \mathcal{F}_\mathcal{Y}) : \mathbb{F} \times \mathbb{F} \to \mathbb{R}_\mathbb{F} \quad .$$

Cuts in $\mathbb{R}_\mathbb{F}$ are taken as Euclidean magnitudes "of a different kind" than those in $\mathbb{R}$. By defining the non-arithmatic real numbers of one kind as simultaneous natural numbers of another kind, we avoid all of the problems associated with the $\mathcal{F}(n)$ notation: the notation $x = \mathcal{F}_\mathcal{X}$ tells us to treat x as a cut in $\mathbb{R}$ and $x = \mathcal{F}(n)$ tells us to treat it as a cut in $\mathbb{R}_\mathbb{F}$. In either case, x is a cut in the real number line.

Now we will conclude the proof of Main Theorem 7.5.26. Regarding the Archimedes property for $x = \mathcal{F}_\mathcal{X}$ and $y = \mathcal{F}_\mathcal{Y}$ (the third bulleted case above), let

$$\mathcal{F}_\mathcal{X} = \mathcal{F}(n) \quad , \quad \text{and} \qquad \mathcal{F}_\mathcal{Y} = \mathcal{F}(m) \quad ,$$

so that $\mathcal{F}(n) < \mathcal{F}(m)$. Choose $z_1 = \mathcal{F}(k)$ such that $nk > m$, and $z_2 = \mathcal{F}(1)$ so that Axiom 7.5.24 gives

$$\mathcal{F}(k) \cdot \mathcal{F}(n) = \mathcal{F}(kn) \quad > \quad \mathcal{F}(m) \quad .$$

We have demonstrated the main cases and conclude the proof with an assumption that the other cases follow directly. Non-arithmatic, immeasurable numbers satisfy the ancient Archimedes property of real numbers. ∎

Remark 7.5.28 Due to the explicit requirement for multiplication by n or $\aleph_n$ in the Archimedes property of 1D transfinitely continued real numbers (Axiom 6.3.12), we cannot directly demonstrate the compliance of $\mathcal{F}_\mathcal{X} \in \mathbb{F}$. The problem lies with the inadmissible domain of

$$f_\times : \mathbb{F} \times \mathbb{R}^{\cup} \to \mathbb{R}_\mathbb{F} \quad .$$

To the extent that Axiom 6.3.12 was proposed only to simplify the demonstration of the compliance of numbers in the neighborhood of infinity with the Archimedes property, we could more precisely call this axiom the Archimedes property of 1D transfinitely continued arithmatic real numbers.

Theorem 7.5.29 *The x and $x_\mathbb{F}$ charts cannot share an origin.*

Proof. The radius of the neighborhood of the origin $\mathbb{R}_0$ is one half the length of any element of $\{\mathbb{R}^{\mathcal{X}}\}$. The origin of the x chart lies one half as far to the left of $\mathcal{F}(1) = 1 \in \mathbb{N}_{\mathbb{F}}$ as $\mathcal{F}(2) = 2 \in \mathbb{N}_{\mathbb{F}}$ lies to the right of it. Now that we have given arithmetic for the immeasurables, we may quantify the notion of immeasurables lying to the left and right of each other with the ordinary Euclidean metric $d(x_{\mathbb{F}}, y_{\mathbb{F}}) = |y_{\mathbb{F}} - x_{\mathbb{F}}|$. This metric requires that distance is uniform over the line, so, therefore, the cut $0 \in \mathbb{R}$ cannot have a simultaneous algebraic definition as $0 \in \mathbb{R}_{\mathbb{F}}$. Such a definition would require a non-Euclidean notion of distance rendering the line in question something other than a number line as given by Definition 2.1.2. With or without a metric, numbers must be spaced evenly along a number line for the purposes of Euclidean geometry. ∎

§7.6 The Topology of the Real Number Line

The thesis of the fractional distance analysis presented here has been to preserve the Euclidean geometric construction of $\mathbb{R}$ through an algebraic construction which does not preclude the existence of the neighborhood of infinity. We began with Axiom 2.1.7 stating that real numbers are represented in algebraic interval notation as $\mathbb{R} = (-\infty, \infty)$, and this axiom is totally equivalent to a requirement that $\mathbb{R}$ has the usual topology. In this section, we will define two different topologies on $\mathbb{R}$: the usual topology and another one which we will call the fractional distance topology.

Definition 7.6.1 A topology on a set S is a collection $\mathcal{T}$ of open subsets of S with the following properties.

- $\mathcal{T}$ contains S and $\varnothing$.
- $\mathcal{T}$ contains the union of any of the elements of $\mathcal{T}$, namely $\bigcup \tau_k \in \mathcal{T}$.
- $\mathcal{T}$ contains the finite intersection of the elements of $\mathcal{T}$. If $n < \infty$, then $\bigcap_{k=1}^{n} \tau_k \in \mathcal{T}$.

The open sets in $\mathcal{T}$ are called the basis of the topology. The topology is the set of all unions of the sets in its basis. Together, the pair $(S, \mathcal{T})$ is called a topological space.

Definition 7.6.2 A basis $\mathcal{B}$ for a topology $\mathcal{T}$ on S is a set $\{\mathcal{B}_k\}$ of subsets $\mathcal{B}_k \subset S$ such that (i) for every $s \in S$ there is at least one basis element which contains s, and such that (ii) if $s \in \mathcal{B}_1 \cap \mathcal{B}_2$, then there exists $\mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$ such that $s \in \mathcal{B}_3$.

Definition 7.6.3 The *usual* basis $\mathcal{B}_0$ for the usual topology on $\mathbb{R}$ is the collection of all 1D open intervals such that

$$\mathcal{B}_0 = \big\{(a, b) \;\big|\; [a], [b] \subset C_{\mathbb{Q}},\ a < b\big\} \quad ,$$

The topology generated by $\mathcal{B}_0$ is called $\mathcal{T}_0$.

Theorem 7.6.4 *$\mathcal{B}_0$ is not a basis for a topology on $\mathbb{R}$.*

Proof. Definition 7.6.2 requires that

$$\forall x \in \mathbb{R} \quad \exists (a,b) \in \mathcal{B}_0 \quad \text{s.t.} \quad x \in (a,b) \quad .$$

Consider $x = \aleph_{0.5} \in \mathbb{R}$. We have

$$\forall a = [a] \quad \forall b = [b] \quad \text{s.t.} \quad [a],[b] \in C_{\mathbb{Q}} \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad a, b < n \quad .$$

It follows that there is no interval $(a,b) \in \mathcal{B}_0$ which contains $x > n$. This proves the theorem. ∎

Theorem 7.6.5 *The pair $(\mathbb{R}_0, \mathcal{T}_0)$ is a topological space when $\mathcal{T}_0$ is the topology generated by $\mathcal{B}_0$.*

Proof. Proof follows from Definitions 7.6.1, 7.6.2, and 7.6.3. ∎

Remark 7.6.6 The usual topology is granted by Axiom 2.1.7 giving $\mathbb{R} = (-\infty, \infty)$. If all of $\mathbb{R}$ is axiomatically taken to reside within the neighborhood of the origin, then $(\mathbb{R}, \mathcal{T}_0)$ is a well-defined topological space, and $\mathcal{T}_0$ is the usual topology on $\mathbb{R}$. This is a restatement of Theorem 7.6.5 with an alternative definition for $\mathbb{R}$. Since we have included the neighborhood of infinity in $\mathbb{R}$, rightly, we cannot construct (generate) the usual topology with its *usual* basis $\mathcal{B}_0$. We need to adjust the basis to reflect the results of fractional distance analysis. Only then will we be able to put $\mathbb{R} = \mathbb{R}_0 \cup \mathbb{R}_\infty = \mathbb{R}^\cup \cup \mathbb{F}$ into a topological space with the usual topology. For breadth in this section, we will also define a fractional distance topology $\mathcal{T}_{\text{FD}}$ separate from the usual topology $\mathcal{T}_{\text{U}}$ on $\mathbb{R}$. To contrast these topologies, we will provide two new axioms regarding possible constructions for $\mathbb{R}$, and then we will show that the axiomatic requirements are satisfied by one topology or the other. However, in the way that an axiom about all real numbers existing in the neighborhood of the origin was mentioned above without overwriting the fractional distance axioms, the following axioms are given only for consideration. By stating them, we will continue the program of contrast between geometry and algebra which is a theme of this analytical work. Namely, we will show how the two topologies are tailored to one axiomatic system or another: one emphasizing geometry or one emphasizing algebra.

Axiom 7.6.7 ***The fundamental axiom of geometric construction***. Up to a sign, real numbers are the finite elements of algebraic representations of points in the infinitely long line segment **AB**, *i.e.*:

$$\mathbb{R} = \left\{ \pm x \;\middle|\; x \in X \in \mathbf{AB},\ 0 \leq x < \widehat{\infty} \right\} \quad .$$

Axiom 7.6.8 ***The fundamental axiom of algebraic construction***. The topology of the real number line is the usual one.

Remark 7.6.9 Axiom 7.6.7 posits the existence of every measurable number $x \in \mathbb{R}^{\cup}$. Axiom 7.6.8 posits the further existence of the immeasurables $x \in \mathbb{F}$ so that there are no gaps in $(-\infty, \infty)$ to prevent us from taking the usual topology $\mathcal{T}_{\mathrm{U}}$ as the set of all unions of $(a, b) \subset (-\infty, \infty)$. Without granting $\{\mathcal{F}_{\mathcal{X}}\} \subset \mathbb{R}$, intervals of the form $(\aleph_{\mathcal{X}}, \aleph_{\mathcal{X}+\delta})$ with $0 < \delta < (1 - \mathcal{X})$ would not be connected subsets of $\mathbb{R}$. They would be disconnected at the immeasurable extrema of the $\mathbb{R}^{\mathcal{X}}$ neighborhoods. In fact, they would not be intervals at all because intervals are necessarily connected.

Definition 7.6.10 The fractional distance topology on $\mathbb{R}^{\cup}$ is $\mathcal{T}_{\mathrm{FD}}$ generated by a basis $\mathcal{B}_{\mathrm{FD}} = \mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\infty}$ such that

$$\mathcal{B}_{\mathcal{X}} = \left\{ (\aleph_{\mathcal{X}} + a, \aleph_{\mathcal{X}} + b) \;\middle|\; [\mathcal{X}], [a], [b] \subset C_{\mathbb{Q}},\ a < b,\ -1 < \mathcal{X} < 1 \right\}$$
$$\mathcal{B}_{\infty} = \left\{ (\pm\widehat{\infty} \mp a, \pm\widehat{\infty} \mp b) \;\middle|\; [a], [b] \subset C_{\mathbb{Q}},\ a > b > 0 \text{ if } +\widehat{\infty},\ 0 < a < b \text{ if } -\widehat{\infty} \right\} .$$

Theorem 7.6.11 *The topological space* $(\mathbb{R}^{\cup}, \mathcal{T}_{FD})$ *satisfies Axiom 7.6.7.*

Proof. Proof follows from Definition 7.6.10. Every x required by Axiom 7.6.7 appears in the basis $\mathcal{B}_{\mathrm{FD}}$ of the topology $\mathcal{T}_{\mathrm{FD}}$. ∎

Remark 7.6.12 If an almost-usual topological space that excludes immeasurable numbers is desired, then $(\mathbb{R}^{\cup}, \mathcal{T}_{\mathrm{FD}})$ is suggested. It is consistent with the fundamental axiom of geometric construction, and topological applications devoid of non-arithmatic numbers are easy to imagine. However, $\mathcal{T}_{\mathrm{FD}}$ is not a topology on $\mathbb{R}$. In the present conventions, $(\mathbb{R}^{\cup}, \mathcal{T}_{\mathrm{FD}})$ is a well-defined topological space but $(\mathbb{R}, \mathcal{T}_{\mathrm{FD}})$ is not. $\mathcal{T}_{\mathrm{FD}}$ does not contain any $x \in \mathbb{F}$. If desired, however, one could axiomatize $(\mathbb{R}, \mathcal{T}_{\mathrm{FD}})$ as a topological space such that every real number is measurable. This would necessarily overwrite the fundamental axiom of algebraic construction, but one might find an analytical framework devoid of immeasurables more useful in certain circumstances.

Definition 7.6.13 The fractional distance basis $\mathcal{B}_{\mathrm{U}}$ for the usual topology $\mathcal{T}_{\mathrm{U}}$ on $\mathbb{R}$ is the collection of all 1D open intervals such that

$$\mathcal{B}_{\mathrm{U}} = \left\{ (a, b) \;\middle|\; a, b \in \overline{\mathbb{R}},\ a < b \right\} \quad .$$

Remark 7.6.14 The *usual* basis $\mathcal{B}_0$ for the usual topology (in its incarnation $\mathcal{T}_0$, as per Definition 7.6.3) assumes that all real numbers are identically Cauchy equivalence classes of rationals $[x] \subset C_{\mathbb{Q}}$. The fractional distance basis $\mathcal{B}_{\mathrm{U}}$ for the usual topology $\mathcal{T}_{\mathrm{U}}$ takes into account the current convention that

real numbers are either extended equivalence classes $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ or extended Dedekind partitions $x = (L, R)$.

Theorem 7.6.15 *The topological space $(\mathbb{R}, \mathcal{T}_U)$ satisfies the fundamental axiom of algebraic construction (Axiom 7.6.8). In other words, $\mathcal{T}_U$ is the usual topology on $\mathbb{R}$.*

Proof. Proof follows from Definition 7.6.13. Every open set $(a, b) \subset \mathbb{R}$ required by Axiom 7.6.8 appears in the basis sets of $\mathcal{T}_{\mathrm{U}}$. ∎

§8 The Riemann Hypothesis

§8.1 The Riemann Zeta Function

The Riemann hypothesis dates to Riemann's 1859 paper [16]. Since the axioms of a complete ordered number field date to Hilbert's 1899 paper [4] at the earliest, it would be improper to claim that the Riemann hypothesis is formulated in terms of the ordered field definition of $\mathbb{R}$. Likewise, Dedekind's partition definition of real numbers [3] and Cantor's definition as equivalence classes of rationals [2] date to a pair of 1872 papers. It follows, then, that the Riemann hypothesis cannot be understood as being phrased in the language of real numbers as Dedekind cuts or Cauchy equivalence classes. The topological space did not exist as a mathematical concept until well into the 20th century, so it would be similarly absurd to claim that the Riemann's hypothesis is formulated in terms of the usual topology on $\mathbb{R}$. While we cannot directly show what definition of $\mathbb{R}$ Riemann had in mind when formulating his hypothesis, we may glean very much from the plain fact that Riemann made no comment or nod toward any definition of $\mathbb{R}$ whatsoever. This should be taken to mean that Riemann felt his definition of $\mathbb{R}$ would have been known *a priori*, absolutely, and unambiguously to his intended audience. The only possible definition which might have been available to satisfy this condition in 1859 was Euclid's definition of real numbers as geometric magnitudes. Indeed, Riemann's program of Riemannian geometry is a direct extension of Euclidean geometry so, to a very high degree, this qualitatively supports the notion that Riemann had in mind the cut-in-a-number-line definition of $\mathbb{R}$ given by Euclid in *The Elements*.

When one examines *The Elements* [1], the many diagrams, definitions, and postulates make it exceedingly obvious that Euclid's definition of a real number as a magnitude, one having a proportion and ratio with respect to all other magnitudes of the same kind, is exactly the one given here as Definition 2.1.4. This definition gives

$$\mathbb{R} \setminus x = (-\infty, x) \cup (x, \infty) \quad ,$$

as an alternative, identical formulation of the Euclidean statement

$$x \in \mathbb{R}^+ \quad \iff \quad (0, \infty) = (0, x] \cup (x, \infty) \quad .$$

It is reasonable to conclude that Riemann formulated his hypothesis with it in mind that any definition of $\mathbb{R}$ consistent with the Euclidean magnitude would be sufficient. The domain of $\zeta(z)$, namely $\mathbb{C}$, would be constructed from two orthogonal copies of $\mathbb{R}$, one of them having the requisite phase factor i. Rather than the underlying definition of $\mathbb{R}$, however, the object of relevance in the Riemann hypothesis should be the behavior of $\zeta(z)$ at various z.

Definition 8.1.1 Complementing Definition 6.2.8 which gave $z \in \mathbb{C}$ as any $z = x + iy$ such that $x, y \in \mathbb{R}$, we define the complex neighborhood of the origin as

$$\mathbb{C}_0 = \left\{ x + iy \;\middle|\; x, y \in \mathbb{R}_0 \right\} \quad .$$

Definition 8.1.2 The arithmatic subset of the complex plane $\mathbb{C}^{\cup} \subset \mathbb{C}$ is such that

$$\mathbb{C}^{\cup} = \left\{ x + iy \;\middle|\; x, y \in \mathbb{R}^{\cup} \right\} \quad .$$

Definition 8.1.3 For $x \in \mathbb{R}$, the Dirichlet series is

$$\mathfrak{D}(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad .$$

This series converges absolutely for all $x \in \mathbb{R}^{\cup}$ such that $x > 1$. Formally, $\mathfrak{D}(x)$ is a special case of Dirichlet series, but we will simply call it *the* Dirichlet series.

Definition 8.1.4 For $z \in \mathbb{C}$, the Dirichlet form of the Riemann ζ function is

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad .$$

This series converges absolutely for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 1$.

Remark 8.1.5 The Riemann hypothesis primarily concerns the behavior of $\mathfrak{D}$ continued onto the region of $\mathbb{C}$ with real parts exceeding the domain of convergence of $\mathfrak{D}$, *i.e.*: the region $(-\infty, 1]$. Riemann's famous functional equation for $\zeta(z)$ [16–28] (Definition 8.1.6) enforces the absolute convergence of ζ on regions of $\mathbb{C}$ whose real parts exceed the domain of convergence of $\mathfrak{D}$. In Theorem 8.1.8, we will prove that $\mathfrak{D}(x)$ converges even for $x \in \mathbb{F}$. Then it will follow trivially that the convergent behavior of $\mathfrak{D}$ at non-arithmatic $x \in \mathbb{R} \setminus \mathbb{R}^{\cup}$ carries over to ζ at non-arithmatic $z \in \mathbb{C} \setminus \mathbb{C}^{\cup}$. This follows because (i) Definition 8.1.4 sets the ζ function exactly equal to $\mathfrak{D}(x)$ when $\mathrm{Re}(z) > 1$ and $\mathrm{Im}(z) = 0$, and (ii) the functional form given in Definition 8.1.6 is exactly equal to the Dirichlet form of ζ on its domain of convergence.

Definition 8.1.6 Riemann's functional equation for the absolutely convergent analytic continuation of $\mathfrak{D}$ is

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

In the region $\mathrm{Re}(z) > 1$, this form of ζ is exactly equal the Dirichlet form given in Definition 8.1.4.

Remark 8.1.7 Under the classical assumption that $\mathbb{C} \setminus \mathbb{C}_0 = \varnothing$, it is said that Riemann's functional equation converges absolutely on $\mathbb{C} \setminus Z_1$ where

$$Z_1 = z(x,y) = z(r,\theta) = 1 \quad ,$$

is such that $\zeta(Z_1) \notin \mathbb{C}$. Presently, the many historical demonstrations of the convergence of Riemann's functional equation for ζ are only valid on $\mathbb{C}^{\cup} \setminus Z_1$ because the immeasurables have not been previously considered in this context. In fact, they are only valid on $\mathbb{C}_0 \setminus Z_1$ because $\mathbb{R}_\infty$ has not been considered, but we will assume such results hold on $\mathbb{C}^{\cup} \setminus Z_1$. Terms like n^{-z} vanish when $z > 0$ and $\mathrm{Re}(z) \notin \mathbb{R}_0$, and convergence for ζ when $z < 0$ (and $\mathrm{Re}(z) \notin \mathbb{R}_0$) should follow from Definition 8.1.6. So, although $\mathfrak{D}(\mathcal{F}_\mathcal{X})$ is not yet clarified, we can say with certainty that Riemann's functional equation is absolutely convergent on $\mathbb{C}^{\cup} \setminus Z_1$. Aside from the behavior of ζ on $\mathbb{C} \setminus \mathbb{C}^{\cup}$, however, we already have all the tools needed to determine its behavior on $\mathbb{C}^{\cup}$. These tools are sufficient to close the book on the Riemann hypothesis as an open question. The relevant analyses are presented in Sections 8.2 and 8.3. First, it will be good for thoroughness to understand the behavior of $\mathfrak{D}(x)$ at $x \in \mathbb{F}$.

It is understood that $\mathfrak{D}(x)$ is absolutely convergent on the ray $x > 1$. This ray includes immeasurable, non-arithmatic numbers $\{\mathcal{F}_\mathcal{X}\} \subset (1,\infty)$, but we have not yet given a concise definition for the $x^{\mathcal{F}_\mathcal{X}}$ needed to compute $\mathfrak{D}(\mathcal{F}_\mathcal{X})$. The absolute convergence of $\mathfrak{D}$ everywhere on this ray is usually framed in the context of every real number being less than some natural number, so we should give due consideration to the numbers in the neighborhood of infinity. It follows from the arithmetic axioms that $\mathfrak{D}(x) = 1$ for any $x \in \mathbb{R}^{\cup} \setminus \mathbb{R}_0$, but the case of

$$\mathfrak{D}(\mathcal{F}_\mathcal{X}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathcal{F}_\mathcal{X}}} \quad ,$$

remains to be clarified. We have required with Axiom 7.5.8 that arithmetic operations between measurable and immeasurable numbers are not defined, but the definition of exponentiation is such that $n^{\mathcal{F}_\mathcal{X}}$ involves arithmetic operations between natural numbers only:

$$n^{\mathcal{F}_\mathcal{X}} = \underbrace{n \cdot n \cdot n \cdot n \cdot n \cdot n \ldots n}_{\text{product of } \mathcal{F}_\mathcal{X}\, n\text{s}} \quad .$$

Raising a natural number to the power of an immeasurable number only makes a call to the immeasurables when one wants to count the number of products of

naturals. Therefore, there is no obvious reason to discount $n^{\mathcal{F}_\mathcal{X}}$ as an undefined operation.

Theorem 8.1.8 *If $0 \leq \mathcal{X} < 1$, then the quantity $\mathfrak{D}(\mathcal{F}_\mathcal{X})$ is equal to one.*

Proof. The Dirichlet series is

$$\mathfrak{D}(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \left(e^{\ln n}\right)^{-x} = \sum_{n=1}^{\infty} e^{-x \ln n} \quad ,$$

so we have

$$\frac{d}{dx}\mathfrak{D}(x) = \sum_{n=1}^{\infty} -\ln n \left(e^{-x \ln n}\right) \quad .$$

The derivative of $\mathfrak{D}(x)$ is non-positive on $x > 1$ due to the overall minus sign. Since the function never increases on the ray, it follows that $\mathfrak{D}(x)$ is monotonic decreasing or constant on the ray. The arithmetic axioms give

$$x_0 \in \left\{\mathbb{R}^\mathcal{X}\right\} \quad \Longrightarrow \quad \mathfrak{D}(x_0) = \sum_{n=1}^{\infty} \frac{1}{n^{\aleph_\mathcal{X}+b}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{\aleph_\mathcal{X}+b}} = 1 \quad ,$$

so $\mathfrak{D}(x)$ is constant on $(\aleph_\mathcal{X}, \aleph_\mathcal{Y}) \subset (1, \infty)$. Since $\mathcal{F}_\mathcal{X} \in (\aleph_\mathcal{X}, \aleph_\mathcal{Y})$, and $\mathfrak{D}$ is constant on this interval, it follows that

$$\mathcal{X} > 0 \quad \Longrightarrow \quad \mathfrak{D}(\mathcal{F}_\mathcal{X}) = 1 \quad .$$

It follows, therefore, that

$$n > 1 \quad , \quad \mathcal{X} > 0 \quad \Longrightarrow \quad \frac{1}{n^{\mathcal{F}_\mathcal{X}}} = 0 \quad .$$

To complete the proof of the present theorem, we need to show

$$n > 1 \quad \Longrightarrow \quad \frac{1}{n^{\mathcal{F}_0}} = 0 \quad .$$

We will prove this remaining case by contradiction. If $n^{-\mathcal{F}_0} > 0$, then

$$\exists m \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{n^{\mathcal{F}_0}} > \frac{1}{m} \quad \Longrightarrow \quad n^{\mathcal{F}_0} < m \quad .$$

Since $n \in \mathbb{N}$, the condition $n > 1$ is equivalent to the condition $n \geq 2$. Since $\mathcal{F}_0 > 1$, it follows that

$$\frac{1}{n^{\mathcal{F}_0}} < \frac{1}{2} \quad \Longrightarrow \quad m > 2 \quad .$$

Every such m may be expressed as $m = n^k$ for some $k \in \mathbb{R}_0$. Then

$$n^{\mathcal{F}_0} < m \quad \Longrightarrow \quad n^{\mathcal{F}_0} < n^k \quad \Longrightarrow \quad \mathcal{F}_0 < k \quad .$$

Since $k \in \mathbb{R}_0$, we have obtained a contradiction. The theorem is proven. ∎

Remark 8.1.9 In Theorem 6.2.5, we used the result

$$\big(\aleph_{\mathcal{X}} \pm k\big)! = \aleph_{\aleph_{\aleph_{...}}} = \infty \quad ,$$

to prove that the big exponential function E^x is equal to the regular exponential function e^x for any $x \in \mathbb{R}_0$. Similarly, Theorem 8.1.8 suggests

$$x^{\mathcal{F}_{\mathcal{X}}} = \infty \quad ,$$

which is not unlike Axiom 5.1.9 giving

$$x^{\widehat{\infty}} = \infty \quad .$$

Since the factorial grows faster than the exponential in the sense that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad ,$$

all three results are consistent. Exponentiation by a number greater than any natural number yields ∞, *c.f.*: Remark 2.2.5.

§8.2 Non-trivial Zeros in the Critical Strip

In this section, we will prove the negation of the Riemann hypothesis.

Remark 8.2.1 The Riemann ζ function is holomorphic on $\mathbb{C} \setminus Z_1$. It is a well-known property of holomorphic functions that their zeros are isolated on a domain, or else the function is constant on that domain. However, this property relies on the implicit axiom that all pairs of points (z_1, z_2) in any subdomain $D \subset \mathbb{C}$ are such that the distance between them is $d(z_1, z_2) \in \mathbb{R}_0$. When we do not take this implicit axiom limiting D to a single neighborhood of fractional distance, further specification is required. With a requisite caveat, the property becomes the following. If the zeros of a holomorphic function are not isolated, then the function is constant everywhere on a disc of radius $r \in \mathbb{R}_0$ about any of the non-isolated zeros.

Proposition 8.2.2 If (i) f is a holomorphic function defined everywhere on an open connected set $D \subset \mathbb{C}$, and (ii) there exists more than one $z_0 \in D$ such that $f(z_0) = 0$, then f is constant on D or the set containing all $z_0 \in D$ is totally disconnected.

Refutation. This proposition is usually proven by a line of reasoning starting with the following. By the holomorphism of f and the property $f(z_0) = 0$, we know there exists a convergent Taylor series representation of $f(z)$ for all $|z - z_0| < r_0$ with $r_0 \in \mathbb{R}$. Here, however, the proposition immediately fails pseudo-trivially because we can select $r_0 \in \{\mathbb{R}^{\mathcal{X}}\}$ and assume

$$\big|z - z_0\big| > \big(\aleph_{\mathcal{X}} + a\big) \quad ,$$

to show that the Taylor series does not converge when $\mathcal{X} > 0$. We have

$$f(z) = f(z_0) + f'(z_0)\big(z - z_0\big) + \frac{f''(z_0)}{2!}\big(z - z_0\big)^2 + \dots \quad .$$

The first term in the series vanishes by definition so, therefore, we have by assumption

$$f(z) > f'(z_0)\big(\aleph_{\mathcal{X}} + b\big) + \sum_{n=2}^{\infty} \frac{f^{(n)}(z_0)}{n!}\big(\aleph_{\mathcal{X}} + b\big)^n \quad .$$

The Taylor series expansion of f does not converge in $\mathbb{R}$ for $|z - z_0| \in \mathbb{R}_0^{\mathcal{X}}$. This follows from $(\aleph_{\mathcal{X}} + b)^n > \aleph_1$ for all $n \geq 2$, as per Axiom 5.2.6. ✐

Theorem 8.2.3 *If (i) f is a holomorphic function defined everywhere on an open connected set $D \subset \mathbb{C}$, (ii) there exists more than one $z_0 \in D$ such that $f(z_0) = 0$, and (iii) every $p \in D$ is such that $|z_0 - p| \in \mathbb{R}_0$, then f is constant on D or the set containing all $z_0 \in D$ is totally disconnected.*

Proof. Various proofs of Theorem 8.2.3 are well known. They are taken for granted. ✐

Main Theorem 8.2.4 *If $\{\gamma_n\}$ is an increasing sequence containing the imaginary parts of the non-trivial zeros of the Riemann ζ function in the upper complex half-plane, then*

$$\lim_{n \to (\aleph_{\mathcal{X}} + b)} \big|\gamma_{n+1} - \gamma_n\big| = 0 \quad .$$

Proof. To prove the theorem, we will follow Titchmarsh's proof [17] of a theorem of Littlewood [29]. The original theorem is stated as follows.

"For every large T, $\zeta(s)$ has a zero $\beta + i\gamma$ satisfying

$$\big|\gamma - T\big| < \frac{A}{\log\log\log T} \quad ."$$

Note that A is some constant $A \in \mathbb{R}_0$. For proof by contradiction, assume

$$\lim_{n \to (\aleph_{\mathcal{X}} + b)} \big|\gamma_{n+1} - \gamma_n\big| \neq 0 \quad .$$

Then there exists some $m(n)$ and some $a \in \mathbb{R}_0^+$ such that

$$\lim_{m(n) \to (\aleph_{\mathcal{X}} + b)} \big|\gamma_{m(n)+1} - \gamma_{m(n)}\big| > 2a \quad .$$

Let T_n be the average of $\gamma_{m(n)+1}$ and $\gamma_{m(n)}$ so

$$T_n = \frac{\gamma_{m(n)+1} + \gamma_{m(n)}}{2} \quad .$$

Now we have

$$\lim_{T_n \to (\aleph_{\mathcal{X}} + b)} \left| \gamma - T_n \right| > a \quad ,$$

because T_n is centered between the next greater and next lesser γ_n. We have shown that this pair of γ_n are separated by more than $2a$. This contradicts Littlewood's result

$$\left| \gamma - T_n \right| < \frac{A}{\log\log\log T_n} \quad , \quad \text{whenever} \qquad \frac{A}{\log\log\log T_n} < a \quad .$$

The limit $T_n \to \aleph_{\mathcal{X}} + b$ is exactly such a case because

$$\log(\aleph_{\mathcal{X}} + b) = \log(\mathcal{X}\widehat{\infty}) + \log(b) = \log(\mathcal{X})\log(\widehat{\infty}) + \log(b) \quad .$$

If we take $\log(\widehat{\infty}) = \widehat{\infty}$ or $\log(\widehat{\infty}) = \infty$, evaluating the log a few more times will yield

$$\frac{A}{f(\log(\widehat{\infty}))} = 0 \quad .$$

This shows that the expression is always less than $a \in \mathbb{R}_0^+$. Therefore, the elements of $\{\gamma_n\}$ form an unbroken line when $|\mathrm{Im}(z)| \in \mathbb{R}_\infty$. This proves the theorem. ∎

Remark 8.2.5 Note that $\{\gamma_n\}$ is not such that each element can be labeled with $n \in \mathbb{N}$ because the zeros become uncountably infinite in the neighborhood of infinity. Rather, $\{\gamma_n\}$ must be a sequence in the sense that it is an ordered set of mathematical objects, some of which may be assembled as intervals. Also note that $\{\gamma_n\}$ is a proper sequence in the usual sense when we take $n \in \mathbb{N}_\infty$, as in Definition 6.2.3. The extended natural numbers are uncountably infinite.

Corollary 8.2.6 *The Riemann ζ function has zeros within the critical strip yet off the critical line.*

Proof. Proof follows from Theorem 8.2.3 and Main Theorem 8.2.4. If the imaginary parts of the zeros form an unbroken line in the neighborhood of infinity, then the zeros are not isolated. Since ζ is holomorphic on $\mathbb{C}^\cup \setminus Z_1$, it must have zeros everywhere on a disc of radius $r_0 \in \mathbb{R}_0$ about any of the zeros on the critical line in the neighborhood of infinity. Some of these zeros, obviously, are within the critical strip yet not on the critical line. ∎

Remark 8.2.7 The Riemann hypothesis is false.

§8.3 Non-trivial Zeros in the Neighborhood of Minus Infinity

The trivial zeros of the Riemann ζ function are the negative even integers $z = -2, -4, -6, \ldots$ [30]. In this section, we will prove that ζ has non-trivial

zeros outside of the critical strip. The theorem of Hadamard and de la Vallée-Poussin [31, 32] is usually taken to rule out the existence of such zeros, so here we will conjecture that the theorem fails in the neighborhood of infinity. Indeed, it follows from Corollary 8.2.6 that ζ has zeros on the line $\mathrm{Re}(z) = 1$, and this is something else which contradicts the theorem of Hadamard and de la Vallée-Poussin. We will conjecture that their result fails in the neighborhood of infinity, most likely due to something about quotients of the form $\mathbb{R}_0 \div \mathbb{R}^{\mathcal{X}}$ being surprising identical zeros.

Theorem 8.3.1 *The Riemann ζ function is equal to one whenever* $\mathrm{Re}(z) \in \mathbb{R}^{\mathcal{X}}$ *such that* $0 < \mathcal{X} \leq 1$.

Proof. Observe that the Dirichlet form of ζ

$$\zeta(z) = \sum_{n \in \mathbb{N}} \frac{1}{n^z} \quad ,$$

takes $z_0 = \big(\aleph_{\mathcal{X}} + b\big) + iy$ as

$$\begin{aligned}
\zeta(z_0) &= \sum_{n=1} \frac{1}{n^{(\aleph_{\mathcal{X}}+b)+iy}} \\
&= \sum_{n=1} \frac{n^{-b} n^{-iy}}{n^{\aleph_{\mathcal{X}}}} \\
&= \sum_{n=1} \frac{n^{-b}}{\big(n^{\mathcal{X}}\big)^{\widetilde{\infty}}} \bigg(\cos(y \ln n) - i \sin(y \ln n) \bigg) \\
&= 1 + \sum_{n=2} \frac{n^{-b}}{\infty} \bigg(\cos(y \ln n) - i \sin(y \ln n) \bigg) \\
&= 1 \quad . \qquad \text{🍃}
\end{aligned}$$

Main Theorem 8.3.2 *The Riemann ζ function has non-trivial zeros z_0 such that $-\mathrm{Re}(z_0) \in \mathbb{R}^{\mathcal{X}}$ for $0 < \mathcal{X} \leq 1$. In other words, ζ has non-trivial zeros in the neighborhood of minus real infinity.*

Proof. Definition 8.1.6 gives Riemann's functional form of ζ [16] as

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

Theorem 8.3.1 gives $\zeta(\aleph_{\mathcal{X}} + b) = 1$ when we set $y = 0$, so we will use Riemann's equation to prove the present theorem by computing $\zeta(z)$ at $z_0 = -(\aleph_{\mathcal{X}} + b) + 1$. (This value for z_0 follows from $1 - z_0 = \aleph_{\mathcal{X}} + b$.) We have

$$\zeta\big[-(\aleph_{\mathcal{X}} + b) + 1\big] = \lim_{z \to -(\aleph_{\mathcal{X}}+b)+1} \left(\frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \right) \lim_{z \to (\aleph_{\mathcal{X}}+b)} \Big(\Gamma(z)\zeta(z) \Big) \tag{8.1}$$

$$= \lim_{z \to -(\aleph_{\mathcal{X}}+b)+1} \left(2 \sin\left(\frac{\pi z}{2}\right) \right) \lim_{z \to (\aleph_{\mathcal{X}}+b)} \left((2\pi)^{-z} \Gamma(z) \zeta(z) \right) \quad .$$

For the limit involving Γ, we will compute the limit as a product of two limits. We separate terms as

$$\lim_{z \to (\aleph_{\mathcal{X}}+b)} \left((2\pi)^{-z} \Gamma(z) \zeta(z) \right) = \lim_{z \to (\aleph_{\mathcal{X}}+b)} \left((2\pi)^{-z} \Gamma(z) \right) \lim_{z \to (\aleph_{\mathcal{X}}+b)} \zeta(z) \quad .$$

From Theorem 8.3.1, we know the limit involving ζ on the right is equal to one. For the remaining limit, we will insert the identity and again compute it as the product of two limits. If z approaches $(\aleph_{\mathcal{X}} + b)$ along the real axis, then it follows from Axiom 5.2.13 that

$$1 = \frac{z - (\aleph_{\mathcal{X}} + b)}{z - (\aleph_{\mathcal{X}} + b)} \quad .$$

Inserting the identity yields

$$\lim_{z \to (\aleph_{\mathcal{X}}+b)} \left((2\pi)^{-z} \Gamma(z) \right) = \lim_{z \to (\aleph_{\mathcal{X}}+b)} \left((2\pi)^{-z} \Gamma(z) \frac{z - (\aleph_{\mathcal{X}} + b)}{z - (\aleph_{\mathcal{X}} + b)} \right) \quad .$$

Let

$$\alpha = \Gamma(z) \Big(z - (\aleph_{\mathcal{X}} + b) \Big) \quad , \quad \text{and} \qquad \beta = \frac{(2\pi)^{-z}}{z - (\aleph_{\mathcal{X}} + b)} \quad .$$

To get the limit of α into workable form, we will use the property $\Gamma(z) = z^{-1}\Gamma(z+1)$ to derive an expression for $\Gamma[z - (\aleph_{\mathcal{X}} + b) + 1]$. If we can write $\Gamma(z)$ in terms of $\Gamma[z - (\aleph_{\mathcal{X}} + b) + 1]$, then the limit as z approaches $(\aleph_{\mathcal{X}} + b)$ will be very easy to compute. Observe that

$$\Gamma\big[z - (\aleph_{\mathcal{X}} + b) + 1\big] = \Gamma\big[z - (\aleph_{\mathcal{X}} + b) + 2\big] \Big(z - (\aleph_{\mathcal{X}} + b) + 1 \Big)^{-1} \quad ,$$

where we have written $\Gamma(z)$ as $\Gamma[z]$. On the RHS, we see that Γ's argument is increased by one with respect to the Γ function that appears on the LHS. The purpose of inserting the identity

$$\frac{z - (\aleph_{\mathcal{X}} + b)}{z - (\aleph_{\mathcal{X}} + b)} = 1 \quad ,$$

was precisely to exploit this self-referential identity of the Γ function, which is most generally expressed as

$$\Gamma\big(z\big) = \Gamma\big(z+1\big) z^{-1} \quad .$$

By taking a limit of recursion, we will let z approach a number in the neighborhood of infinity. Then through the axiomatized addition of such numbers

(Axiom 5.2.3), we will cast the argument of Γ into the neighborhood of the origin where Γ's properties are well known. The limit is

$$\Gamma\big[z-(\aleph_{\mathcal{X}}+b)+1\big]=\Gamma(z)\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=1}^{n}\bigg(z-(\aleph_{\mathcal{X}}+b)+k\bigg)^{-1} \quad .$$

Moving the infinite product to the other side yields

$$\Gamma(z)=\Gamma\big[z-(\aleph_{\mathcal{X}}+b)+1\big]\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=1}^{n}\bigg(z-(\aleph_{\mathcal{X}}+b)+k\bigg) \quad .$$

We have let $\alpha=\Gamma(z)(z-(\aleph_{\mathcal{X}}+b))$ where the coefficient $z-(\aleph_{\mathcal{X}}+b)$ can be expressed as the $k=0$ term in the infinite product. It follows that

$$\alpha=\Gamma\big[z-(\aleph_{\mathcal{X}}+b)+1\big]\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=0}^{n}\bigg(z-(\aleph_{\mathcal{X}}+b)+k\bigg) \quad .$$

To evaluate the limit of $\alpha\beta$, we will take the limits of α and β separately. The limit of α is

$$\begin{aligned}\lim_{z\to(\aleph_{\mathcal{X}}+b)}\alpha=\Gamma\big[(\aleph_{\mathcal{X}}+b)-(\aleph_{\mathcal{X}}+b)+1\big]\times\\ \times\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=0}^{n}\bigg((\aleph_{\mathcal{X}}+b)-(\aleph_{\mathcal{X}}+b)+k\bigg) \quad .\end{aligned}$$

Axiom 5.2.3 gives $(\aleph_{\mathcal{X}}+b)-(\aleph_{\mathcal{X}}+b)=0$ so

$$\lim_{z\to(\aleph_{\mathcal{X}}+b)}A=\Gamma(1)\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=0}^{n}k=0 \quad .$$

Direct evaluation of the $z\to(\aleph_{\mathcal{X}}+b)$ limit of $\beta=(2\pi)^{-z}(z-(\aleph_{\mathcal{X}}+b))^{-1}$ gives $\frac{0}{0}$ so we need to use L'Hôpital's rule. Evaluation yields

$$\begin{aligned}\lim_{z\to(\aleph_{\mathcal{X}}+b)}\beta&\overset{*}{=}\lim_{z\to(\aleph_{\mathcal{X}}+b)}\left(\frac{\frac{d}{dz}(2\pi)^{-z}}{\frac{d}{dz}\bigg(z-(\aleph_{\mathcal{X}}+b)\bigg)}\right)\\ &=\lim_{z\to(\aleph_{\mathcal{X}}+b)}\frac{d}{dz}e^{-z\ln(2\pi)}\\ &=-\ln(2\pi)\ e^{-(\aleph_{\mathcal{X}}+b)\ln(2\pi)}\\ &=-\ln(2\pi)\ \frac{e^{-b\ln(2\pi)}}{\big(e^{\mathcal{X}\ln(2\pi)}\big)^{\widetilde{\infty}}}\\ &=-\ln(2\pi)\frac{e^{-b\ln(2\pi)}}{\infty} \quad .\end{aligned}$$

By Axiom 5.1.3 giving $\frac{1}{\infty} = 0$, we find that the limit of $\alpha\beta$ is 0. It follows from Equation (8.1) that

$$\zeta\big[-(\aleph_{\mathcal{X}} + b) + 1\big] = \lim_{z \to -(\aleph_{\mathcal{X}}+b)+1} 2\sin\left(\frac{\pi z}{2}\right) \cdot 0 = 0 \quad .$$

This proves the theorem. 

Example 8.3.3 In Theorem 8.3.1, we found that $\zeta(z) = 1$ for any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \in \mathbb{R}^{\mathcal{X}}$ and $0 < \mathcal{X} \leq 1$. Riemann's functional equation

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad , \tag{8.2}$$

allows us to compute the value of ζ at any z in the leftward region of the complex plane whenever an appropriate $\zeta(1-z)$ is known (Definition 8.1.6). In Main Theorem 8.3.2, we used Theorem 8.3.1 to set

$$\zeta(1 - z_0) = \zeta(\aleph_{\mathcal{X}} + b) = 1 \quad ,$$

so that we were able to compute

$$\zeta(z_0) = \zeta(-\aleph_{\mathcal{X}} - b + 1) = 0 \quad ,$$

in the neighborhood of minus real infinity. In this example, we will demonstrate that Riemann's functional form of ζ is robust by checking for consistency when the orientation of z and $1 - z$ is reversed so that $1 - z$ is in the left complex half-plane. Namely, we will examine whether the formula disagrees with $\zeta(\aleph_{\mathcal{X}}) = 1$ (Theorem 8.3.1) when we choose

$$\zeta(1 - z_0) = \zeta(-\aleph_{\mathcal{X}} + 1) = 0 \quad .$$

With the $\zeta(1-z)$ term on the right of the Equation (8.2) already determined, we will use the formula

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{z} \right] \quad ,$$

to compute the Γ function as

$$\Gamma(-\aleph_{\mathcal{X}} + 1) = \frac{1}{-\aleph_{\mathcal{X}} + 1} \prod_{n=1}^{\infty} \left[\left(1 - \aleph_{\left(\frac{\mathcal{X}}{n}\right)} + \frac{1}{n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{-\aleph_{\mathcal{X}}+1} \right] = 0 \quad .$$

Evaluation of Riemann's equation yields

$$\zeta(\aleph_{\mathcal{X}}) = \pi^{-1} \underbrace{(2\pi)^{\aleph_{\mathcal{X}}}}_{\infty} \sin\left(\frac{\pi \aleph_{\mathcal{X}}}{2}\right) \underbrace{\Gamma(-\aleph_{\mathcal{X}} + 1)}_{0} \underbrace{\zeta(-\aleph_{\mathcal{X}} + 1)}_{0} \quad .$$

This equation is undefined due to the product $\infty \cdot 0$. We cannot obtain a contradiction. We have not taken the limit $z \to \aleph_{\mathcal{X}}$ following the form of Main Theorem 8.3.2 because the relevant quantities are easily computed. A more nuanced treatment, however, one befitting a theorem, might resolve the intractable form dependent on $\infty \cdot 0$.

This example has demonstrated the robust character of Riemann's functional equation in the neighborhood of infinity. It has also demonstrated why we must take $x^{\widehat{\infty}} = \infty$ for $x > 1$ (Axiom 5.1.9). If this expression was said to be equal to algebraic infinity, as in $x^{\widehat{\infty}} = \widehat{\infty}$, then Axiom 5.1.6 giving $\widehat{\infty} \cdot 0 = 0$ would produce a contradiction in Riemann's functional equation under the reversal of z and $1 - z$.

Remark 8.3.4 If one requires that Riemann's functional equation can never be undefined, meaning that it is not sufficient for the equation to simply determine ζ on the left complex half-plane from its behavior on the right, but that it must determine one equally from the other, then we must introduce a convention such that $\infty \cdot 0 = 1$. With this definition, the derivation followed in Example 8.3.3 would confirm Theorem 8.3.1 giving $\zeta(\aleph_{\mathcal{X}}) = 1$. For many reasons, the product $0 \cdot \infty$ is taken as undefined, and yet there are certain realms of mathematics in which it is given the definition $0 \cdot \infty = 1$. Therefore, one would explore whether or not a scheme of transfinite numbers as the 1D longitudinal analytic continuation of $\mathbb{R}$ onto $\mathbb{R}_{\mathbb{F}^k}$, or onto $\mathbb{T}$ via the order relation $|\widehat{\infty}| < \infty$, might allow for $0 \cdot \infty = 1$. For the purposes of the Riemann hypothesis, however, it is sufficient that the functional equation determines ζ on the left complex half-plane without invoking a contradiction.

Conjecture 8.3.5 The theorem of Hadamard and de la Vallée-Poussin [31,32] showing that ζ never vanishes on the line $\mathrm{Re}(z) = 1$ should fail along the portions of that line lying in the neighborhood of infinity. A corollary of this theorem is refuted in Reference [8]. The corollary is said to prove that the Riemann ζ function cannot have non-trivial zeros beyond the critical strip, such as those demonstrated in Main Theorem 8.3.2, but the corollary fails due to the arithmetic of numbers in the neighborhood of infinity. The theorem of Hadamard and de la Vallée-Poussin itself should fail for the same reason.

Remark 8.3.6 Patterson writes the following in reference [18].

> "There is a second representation of ζ due to Euler in 1749 which [...] is the reason for the significance of the zeta-function. This is
>
> $$\zeta(s) = \prod_{p \in \text{primes}} \left(1 - p^{-s}\right)^{-1} \quad ,$$
>
> where the product is taken over all prime numbers p. This is called

the Euler Product representation of the zeta-function and gives analytic expression to the fundamental theorem of arithmetic."

The fundamental theorem of arithmetic is given in *The Elements* [1] as Book 7, Propositions 30, 31, and 32. A modern statement of the fundamental theorem of arithmetic is that every natural number greater than one is a prime number, or it is a product of prime numbers. With the ultimate goal of number theory being concerned with the distribution of the prime numbers, now we will demonstrate as a corollary result that the Euler product form of ζ [18, 33] shares at least some zeros with the Riemann ζ function in the left complex half-plane where the absolute convergence of the Euler product to the Riemann ζ function is not historically proven.

Corollary 8.3.7 *The Euler product from of ζ has non-trivial zeros with negative real parts in $\mathbb{R}_\infty$.*

Proof. Consider a number $z_0 \in \mathbb{C}$ such that

$$z_0 = -(\aleph_{\mathcal{X}} + b) + iy \quad , \quad \text{where} \qquad b, y \in \mathbb{R}_0 \quad .$$

Observe that the Euler product form of ζ [33] takes z_0 as

$$\begin{aligned}
\zeta(z_0) &= \prod_p \frac{1}{1 - p^{(\aleph_{\mathcal{X}}+b)-iy}} \\
&= \left(\frac{1}{1 - P^{(\mathcal{X}\aleph_1+b)-iy}}\right) \prod_{p\neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}}+b)-iy}} \\
&= \left(\frac{1}{1 - P^b\left(P^{\mathcal{X}}\right)^{\widehat{\infty}} \left[\cos(y \ln P) - i \sin(y \ln P)\right]}\right) \prod_{p\neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}}+b)-iy}} \quad .
\end{aligned}$$

Let $y \ln P = 2n\pi$ for some prime P and $(n+1) \in \mathbb{N}$. Then

$$\zeta(z_0) = \frac{1}{\infty} \prod_{p\neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}}+b)-iy}} = 0 \quad . \qquad \blacksquare$$

§A Developing Mathematical Systems Historically

Because this treatise so concisely follows a *very* long trail of preexisting philosophical pursuits in mathematics, we present here as an appendix a concise summary of some of the important questions which motivated the modernist approach to complementing Euclid as the foundation of real analysis. In the article *The Real Numbers: From Stevin to Hilbert*, O'Connor and Robertson write the following [34].

> "By the time Stevin proposed the use of decimal fractions in 1585, the concept of a real number had developed little from that of Euclid's *Elements*. Details of the earlier contributions are examined in some detail in our article *The real numbers: from Pythagoras to Stevin*."

This appendix summarizes two articles by O'Connor and Robertson which outline the history of what are called "the real numbers" today [34, 35]. With this summary, we will essentialize the trail of facts supporting the present axiom-constructive fractional distance approach to the real number system. Setting the stage for the theme, O'Connor and Robertson write the following.

> "By the beginning of the 20th century then, the concept of a real number had moved away completely from the concept of a number which had existed from the most ancient times to the beginning of the 19th century, namely its connection with measurement and quantity."

Pertaining to the introduction of decimal fractions, O'Connor and Robertson cite Wallis as writing the following about irrational numbers in 1684 [34].

> "[S]uch proportion is not to be expressed in the commonly received ways of notation."

Wallis makes a wholehearted declaration of the matter contended in fractional distance analysis. Sometimes it is necessary in mathematics to introduce new notations such as $\aleph_{\mathcal{X}}$, $\widehat{\infty}$, and $\mathbb{F}^{\mathcal{X}}$. Should it be claimed that one may not simply declare a thing such as $\aleph_{\mathcal{X}}$, Wallis is cited as evidence that one may, and that, at times, one must. The program by which we have introduced infinity hat for the algebraic representation and analysis of numbers in the neighborhood of infinity is a seamless continuation of the most proper traditions in mathematical scholarship. Further emphasizing the importance of the influx of new notations into contemporary mathematics, O'Connor and Robertson write the following [35].

> "A major advance was made by Stevin in 1585 in *La Thiende* when he introduced decimal fractions. One has to understand here that in fact it was in a sense fortuitous that his invention led to a much

> deeper understanding of numbers for he certainly did not introduce the notation with that in mind. Only finite decimals were allowed, so with his notation only certain rationals [*were*] to be represented exactly. Other rationals could be represented approximately and Stevin saw the system as a means to calculate with approximate rational values. His notation was to be taken up by Clavius and Napier but others resisted using it since they saw it as a backwards step to adopt a system which could not even represent $\frac{1}{3}$ exactly."

Still yet further emphasizing the rightful place of new notations in mathematics, O'Connor and Robertson write the following [34].

> "[*T*]he question of the existence of numbers can only refer to the thinking subject or to those objects of thought whose relations are represented by numbers. Strictly speaking, only that which is logically impossible (i.e.: which contradicts itself) counts as impossible for the mathematician."

New numbers require new notations. As it was from the naturals to the rationals, irrationals, and complex numbers, *etc.*, so it is to the neighborhood of infinity. All progress in mathematics must be predicated from time to time upon the introduction of new notations such as $\aleph_{\mathcal{X}}$ and $\widehat{\infty}$. Stevin introduced decimal fractions, and now we have introduced infinity hat. Leibniz gave us the integral symbol, and now there exists a real number $\aleph_{0.5}$ (which was already known at least as long ago Euler who wrote $\frac{i}{2}$ in the 18th century [7].) So, now we have shown the qualitative likeness of the present course to the previous course, and we will emphasize that the course in question has always been the means by which to unify algebra and geometry.

O'Connor and Robertson write the following [34].

> "Similarly Cantor realized that if ***he wants the line to represent the real numbers*** [*emphasis added*] then he has to introduce an axiom to recover the connection between the way real numbers are now being defined and the old concept of measurement."

O'Connor and Robertson specifically identify Cantor's motivations as the same given here: how can we best preserve the geometric notion of an infinite line in the algebraic arena? If one supposes that "infinity is not allowed" and lets that be the end of inquiry into the preservation of the notion of infinite geometric extent, then it is unlikely that the resulting mathematical system will make sufficient provisions for this fundamental notion. Indeed, the entire theme of this work has been to modify existing mathematical systems so as to *better* accommodate the notion of infinite geometric extent.

Cantor himself writes the following [2].

> "[*One may*] add an axiom which simply says that every numerical quantity also has a determined point on the straight line whose coordinate is equal to that quantity."

In the present treatise, we have established Cantor's suggested axiom as the definition of a real number with Definitions 2.1.4 and 2.1.5. However, we have extended Cantor to separately consider the "determined" geometric point from the numbers in the algebraic representation of that point. Indeed, this is the main distinction between our own approach and Cantor's approach. This issue fairly well represents the issue cited in Remark 3.1.20 as the source of "much pathology" in modern analysis: the presumption of a one-to-one correspondence between numbers and geometric points is a fair proxy for one's choice to distinguish algebraic FDFs of the first and second kinds (Definitions 3.1.9 and 3.1.11). Cantor's implied concept of fractional distance seems to favor $\mathcal{D}^{\dagger}_{AB} = \mathcal{D}''_{AB}$, whereas we have demonstrated the philosophical superiority of $\mathcal{D}^{\dagger}_{AB} = \mathcal{D}'_{AB}$. We can glean from Cantor's words that he likely associated only one number with each point, but we have shown that this is only best on a line segment with length in the neighborhood of the origin. If the determined point is in an infinitely long line or a line segment with non-vanishing fractional distance such as $X \in \mathbf{AB}$, then we have proven that the determined point does not have one uniquely associated real number.

We have sought to nourish the continuing synthesis of algebra and geometry that is the best fruit in the garden of modern mathematics, and we have also identified a weed in that garden that must be pulled up by its roots. What is called the Archimedes property of real numbers today disallows a corner case allowed by the property as it was recorded in *The Elements*. Where Bugs Bunny might lament that he *should have* taken a left turn at Albuquerque, this writer demands that the mathematical establishment *must* retrace its steps and make the right turn at Archimedes. Pointing to the proper course, we have *properly* restated the ancient Archimedes property with English verbiage and Latin mathematical symbols (Definition 6.3.9), and we have also given a separate statement as the Archimedes property of 1D transfinitely continued real numbers (Axiom 6.3.12).

In a similar program, Hilbert gave his own modernized restatement of the Archimedes property in *The Foundations of Geometry* [4]. O'Connor and Robertson write the following [34].

> "[*Hilbert's statement of the Archimedes property was*] that given positive numbers a and b then it is possible to add a to itself a finite number of times so that the sum exceeds b."

Townsend translates Hilbert's original German as follows [36].

> "Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points A_2, A_3, A_4, ... so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments
>
> $$AA_1,\ A_1A_2,\ A_2A_3,\ A_3A_4, \ldots$$

> be equal to one another. Then, among this series of points, there always exists a certain A_n such that B lies between A and A_n."

Hilbert's reliance on the AB notation to give a statement of the Archimedes property for a Euclidean line segment powerfully highlights the historical similitude of the present fractional distance approach to a modernizing algebraic capstone on Euclidean geometry. Hilbert's axioms of geometry applied to Dedekind cuts give us the highly regarded field axioms, more or less, so it is remarkable that this writer was likewise called—while working to the same ends as Hilbert—to give a restatement of what Euclid meant when he said he had it on good authority that Archimedes had heard from Eudoxus that such and such was the real Archimedes property of real numbers. In the case of Hilbert's statement of the property, we see that Hilbert gave a finite multiplier which is implicitly, but not *explicitly*, $n \in \mathbb{N}$. The extended natural numbers $n \in \mathbb{N}_\infty$ will provide the multipliers needed to preserve Hilbert's statement of the property in fractional distance analysis. As O'Connor and Robertson have paraphrased him, Archimedean multipliers are constrained as finite $m < \infty$, not $n \in \mathbb{N}$.

Regarding the very ancient history, O'Connor and Robertson write the following [35].

> "It seems clear that Pythagoras would have thought of $1, 2, 3, 4 \ldots$ (the natural numbers in the terminology of today) in a geometrical way, not as lengths of a line as we do, but rather in the form of discrete points. Addition, subtraction, and multiplication of integers are natural concepts with this type of representation but there seems to have been no notion of division."

Even as long ago as Pythagoras, the open question of the separation of algebraic numbers from geometric magnitudes was already one of import. Most interestingly, O'Connor and Robertson present an apt and distinct likeness here. We might give the regularly-spaced, disconnected immeasurable numbers $\mathcal{F}_\mathcal{X} \in \mathbb{F}$ an arithmetic such that they behave like the natural numbers on another copy of the real line, as in Remark 7.5.21. An immeasurable number has certain difficulties as the length of a line segment, but no such difficulties exist when we consider $\{\mathcal{F}(n)\}$ as a set of discrete points.

In the present treatise overall, like Hilbert very recently and those who came after Pythagoras in antiquity, we have sought to build a hybrid framework for mathematical analysis which maximizes the synergy between algebra and geometry. O'Connor and Robertson write the following [35].

> "[I]t should be mentioned at this stage that the Egyptians and the Babylonians had a different notion of a number to that of the ancient Greeks. The Babylonians looked at reciprocals and also at approximations to irrational numbers, such as $\sqrt{2}$, long before Greek mathematicians considered approximations. The Egyptians also looked

at approximating irrational numbers.

"Let us now look at [...] Euclid's *Elements*. This is an important stage since it would remain the state of play for nearly the next 2000 years. In Book 5 Euclid considers magnitudes and the theory of proportion of magnitudes. It is probable (and claimed in a later version of *The Elements*) that this was the work of Eudoxus. Usually when Euclid wants to illustrate a theorem about magnitudes he gives a diagram representing the magnitude by a line segment. However magnitude is an abstract concept to Euclid and applies to lines, surfaces and solids. Also, more generally, Euclid also knows that his theory applies to time and angles.

"Given that Euclid is famous for an axiomatic approach to mathematics, one might expect him to begin with a definition of magnitude and state some unproved axioms. However he leaves the concept of magnitude undefined and his first two definitions refer to the part of a magnitude and a multiple of a magnitude."

O'Connor and Robertson proceed to break down Euclid's Book 5 as we have when examining the Archimedes property in Section 6.3. Therefore, we will list the properties of magnitudes again and present comments on them in consolidated form. Fitzpatrick's comments on Euclid's original text are labeled (RF), our own comments are labeled (JT), and the comments of O'Connor and Robertson are labeled (OR).

Book 5, Definition 1 A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.

(RF) In other words, α is said to be a part of β if $\beta = m\alpha$.

(JT) The first definition makes it obvious that the Archimedean multiplier is not meant to be a natural number. If the magnitude of ten units of geometric length is to be greater than one of nine, meaning that nine is to be a part of ten, then there must exist non-integer multipliers such as ten ninths.

(OR) Again the term "measures" here is undefined but clearly Euclid means that (in modern symbols) the smaller magnitude x is a part of the greater magnitude y if $nx = y$ for some natural number $n > 1$.

When O'Connor and Robertson cite $n \in \mathbb{N}$, they do not take into consideration numbers having non-integer quotients, *e.g.*: $10 : 9$, or else they are only giving a subcase of what is meant in the original context. Even if they meant to write $nx > y$, which is not unlikely, using the natural numbers to demonstrate the property only makes sense if one takes the auxiliary axiom that there are no real numbers greater than every natural number. In that case, the Archimedes property is irreducibly represented in the natural number statement of the Archimedean multiplier, and one notes that the supposition

that every real number is in the neighborhood of the origin was a normal axiom at the time of the publication of References [34, 35].

The primary theme of the past centuries' mathematics has been one of attempting to subordinate geometry to algebra. Many historical approaches have assumed some algebraic axioms and tried to fit everything inside those axioms by ignoring geometric infinity and making a rule that one must never mention it. In taking the fractional distance tack, we have eschewed such efforts. The main difference between the present approach and the usual (or historical) approaches to merging geometry and algebra is that we have not tried to squeeze the notion of geometric infinity into the algebraic sector. Rather, $\widehat{\infty}$ is such that the algebraic structure is totally subordinated to the geometric structure.

Note the equal weighting of the gravity of the matters in the choice to suppose one of the two following axioms.

Axiom A.1 There exists a non-empty set of real numbers greater than any natural number.

Axiom A.2 There does not exist any real number greater than every natural number.

An assigned superiority in the algebraic sector might make Axiom A.2 the more attractive axiom because it allows everything to be written with the field axioms. By assigning the superior quality as the historical geometric conception of numbers, however, we are drawn toward Axiom A.1 as the preferable axiom. Additionally, we have proven multiply that Axiom A.2 causes undesirable contradictions with the geometric notion of fractional distance and the concise statement $\mathbb{R} = (-\infty, \infty)$. Even when algebraic considerations are chosen as superior to geometric ones, the superior axiom must not contradict its inferior complement. The neighborhood of infinity does exist; fractional distance requires it. The question is nothing more than whether or not we should adopt an algebraic convention which reflects the geometric reality.

Book 5, Definition 2 And the greater is a multiple of the lesser whenever it is measured by the lesser.

> (JT) This definition makes it explicitly clear that the manner in which one magnitude may measure another is such that, for example, nine can measure ten by 10 : 9.
>
> (OR) Then comes the definition of ratio.

Book 5, Definition 3 A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.

> (RF) In modern notation, the ratio of two magnitudes, α and β, is denoted $\alpha : \beta$.

(JT) This definition tells us that $\mathbb{R}$ is equipped with the $\leq$ relation. The specification of two magnitudes of the same kind tells us, essentially, that Euclid does not want his reader to compare lengths with areas, volumes, angles, time, hypervolumes, *etc.* Likewise, once we have conjured $\mathbb{R}_{\mathbb{F}}$ from an interpretation of $\mathcal{F}(n) \in \mathbb{R}$ as $n \in \mathbb{N}_{\mathbb{F}}$, we must be careful to distinguish the underlying magnitudes as different kinds.

(OR) This is an exceptionally vague definition of ratio which basically fails to define it at all. [*Euclid*] then defines when magnitudes have a ratio, which according to the definition is when there is a multiple (by a natural number) of the first which exceeds the second and a multiple of the second which exceeds the first.

Book 5, Definition 4 (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.

(RF) In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n.

(JT) The Archimedes property of real numbers requires that for every real number, there is a greater real number. In other words and in a general way, there is no largest real number because $\aleph_1 \notin \mathbb{R}$. Usually it is said that a smallest real number is also precluded by the inverse of the unbounded large number, but, surprisingly, the usual topology requirement of the fundamental axiom of algebraic construction (Axiom 7.6.8 and/or Axiom 2.1.7) seems to indicate that a smallest real number must exist (Proposition 7.2.12). This is the $\mathcal{X}_{\min}$ value of $\aleph(2) = \aleph_{\mathcal{X}_{\min}}$. As the issue of a smallest positive real number might vex one's intuition, it may be considered an exciting topic for new inquiry.

Regarding the intuitive notion of a smallest real number, consider the following. If every interior point in a connected interval $(-1, 1) \subset \mathbb{R}$ is left- and right-adjacent to another point, meaning the interval is not disconnected, then writing

$$(-1, 1) = (-1, 0] \cup (0, 1) \quad ,$$

suggests, in an intuitive way at least, that zero must be left-adjacent to the smallest positive real number. However, the protocols of mathematics override intuition, and it is said that zero is not left-adjacent to any element of $(0, 1)$ because every element of $(0, 1)$ has a δ-neighborhood lying totally within $(0, 1)$. So, if some way is found to claw a least positive real number from the analysis of fractional distance functions, then the concept of no greatest real number would

also have to be done away with due to the invariance of **AB** under permutations of the labels of its endpoints. Infinity minus the least positive real number would be the greatest real number.

(OR) The Archimedean axiom stated that given positive numbers a and b then it is possible to add a to itself a finite number of times so that the sum exceed b.

Book 5, Definition 5 Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and third both exceed, are both equal to, or are both less than, equal multiples of the second and fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.

(RF) In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\alpha > n\beta$ whenever $m\gamma > n\delta$, $m\alpha = n\beta$ whenever $m\gamma = n\delta$, and $m\alpha < n\beta$ whenever $m\gamma < n\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, *etc.*, are irrational.

(JT) This definition gives the trichotomy of the $\leq$ relation. Note well, the ratio of ratios is like the ratio of two fractional distances.

(OR) Then comes the vital definition of when two magnitudes are in the same ratio as a second pair of magnitudes. As it is quite hard to understand in Euclid's language, let us translate it into modern notation. It says that $a : b = c : d$ if given any natural numbers n and m we have

$$\begin{aligned} na > mb \quad &\text{if and only if} \quad nc > md \\ na = mb \quad &\text{if and only if} \quad nc = md \\ na < mb \quad &\text{if and only if} \quad nc < md \quad . \end{aligned}$$

Euclid then goes on to prove theorems which look to a modern mathematician as if magnitudes are vectors, integers are scalars, and he is proving the vector space axioms.

The main hurdle in the vector space conception of $\mathbb{R}$ is that the product of two vectors is a scalar, while the product of two real numbers is another real number. Even in the transfinite continuation beyond algebraic infinity, and even when the product of two things in the line always remains within the geometrically infinite line as if it were a vector space, the problem remains that the product of any two $x \in \mathbb{T}$ will be another $x \in \mathbb{T}$. There is no distinguishing a vector from a scalar. Despite this difficulty, one easily imagines $\widehat{\infty}$ as an anchor point for 1D vectors $x \in \mathbb{R}$ different than the anchor point at the origin, which we might label $\hat{0}$. Vectors anchored in the neighborhood of the origin look like $\hat{0} + \vec{b}$, and those anchored in the neighborhood of positive

infinity look like $\widehat{\infty} - \vec{b}$. The 1D vector space picture is very becoming the notion of a 1D geometric space, but the lack of distinction among vectors and scalars forbids any approach to the commonly stated, modern vector space axioms.

§B Toward Mathematical Physics

To the extent that the present fractional distance work in pure mathematics grew from, and was motivated by, a preexisting research program in theoretical physics, some things should be pointed out. Beyond this appendix, further details and other physical applications for fractional distance analysis are given throughout Reference [37].

Interaction Ranges

The first salient point regards the finite or infinite interaction ranges of the fundamental forces. Gravity and electromagnetism are said to have infinite ranges because the relevant classical forces between massive or charged particles go as $\frac{1}{r^2}$. When all finite numbers are assumed to be less than some natural number, it is a direct consequence that such inverse square force laws can never go to zero for any $r \in \mathbb{R}$. The arithmetic axioms, however, allow these forces to go to zero for finite separations in the neighborhood of infinity. Indeed, the full scope of the initial inquiry into infinity which resulted in the fruits presented here was the following question. How might we have two physical objects (where physicality requires that they are separated in spacetime by finite spacetime interval) whose mutual gravitational interaction is precisely zero? Now $r \in \mathbb{R} \setminus \mathbb{R}_0$ provides the requisite finite scale.

The Radius of the Universe

As an example for how the neighborhood of infinity might be worked into the progenitive cosmological scenario which spurred this research—the Modified Cosmological Model (MCM) [37–39]—we could set the scale of $\mathcal{F}_0$ as the $\sim$13.7Gcy radius of the observable universe. Nothing beyond the cosmic microwave background (CMB) at that distance can be observed, yet it might be helpful for the development of new theories if we could set the interactions between the local frame and the occulted region beyond the CMB to an identical zero rather than the *almost zero* value which results from forces with infinite ranges. "Almost zero" may be negligible in an application, but will always have a big impact on quantum mechanics, and usually across cosmological timescales.

Wavepackets

Another good use for the neighborhood of infinity is the mathematical description of wavepackets. In the efforts of physicists to describe the wave-particle duality of quantum particles, many quantum states are formally rendered as enveloped wavepackets whose tails extend to infinity. Often times, the non-vanishing tails of these wavepackets can be ignored. For instance, if the probability that an electron will be observed in a lab during one moment and then observed 40cy from the Earth in the next moment is on the order of one in

10^{-50}, then we may treat this as zero probability and proceed accordingly without shooting our theories in the foot. However, in the regime of ultra-fast quantum optics, the tails of mathematical wavepackets describing picosecond laser pulses generate notorious discrepancies with what is observed in the lab. Having developed the neighborhood of infinity, one might develop a language for wavepackets whose tails go to zero on the scale of $\mathcal{F}_0$ while the universe itself, or the lab frame, could be said to have a characteristic scale on the order of $\aleph_1$. Although we have spoken of the big parts of numbers as unfathomably huge, we might equally take the small parts of numbers to be unusually small. Any number of such schemes could be developed.

Levels of Aleph

Throughout the cosmology research that led to the present fractional distance analysis, we have developed a requirement for some intermediate scale between the scale of natural numbers and the scale of infinity. Specifically, we have used the concept of "odd and even levels of aleph" to describe the behaviors required for the underlying cosmological model [37–41]. In the language of Robinson's hyperreal analysis, there are "limiteds" and "unlimiteds," but the model which spurred the present analysis generated a requirement for some intermediate scale between them: the odd level of aleph where the hyperreal limited and unlimited scales would represent two successive even levels. In this language, we say that infinity is two levels of aleph higher than the level of aleph upon which resides the origin of the abstract inertial lab frame. Robinson's canonical method for mixing limiteds and unlimiteds does not provide any tools for accessing the odd level of aleph which is expected to be important for the unit cell of the MCM's lattice cosmology. In the present work, we have generated the "super-finite" $\mathcal{F}_0$ scale which can serve as the basis for an odd level of aleph between Robinson's limiteds and unlimiteds. (See Reference [42] for a treatment of the Riemann hypothesis in terms of odd and even levels of aleph.)

(Fractional) Quantization

In the course of developing the MCM, we have shown that the structure of the standard model of particle physics arises as the elementary structure of the unit cell of the model's lattice cosmology [37, 40]. In the present mathematical analysis, we have found yet more of the fundamental quantum numbers in the underlying analytical structure. Quantum mechanics has the curiously measurable half-integer spin quantization of fermions, and now we have demonstrated a half-integer interval of spacing inherent between the origins of the successively transfinite, scaled copies of $\mathbb{R}_{\mathbb{F}^k}$ (Theorem 7.5.29). Furthermore, the number $\mathcal{F}(1)$ lies one third of the way down the interval $[0, \mathcal{F}(2)]$, so we also have structure alike to the asymmetric fractional electric charges of the quarks: $-\frac{1}{3}e$ for three of them and $+\frac{2}{3}e$ for the other three. Since we have previously obtained the spectrum of the fundamental particles of the standard model as the elements of the cosmological unit cell whose worldsheets

are separated by immeasurable distance [37, 40], one would attempt to correlate the more nuanced aspects of elementary particles' quantizations with the fundamentals of fractional distance analysis.

Quantum Mechanics

It is notable that the $[0, 1]$ interval containing big parts of positive real numbers is identically the $[0, 1]$ domain of positive-definite probabilities in quantum mechanics. This likeness suggests a ready home for fractional distance analysis in quantum theory. Furthermore, the instantaneous collapse of the wavefunction has no good mathematical description in the canonical theory where ψ obeys the Schrödinger equation, but the infinite derivate of $f(x) = \aleph_x$ seems well suited to instantaneous processes. Quantum theory, in its current incarnation, is such that time and space domains are often forced into artificial and unphysical restrictions to avoid divergent integrals, and we might avoid such divergences through the arithmetic axioms for $x \in \mathbb{R}^{\cup}$, or through the arithmetic of $\mathbb{R}_0$ under new rules for treating $\mathcal{F}(n) \in \mathbb{R}$ as $n \in \mathbb{R}_{\mathbb{F}}$.

Quantum Theory for the Riemann ζ Function

Beyond applications for fractional distance analysis in physics, there may exist applications for physics in mathematics as well. For instance, there is a widely held (but nebulous) belief in some fundamental (but unknown) connection between the Riemann ζ function and quantum field theory, so one might invoke physical principles in an area of pure mathematics where they usually have no place: the analysis of ζ.

In the absence of an observation to collapse a system's wavefunction, quantum theory dictates that the system must be represented by a superposition of its eigenstates. As we have extensively discussed the lack of any preferred scale for $\mathbb{R}$ in Remark 7.5.21, one might attempt to study the real line as the simultaneous superposition of all possible scales for $\mathbb{R}$. Theorem 7.5.29 proves that the x and $x_{\mathbb{F}}$ charts cannot share an origin ($x_{\mathbb{F}} = \frac{1}{2}$ is pinned to $x = 0$), but we might suppose that all possible scaled charts of $\mathbb{R}$ can share an origin with one of x or $x_{\mathbb{F}}$. In essence, these would be copies of $\mathbb{R}$ on odd and even levels of aleph, and we might then restrict ourselves to a countably infinite set of "linearly independent" $x_{\mathbb{F}^k}$ charts among uncountably many possible copies of $\mathbb{R}$. In fact, we might simplify to just two linearly independent charts: x and $x_{\mathbb{F}}$ associated respectively with the linearly dependent even levels of aleph and the linearly dependent odd levels. In doing so, the idea suggested here (which is relegated to an appendix rather than posed as conjecture in Section 8 because it is *highly* speculative) is that we would try to associate the structure of the critical strip in the domain of $\zeta(z)$ with the origins of the charts on the odd and even levels of aleph. Although the superposition of two linearly independent quantum states does not constitute a third linearly independent state, one might seek to obtain a coordinate system (chart) with its origin on the critical line as a superposition of the x and $x_{\mathbb{F}}$ charts whose origins

are located at at 0 and 1 in an appropriate coordinate system. One might alternatively derive a third origin by an appeal to uncertainty between $(x, x_{\mathbb{F}})$ and $(x_{\mathbb{F}^{-1}}, x)$ or uncertainty in the direction of increase from $x_{\mathbb{F}} = 1$ after it is pinned to $x = \mathcal{F}_0$. Ultimately, the intention would be to make a connection to the time arrow cosmology eigenstates in the MCM. Namely, the critical strip would be asymmetric in $\mathbb{C}$ for the same reason that a universe has to have a time arrow pointing in one direction or the other. Although MCM time is a quantum superposition of time eigenstates because there is no absolute standard against which an observer might determine if time goes "forward" or "backward," the observer does always find a symmetry-breaking arrow of time when measurements are made. Similarly, we cannot know if the critical strip is in the "left" or "right" complex half-plane because the real line might increase in either direction. Barring Galilean coordinate transformations, ζ's critical line is always offset from the imaginary axis in $\mathbb{C}$, toward one direction or the other.

Spinor Transformations

Quantum spinor states do not transform as vectors. This, more than almost anything, underpins the inability of physicists to describe quantum mechanical processes with classical analogues. To develop fractional distance analysis, we have made extensive appeal to the invariance of line segments under permutations of the labels of their endpoints [9], but we have not studied what happens to functions when their domain in the neighborhood of the origin is replaced with a domain in the neighborhood of infinity. As it is usual in physics to artificially restrict a wavefunction's domain to some finite region, we should consider the behavior of

$$\psi : D \to \mathbb{C} \quad , \quad \text{with} \qquad D = (-b, b) \quad ,$$

when the permutation of $\hat{0}$ and $\widehat{\infty}$ results in

$$\psi : D \to \mathbb{C} \quad , \quad \text{with} \qquad D = (\widehat{\infty} - b, \widehat{\infty}] \cup [-\widehat{\infty}, -\widehat{\infty} + b) \quad .$$

The change in the structure of D is at least superficially suggestive of the out-of-phase variation of the components of a spinor relative to the in-phase variation of the components of a vector. As "supersymmetry transformations" swap vectorial bosons for spinorial fermions, one would attempt to liken supersymmetry transformations with permutations of the labels of the endpoints of Euclidean line segments.

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