A modified belief functions distance measure for orderable set

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Abstract

This paper proposes a new method of measuring the distance between conflicting order sets, quantifying the similarity between focal elements and their own size. This method can effectively measure the conflict of belief functions on an ordered set without saturation due to the non-overlapping focus elements. It has proven that the method satisfies the property of the distance. Examples of the engineering budget and sensors show that the distance can effectively measure the conflict between ordered sets, and prove the distance we propose to reflect the information of order sets more comprehensively by comparison with existing methods, and the conflict metric between ordered sets is more robust and accurate.

Keywords: Orderable sets, Distance metric, Dempster-Shafer theory, Belief function

1. Introduction

Dempster-Shafer theory, also known as belief function theory, was first proposed by Demster[1] and later improved by Shafer[2] because it can handle information fusion problem well, and is widely used in decision-making[3, 4, 5,

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6, 7, 8], information fusion[9, 10, 11, 4, 12, 13, 14, 15] and uncertain information processing[16, 17, 18, 19, 20, 21]. However, when given that the evidence is highly conflicting with each other, the results of using the Demster combination rule are counterintuitive, as in the case of Zadeh[22]. In recent years, a number of ways have been proposed to solve this problem[23, 24, 25, 26]. One of the most widely used methods is the distance of evidence[27, 28, 29].

In the Deepster-Shafer domain, which uses traditionally similar functions, however, the basic assumption of Jousselme distance and other distances is that the measurement space for building BPA is ”exclusive and exhaustive”[1]. Therefore, the only difference between focus elements is their concurrent or intersectional cardinality. Based on the defined similarity function, if there is no overlap between the two focal elements, the distance obtained by the cardinality-based measure will reach saturation values. In real life, there is a lot of orderly information, for example, professors with ”excellent, good, medium, pass, fail” to give students a score[30, 31]. The Dempster-Shafer domain hypothesis does not apply on an ordered set. In order to effectively measure the conflict of belief functions on an ordered set, Sunberg et al.[32] proposed a Sunberg distance based on Hausdorff distance. By calculating the distance between the focal elements, consider the distance between the two BPAs in the metric space, the effectiveness is proved by continuous (orderly) measuring spaces. In order to overcome the shortcomings of Sunberg distance, Cheng et al.[33] proposed a new ordered sets distance measurement, fully considering a variety of features of data, and comprehensively reflects the information of order sets.

In this paper, a new measure of the distance of the mass distribution conflict of ordered set confidence is proposed, which overcomes some shortcomings of the existing method. Considering not only the distance effect of the two BPA focus elements in measuring space, but also the influence of the focus element's own size on distance, because it considers the multi-faceted characteristics of the data, it can reflect the data information of the ordered set more comprehensively and effectively, and is more robust and flexible than the existing method. Numerical examples show that the proposed correlation coefficients overcome the shortcomings of existing methods and can more effectively measure the degree of conflict between belief functions.

The organizational structure of this paper is as follows. The Section 2 briefly introduces the basic knowledge of D-S theory and the existing method of evidence distance measurement. Section 3 introduces the proposed new distances and uses a simple example to illustrate the shortcomings of Sunberg distances and the validity of the distances we propose. In the Section 4, some properties of the proposed distance and the role of tuning parameters are discussed. Section 5 illustrates the validity of the proposed new distance through some numerical examples and practical applications. Finally, a brief conclusion is given in Section 6.

2. Preliminaries

2.1. Belief function theory

The belief function theory also called the Dempster-Shafer theory or evidence theory, which is proposed by Dempster[1] and supplemented by Shafer[2] to form a framework for data fusion of uncertain or inaccurate data fragments. Let Θ be a set of N mutually exclusive and exhaustive events. Then, Θ is called the frame of discernment and is denoted by[1, 2]

$$\Theta = \{\theta_1, \theta_2, \cdots, \theta_N\} \tag{1}$$

The power set 2^Θ of Θ contains all possible subsets contained in Θ is indicated as:

$$2^\Theta = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_N\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, \Theta\} \tag{2}$$

Definition 1. *A basic probability assignment (BPA) is a mapping m from 2^Θ to $[0, 1]$ defined by[1, 2]:*

$$m : 2^\Theta \to [0, 1] \tag{3}$$

which satisfies the following conditions:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^{\Theta}} m(A) = 1 \tag{4}$$

where m(A) represents the belief to A. In addition, when m(A)>0, A is called a focal element.

2.2. Existing belief function distance

2.2.1. Jousselme's distance

Jousselme et al.[34] proposed a method to measure the distance of belief function based on the distance between the intersection and union cardinality of focal elements.

Definition 2. *Let m_1 and m_2 be two BPAs on the same frame of discernment Θ, which contains N mutually exclusive and exhaustive hypotheses. The distance between m_1 and m_2 is represented as follows:*

$$d_J(m_1, m_2) = \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D_J (\overrightarrow{m_1} - \overrightarrow{m_2})} \tag{5}$$

where, $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ represent two BPAs m_1 and m_2 as vectors respectively, and D_J is a $2^N \times 2^N$ matrix used to represent the similarity of focal elements, and its elements are represented as:

$$D_J(A, B) = \frac{|A \cap B|}{|A \cup B|} \tag{6}$$

where $|A|$ is the cardinality of A, and $A, B \in 2^{\Theta}$ are two subsets of Θ.

2.2.2. Sunberg et al.'s distance

Zachary Sunberg et al.[32] proposed a Hausdorff-based measure for orderable sets. It can accurately measure the conflict between BPAs, and will not be saturated just because the two BPAs have no common focal elements, which is defined as follows:

Definition 3.

$$d_H(m_1, m_2) = \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D_H(\overrightarrow{m_1} - \overrightarrow{m_2})} \tag{7}$$

where D_H is the similarity matrix whose elements are

$$D_H(A_i, A_j) = \frac{1}{1 + KH(A_i, A_j)} \tag{8}$$

where $D_H(A_i, A_j) \in [0, 1]$, represents similarity between A_i and A_j. $K > 0$ is the user-defined tuning parameter, used to adjust measure response of the orderable spatial discretization. $H(A_i, A_j)$ is the Hausdorff distance between A_i and A_j. It is used to quantify the distance of two focal elements, which is defined according to

$$H(A_i, A_j) = max\{\underset{b \in A_i}{sup} \ \underset{c \in A_j}{inf} \ d(b, c), \underset{c \in A_j}{sup} \ \underset{b \in A_i}{inf} \ d(b, c)\} \tag{9}$$

where $d(x, y)$ is the distance between the two elements of the sets, which is any valid measure distance defined on the measured space.

2.2.3. Cheng et al.'s distance

Sunberg et al.'s distance is insensitive to the distribution of the set edge, and Cheng et al.[33] proposed a new distance, which effectively improves the sensitivity of distance to set edge distribution, and overcomes Sunberg et al.'s distance is insensitive to the set edge distribution. The method uses the same structure as Eq.(7),defined as follows:

Definition 4.

$$d_M(m_1, m_2) = \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D_M(\overrightarrow{m_1} - \overrightarrow{m_2})} \tag{10}$$

where D_M is a $n \times n$ matrix, n is the cardinality of the union of two focal elements, and the element D_{ij} in D_α represents the similarity of the ith and jth *element, defined as follows.*

$$D_M ij = D_\alpha(i,j) = \frac{1}{1+\alpha M(A,B)} \tag{11}$$

where $\alpha > 0$, its effect is equivalent to K in Eq.(8). $M(A,B)$ measure the distance between ith element A and ith element B in an ordered set. Suppose all elements in A and B are $\{A_1, \cdots, A_i, B_1, \cdots, B_j\}$, then $M(A,B)$is represented by:

$$M(A,B) = \frac{|min(A) - min(B)| + |max(A) - max(B)|}{max((A_i^+ - A_i^-),(B_i^+ - B_j^-))} \tag{12}$$

Example 1. *Suppose you have two groups of BPA, which have only m_1's focal elements different, as shown below:*

$$Group1: m_1 = \{[4,6],1\}, m_2 = \{[101,103],1\}, m_3 = \{[201,203],1\}$$
$$Group2: m_1' = \{[4.5,5.5],1\}, m_2' = \{[101,103],1\}, m_3' = \{[201,203],1\}$$

Although m_1's focal element $[4,6]$ is greater than m_1''s focal element $[4.5,5.5]$, according to Cheng et al.'s method, $d_M(m_1,m_2) = d_M(m_1',m_2')$, the result is unreasonable.

3. Proposed distance

We propose a new distance that effectively measures the distance between ordered sets. It considers not only the physical distance between the focal elements, but also the size of the focal elements themselves, is proposed as follows:

Definition 5. *Let m_1 and m_2 be two BPAs on the same ordinal frame of discernment Θ_O, we proposed a new distance defined as follow:*

$$d_o(m_1,m_2) = \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D(\overrightarrow{m_1} - \overrightarrow{m_2})} \tag{13}$$

$$where, D_{ij} = \frac{1}{1+\lambda O(A_i,A_j)} \tag{14}$$

where, $\lambda > 0$ *and Sunberg et al.'s* K*, Cheng et al.'s* α *are similar to user-defined tuning parameter that adjust the measurement response based on the discrete adjustment of the orderable space.* A_i*,* A_j *are the* i*th and* j*th focal elements on* Θ_O*.* $O(A_i, A_j)$ *indicates the similarity between* A_i *and* A_j*. It is defined as follows:*

Definition 6.

$$O(A_i, A_j) = \frac{|min(A_i) - min(A_j)| + |max(A_i) - max(A_j)|}{|max(A_i) - min(A_i)| + |max(A_j) - min(A_j)|} \tag{15}$$

A special case is that if both A_i *and* A_j *are real values, then* $O(A_i, A_j) = |A_i - A_j|$.

For the proposed distance to use the Euclidean norm, it can be defined as an element that is an N-dimensional vector discernment frame. Using the proposed similarity coefficient describe the similarity between focal elements, taking into account both the physical distance between the focal elements and the effect of the size of the focal elements themselves. Moreover, the larger the focal element, the closer it should be to the other focal elements. To illustrate the effect of the focal element's own size, an illustrative example is given:

Example 2. *Assume three BPAs* $m_1 = \{[4, 6], 1\}$*,* $m_2 = \{[101, 103], 1\}$*,* $m_3 = \{[201, 203], 1\}$*, keep* m_2*,* m_3 *unchanged, only change the size of the focus element of* m_1*, as follows:*

$$Group1 : m_1 = \{[4.9, 5.1], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$
$$Group2 : m_1 = \{[4.6, 5.4], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$
$$Group3 : m_1 = \{[4.3, 5.7], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$
$$Group4 : m_1 = \{[4, 6], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$
$$Group5 : m_1 = \{[3, 7], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$
$$Group6 : m_1 = \{[2, 8], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$
$$Group7 : m_1 = \{[1, 9], 1\}, m_2 = \{[101, 103], 1\}, m_3 = \{[201, 203], 1\}$$

The results of using Cheng et al.'s distance and proposed new distance, the distance of $d(m_1, m_2)$, $d(m_1, m_3)$ and $d(m_2, m_3)$ are shown in Table 1, Table 2 and Table 3 respectively. As we can see from Fig.1, when the size of the focus element is less than 2, $d(m_1, m_2)$ remains the same, and only when it is greater than 2, it increases as the focus element expands, this result is not reasonable. In Fig.2, the change trend of $d(m_1, m_3)$ is the same as that of $d(m_1, m_2)$. Since the interval size and distance of m_2 and m_3 have not changed, $d(m_2, m_2)$ remains unchanged.

The main reason for this happens is that only the largest focal elements in all focal elements are considered in the Cheng et al.'s distance, and the change in focal elements with small size is not sensitive. The distance we propose fully considers the size of each focal element and improves the sensitivity of the distance to focal element size.

Table 1: The distance $d(m_1, m_2)$ obtained by two different methods, in Example 2

Distance\Group	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7
Cheng et al.'s	0.7017	0.7017	0.7017	0.7017	0.5715	0.4943	0.4418
Proposed	0.6845	0.6398	0.6027	0.5715	0.4943	0.4418	0.4031

Table 2: The distance $d(m_1, m_3)$ obtained by two different methods, in Example 2

Distance\Group	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7
Cheng et al.'s	0.8144	0.8144	0.8144	0.8144	0.7044	0.6296	0.5744
Proposed	0.8011	0.7646	0.7327	0.7044	0.6296	0.5744	0.5316

Table 3: The distance $d(m_2, m_3)$ obtained by two different methods, in Example 2

Distance\Group	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7
Cheng et al.'s	0.7017	0.7017	0.7017	0.7017	0.7017	0.7071	0.7071
Proposed	0.5774	0.5774	0.5774	0.5774	0.5774	0.5774	0.5774

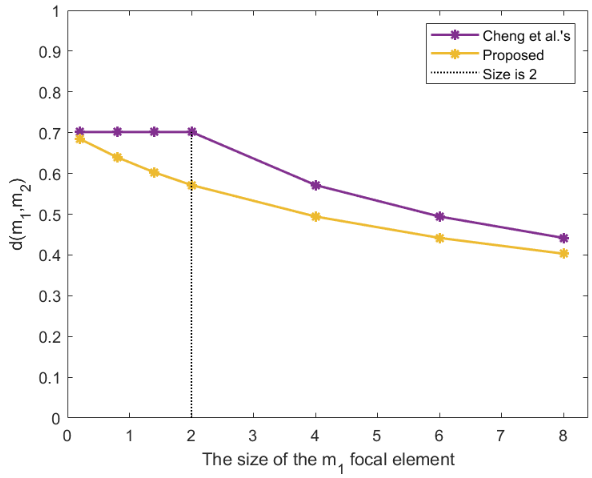

Figure 1: The distance $d(m_1, m_2)$ obtained by two different methods

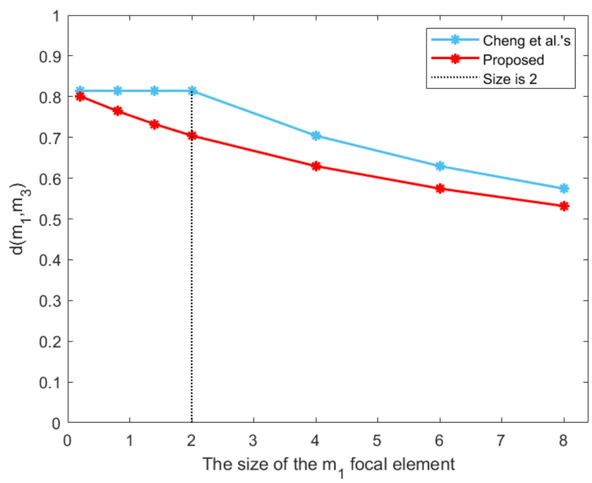

Figure 2: The distance $d(m_1, m_3)$ obtained by two different methods

4. New distance properties

4.1. Distance properties

For a ordinal frame of discernment Θ_O, the properties of proposed new distance are described as follows:

1. Non-negativity: $d_o(m_1, m_2) \geq 0$;
2. Symmetry: $d_o(m_1, m_2) = d_o(m_2, m_1)$;
3. Triangle inequality: $d_o(m_1, m_2) \leq d_o(m_1, m_3) + d_o(m_3, m_2)$.

Non-negativity. $d_o(m_1, m_2) \geq 0$

Proof. Let m_1,m_2 are two valid BPAs, according to the Eq.(13), the $n \times n$ similarity matrix D of the focal elements in m_1 and m_2 is calculated. Where, $D_{ii} = 1 \ \forall i$ and $0 < D_{ij} < 1 \ \forall i, j; i \neq j$. D is defined as

$$D = \begin{bmatrix} 1 & D_{12} & & \cdots & & D_{1n} \\ D_{21} & 1 & & \cdots & & D_{2n} \\ \vdots & \vdots & \ddots & & & \vdots \\ & & & D_{ij} & & \\ \vdots & \vdots & & & \ddots & \vdots \\ D_{n1} & D_{n2} & & \cdots & & 1 \end{bmatrix}$$

D_{ij} measures the distance between the ith and jth focal elements. Assume that there is any three focal elements ith, kth, jth. If the distance between the ith and kth elements D_{ik}, and the distance between the kth and jth elements D_{kj}, has been determined, then the distance between the ith and jth elements D_{ij} is fixed within a certain range.

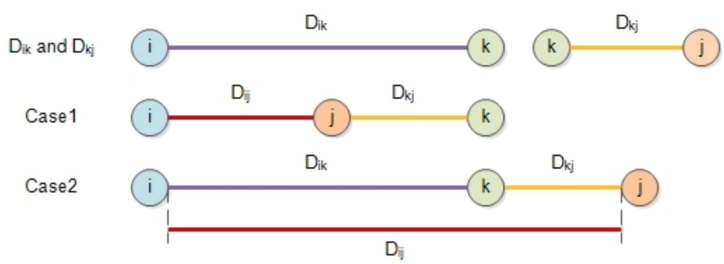

Figure 3: D_{ij} in different Cases

As shown in Fig.3, when j is close to i, D_{ij} is the smallest, is Case1, when j is far away from i, D_{ij} is the largest, is Case2. This geometric constraint is the same as equation (16) in [32].

$$\frac{1-D_{ij}}{D_{ij}} \in \left[\frac{|D_{ik}-D_{kj}|}{D_{ik}D_{kj}}, \frac{D_{ik}+D_{kj}-2D_{ik}D_{kj}}{D_{ik}D_{kj}}\right] \forall k \tag{16}$$

Therefore, we can conclude that D is positive definite[32], and by Cholesky decomposition

$$D = C^T C \tag{17}$$

where C is a $n \times n$ lower triangular matrix. Then Eq.(13) can be transformed as follows:

$$\begin{aligned} d_o(m_1, m_2) &= \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D (\overrightarrow{m_1} - \overrightarrow{m_2})} \\ &= \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T C^T C (\overrightarrow{m_1} - \overrightarrow{m_2})} \\ &= \sqrt{\frac{1}{2}(C(\overrightarrow{m_1} - \overrightarrow{m_2}))^T (C(\overrightarrow{m_1} - \overrightarrow{m_2}))} \\ &= \sqrt{\frac{1}{2}\,||(C(\overrightarrow{m_1} - \overrightarrow{m_2}))||_2^2} \\ &= \frac{\sqrt{2}}{2}\,||(C(\overrightarrow{m_1} - \overrightarrow{m_2}))||_2 \end{aligned} \tag{18}$$

Because of $||(C(\overrightarrow{m_1} - \overrightarrow{m_2}))||_2 \geq 0$, it can be proved $d_o(m_1, m_2) \geq 0$.

Symmetry. $d_o(m_1, m_2) = d_o(m_2, m_1)$

Proof. Because $(\overrightarrow{m_1} - \overrightarrow{m_2})$ can be expressed as $-(\overrightarrow{m_2} - \overrightarrow{m_1})$, so we have

$$\begin{aligned} d_o(m_1, m_2) &= \sqrt{\frac{1}{2}(\overrightarrow{m_1} - \overrightarrow{m_2})^T D (\overrightarrow{m_1} - \overrightarrow{m_2})} \\ &= \sqrt{\frac{1}{2}\,[-(\overrightarrow{m_2} - \overrightarrow{m_1})^T]\, D\, [-(\overrightarrow{m_2} - \overrightarrow{m_1})]} \\ &= \sqrt{\frac{1}{2}(\overrightarrow{m_2} - \overrightarrow{m_1})^T D (\overrightarrow{m_2} - \overrightarrow{m_1})} \\ &= d_o(m_2, m_1) \end{aligned}$$

Triangle inequality. $d_o(m_1, m_2) \leq d_o(m_1, m_3) + d_o(m_3, m_2)$

Proof. Since m_1, m_2, m_3 is defined on the same discernment frame, the $D = C^T C$ equality in $d_o(m_1, m_2)$, $d_o(m_1, m_3)$, and $d_o(m_2, m_3)$ represents the distance between n elements. m_1, m_2, m_3 and C are represented as:

$$\overrightarrow{m_1} = (a_1, a_2, \cdots, a_n)^T; \overrightarrow{m_2} = (e_1, e_2, \cdots, e_n)^T; \overrightarrow{m_3} = (b_1, b_2, \cdots, b_n)^T;$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

$$\begin{aligned} C(\overrightarrow{m_1} - \overrightarrow{m_3}) &= (\sum_{i=1}^{n} c_{1i}(a_i - b_i), \sum_{i=1}^{n} c_{2i}(a_i - b_i), \cdots, \sum_{i=1}^{n} c_{ni}(a_i - b_i))^T \\ &= (\sum_{i=1}^{n} c_{1i}(a_i - e_i + e_i - b_i), \sum_{i=1}^{n} c_{2i}(a_i - e_i + e_i - b_i), \cdots, \sum_{i=1}^{n} c_{ni}(a_i - -e_i + e_i b_i))^T \\ &= (\sum_{i=1}^{n} c_{1i}(a_i - e_i) + \sum_{i=1}^{n} c_{1i}(e_i - b_i), \sum_{i=1}^{n} c_{2i}(a_i - e_i) + \sum_{i=1}^{n} c_{2i}(e_i - b_i), \cdots, \\ &\quad \sum_{i=1}^{n} c_{ni}(a_i - e_i) + \sum_{i=1}^{n} c_{ni}(e_i - b_i))^T \\ &= (\sum_{i=1}^{n} c_{1i}(a_i - e_i), \sum_{i=1}^{n} c_{2i}(a_i - e_i), \cdots, \sum_{i=1}^{n} c_{ni}(a_i - e_i))^T \\ &\quad + (\sum_{i=1}^{n} c_{1i}(e_i - b_i), \sum_{i=1}^{n} c_{2i}(e_i - b_i), \cdots, \sum_{i=1}^{n} c_{ni}(e_i - b_i))^T \\ &= C(\overrightarrow{m_1} - \overrightarrow{m_2}) + C(\overrightarrow{m_2} - \overrightarrow{m_3}) \end{aligned}$$

According to Cauchy-Schwarz inequality, we have

$$||C(\overrightarrow{m_1} - \overrightarrow{m_3})||_2 = ||C(\overrightarrow{m_1} - \overrightarrow{m_2}) + C(\overrightarrow{m_2} - \overrightarrow{m_3})||_2 \leq ||C(\overrightarrow{m_1} - \overrightarrow{m_2})|| + ||C(\overrightarrow{m_2} - \overrightarrow{m_3})||_2$$

According to Eq.(18) can be drawn:

$$\begin{aligned} \frac{\sqrt{2}}{2} ||C(\overrightarrow{m_1} - \overrightarrow{m_3})||_2 &\leq \frac{\sqrt{2}}{2} ||C(\overrightarrow{m_1} - \overrightarrow{m_2})||_2 + \frac{\sqrt{2}}{2} ||C(\overrightarrow{m_2} - \overrightarrow{m_3})||_2 \\ d_o(m_1, m_3) &\leq d_o(m_1, m_2) + d_o(m_2, m_3) \end{aligned}$$

It has been proved that the proposed distance d_o satisfies the triangle inequality.

4.2. Special case analysis

There is a special case where the belief function is completely focused on a single focal element, such as $m_1(A) = 1$ $m_2(B) = 1$.

$$\begin{cases} d_o(m_1, m_2) = 0 & A = B \\ d_o(m_1, m_2) = \sqrt{\frac{\lambda O(A,B)}{1+\lambda O(A,B)}} & Otherwise \end{cases}$$

Only when the structure of the two BPAs is equal, the distance $d_o(m_1, m_2)$ is equal to 0.

When the two BPA are infinitely far apart, $d_o(m_1, m_2)$ is approaching 1.

4.3. Influence of tuning parameters

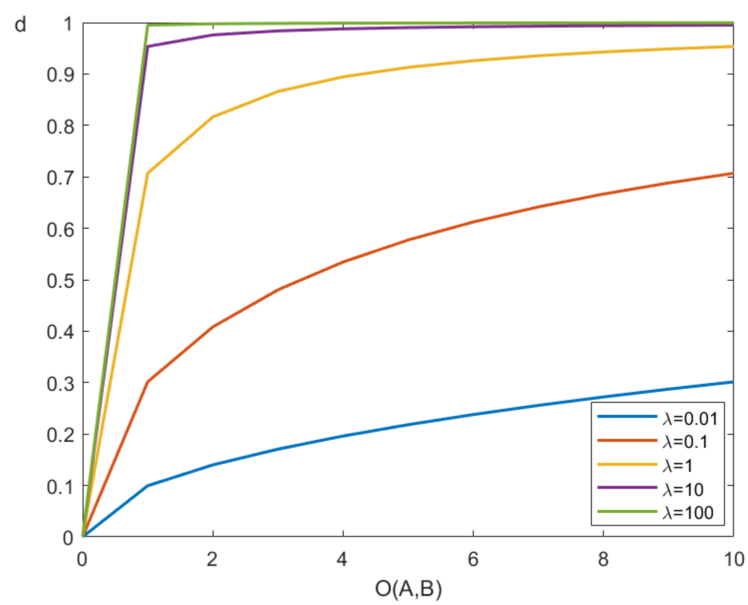

Figure 4: Different tuning parameters affect the same BPAS

The tuning parameters λ is defined by the user to adjust the distance to obtain an output within a specific range, as shown in Fig.(4), which describes the change in the distance between two BPAs including only a single focal element. It is clear that if the distance between elements is small, then $\lambda = 100$ is more appropriate, and if the distance between elements is larger, then $\lambda = 0.01$ is more appropriate. In general, the greater the λ, the less the similarity between focal elements affects distance.

5. Numerical examples

5.1. A case study: project budget

This section discusses the effectiveness of the proposed distance in focal elements through the project budget issues proposed by Cheng et al.[33].

Suppose a construction project is about to start, and six experts are invited to evaluate the budget on E_1, E_2, E_3, E_4, E_5 and E_6. If the budget given by the experts is not consistent with the budget given by most experts, it can be considered unreliable. Similarly, consider two different cases of the interval-value and real number.

5.1.1. Case 1: the budget given by interval-value sets

In this case, interval-value set is used to represent the budget given by experts. The budget given by the six experts is as shown in Table 4. It can be seen that the average budget of six experts is equal to 530.

The distance between the six experts was measured using different methods, and the results were as shown in Table 5(Tuning parameters for each method are 3).

Table 4: The project budget given by six different experts

Cost(1000\$)\Expert	E_1	E_2	E_3	E_4	E_5	E_6
$[500, 520]$	0.0	0.0	0.0	0.2	0.5	0.5
$[510, 530]$	0.0	0.5	0.0	0.3	0.0	0.0
$[500, 530]$	0.5	0.0	0.4	0.0	0.0	0.0
$[530, 550]$	0.0	0.5	0.6	0.3	0.0	0.0
$[540, 560]$	0.0	0.0	0.0	0.2	0.5	0.0
$[530, 560]$	0.5	0.0	0.0	0.0	0.0	0.0
$[545, 555]$	0.0	0.0	0.0	0.0	0.0	0.5

Table 5: The distance between the six experts was measured using different methods

Distance\Method	Jousselme et al.	Sunberg et al.	Cheng et al.	Proposed
$d(E_1, E_2)$	0.40825	0.69658	0.48795	0.43301
$d(E_1, E_3)$	0.33166	0.54792	0.39641	0.34641
$d(E_1, E_4)$	0.29439	0.60718	0.38555	0.33848
$d(E_1, E_5)$	0.40825	0.69610	0.49239	0.43539
$d(E_1, E_6)$	0.50000	0.69798	0.53302	0.49593
$d(E_2, E_3)$	0.27689	0.45074	0.33094	0.28723
$d(E_2, E_4)$	0.23094	0.27844	0.24656	0.22174
$d(E_2, E_5)$	0.57735	0.69609	**0.61640**	0.55435
$d(E_2, E_6)$	**0.60553**	0.69809	**0.61640**	**0.57077**
$d(E_3, E_4)$	0.27689	0.45062	0.33199	0.28780
$d(E_3, E_5)$	0.52599	0.70258	0.58244	0.51740
$d(E_3, E_6)$	0.56273	**0.70479**	0.58244	0.53501
$d(E_4, E_5)$	0.34641	0.41765	0.36984	0.33261
$d(E_4, E_6)$	0.43589	0.51948	0.44360	0.41438
$d(E_5, E_6)$	0.35355	0.48412	0.38730	0.35355

It can be seen from Fig.5, $d(E_1, E_2)$ and $d(E_1, E_5)$ are equal according to the distance of Julesselme et al., indicating that the distance cannot reflect the distribution difference between ordered sets. Based on the distance of Sunberg et al., the distance between E_3 and E_6 is considered to be the largest, which is different from the results of other methods because Sunberg et al.'s method does not take into account the intersection of the distribution intervals of the focal elements of the ordered set. $d(E_2, E_5)$ and $d(E_2, E_6)$ are equal based on the distance of Cheng et al., because Cheng et al.'s method does not take into account the size of the interval of the focal element of the ordered set. The proposed method, not only considers the intersection of the distribution interval of the focal element, but also takes into account the size of the focal element interval itself, which can fully reflect the distribution difference between the ordered sets.

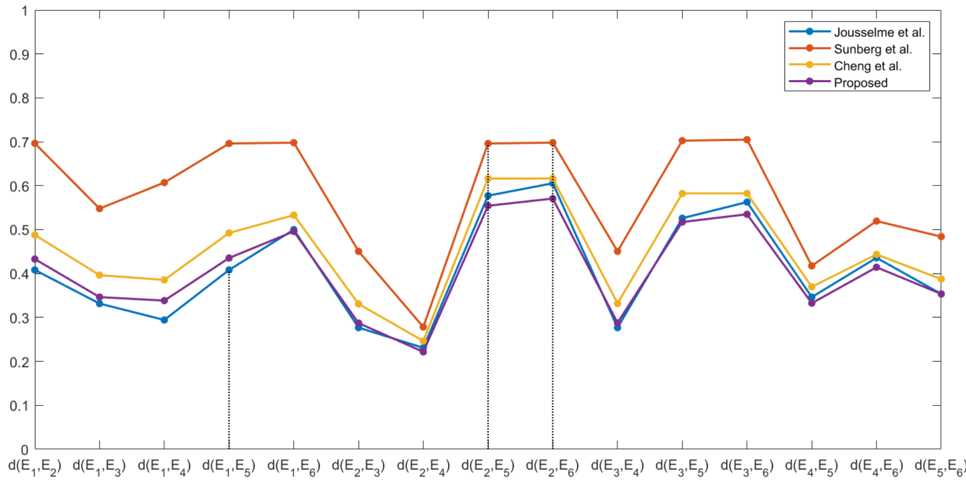

Figure 5: The distance between the six experts was measured using different methods

5.1.2. Case 2: the budget given by real number sets

Based on the results in Table5 and the symmetry of the proposed distance, the total distance from each expert to all other experts can be calculated.

$$d_t(E_1) = d(E_1, E_2) + d(E_1, E_3) + d(E_1, E_4) + d(E_1, E_5) + d(E_1, E_6) = 2.04923$$

$$d_t(E_2) = d(E_1, E_2) + d(E_2, E_3) + d(E_2, E_4) + d(E_2, E_5) + d(E_2, E_6) = 2.06710$$

$$d_t(E_3) = d(E_1, E_3) + d(E_2, E_3) + d(E_3, E_4) + d(E_3, E_5) + d(E_3, E_6) = 1.97385$$

$$d_t(E_4) = d(E_1, E_4) + d(E_2, E_4) + d(E_3, E_4) + d(E_4, E_5) + d(E_4, E_6) = 1.59502$$

$$d_t(E_5) = d(E_1, E_5) + d(E_2, E_5) + d(E_3, E_5) + d(E_4, E_5) + d(E_5, E_6) = 2.19330$$

$$d_t(E_6) = d(E_1, E_6) + d(E_2, E_6) + d(E_3, E_6) + d(E_4, E_6) + d(E_5, E_6) = 2.36964$$

According to the above results, due to the maximum $d_t(E_6)$, expert E_6 was eliminated. The other five experts further made a more precise budget, with budget values given in real number and allows more than one value. Experts' precise budgets are shown as follows:

$$E_1 = \{(530), 1\}, E_2 = \{(520, 540), 1\}, E_3 = \{(540, 550), 1\},$$

$$E_4 = \{(510, 560), 1\}, E_5 = \{(510, 550), 1\}$$

The results obtained through different methods, as shown in Table 6, (Tuning parameters for each method are 3) Jousselme et al. argue that in this case, the belief function is completely conflicting, ignoring the physical distance between elements on an ordered set. In this case, our method is the same as the method of Cheng et al., which both consider the biggest difference between E_1 and E_4. However, the distance of Sunberg et al. is considered to be the largest distance of E_1E_3, E_2E_4, E_3E_4, E_3E_5, because this method only cares about the maximum distance between elements, so the sensitivity is reduced.

Table 6: The distance between the five experts was measured using different methods

Distance\Method	Jousselme et al.	Sunberg et al.	Cheng et al.	Proposed
$d(E_1, E_2)$	1	0.70711	0.81650	0.81650
$d(E_1, E_3)$	1	0.81650	0.86603	0.86603
$d(E_1, E_4)$	1	0.86603	0.91287	0.91287
$d(E_1, E_5)$	1	0.81650	0.89443	0.89443
$d(E_2, E_3)$	0.81650	0.81650	0.86603	0.86603
$d(E_2, E_4)$	1	0.86603	0.86603	0.86603
$d(E_2, E_5)$	1	0.70711	0.81650	0.81650
$d(E_3, E_4)$	1	0.86603	0.89443	0.89443
$d(E_3, E_5)$	0.81650	0.86603	0.86603	0.86603
$d(E_4, E_5)$	0.81650	0.70711	0.70711	0.70711

5.2. Fixed masses of varied BPAs

In this section, we show examples of sensors with different but fixed values through a typical application of an ordered set[32]. Sensor 1 remains stationary, Sensor 2 moves to the right along a solid line, the distance between the two sensors increases, and the two sensors are equivalent to two BPAs:

Sensor 1 is represented as:

$$
\begin{aligned}
& m_1(2) = 0.1 \\
& m_1(2, 2.3) = 0.2 \\
& m_1(2, 2.3, 2.5) = 0.4 \\
& m_1(2, 2.3, 2.5, 2.7) = 0.2 \\
& m_1(2, 2.3, 2.5, 2.7, 3) = 0.1
\end{aligned}
$$

Sensor 2 is represented as:

$$
\begin{aligned}
& m_2(i) = \frac{1}{3} \\
& m_2(i, 0.5 + i) = \frac{1}{3} \\
& m_2(i, 0.5 + i, 1 + i) = \frac{1}{3}
\end{aligned}
$$

Where i is an integer from 2 to 12, indicating the distance between the sensor 1 and the sensor 2.

The distance between m_1 and m_2 calculated using different methods is shown in Fig.(6). This example shows that the distance between Jousselme et al. does not effectively reflect the change in the distance between the two sensors (BPA), the distance between Sunberg et al., the distance between Cheng et al., and the distance we propose can effectively reflect the distance change between the two sensors.

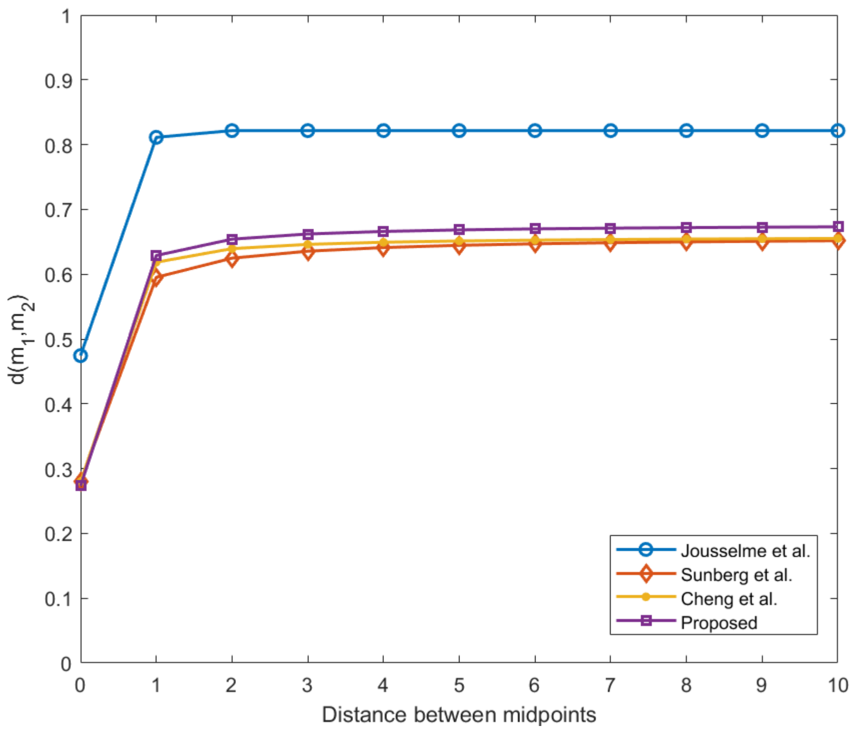

Figure 6: The distance between m_1 and m_2 calculated using different methods

6. Conclusion

This paper proposes a method for measuring the distance between BPAs defined on an ordered set. Measure conflicts between BPAs by measuring the physical distance between focal elements and their own size. Several numerical examples of the ordered measurement space indicate that the proposed method is superior to the distance of the Jousselme et al., and the appearance of saturation can be avoided. In addition, the new distance is more robust and effective, overcoming the shortcomings of the existing method of measuring the distance between ordered sets, and effectively reducing the appearance of violation of intuition.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. P. Dempster, Upper and lower probabilities induced by a multivalued mapping, in: Classic works of the Dempster-Shafer theory of belief functions, Springer, 2008, pp. 57–72.

[2] G. Shafer, A mathematical theory of evidence, Princeton university press, 1976.

[3] R. R. Yager, N. Alajlan, Evaluating belief structure satisfaction to uncertain target values, IEEE transactions on cybernetics 46 (4) (2015) 869–877.

[4] W. Jiang, J. Zhan, A modified combination rule in generalized evidence theory, Applied Intelligence 46 (3) (2017) 630–640.

[5] J. Ma, W. Liu, P. Miller, H. Zhou, An evidential fusion approach for gender profiling, Information Sciences 333 (2016) 10–20.

[6] R. R. Yager, N. Alajlan, Decision making with ordinal payoffs under dempster–shafer type uncertainty, International Journal of Intelligent Systems 28 (11) (2013) 1039–1053.

[7] C. Lian, S. Ruan, T. Denoeux, An evidential classifier based on feature selection and two-step classification strategy, Pattern Recognition 48 (7) (2015) 2318–2327.

[8] C. Huang, X. Mi, B. Kang, Basic probability assignment to probability distribution function based on the shapley value approach, International Journal of Intelligent Systems (2021).

[9] Z. Liu, Q. Pan, J. Dezert, J.-W. Han, Y. He, Classifier fusion with contextual reliability evaluation, IEEE transactions on cybernetics 48 (5) (2017) 1605–1618.

[10] M. S. Islam, R. Sadiq, M. J. Rodriguez, H. Najjaran, M. Hoorfar, Integrated decision support system for prognostic and diagnostic analyses of water distribution system failures, Water resources management 30 (8) (2016) 2831–2850.

[11] U. Mönks, H. Dörksen, V. Lohweg, M. Hübner, Information fusion of conflicting input data, Sensors 16 (11) (2016) 1798.

[12] W. Jiang, A correlation coefficient for belief functions, International Journal of Approximate Reasoning 103 (2018) 94–106.

[13] D. Xinyang, C. Yebi, W. JIANG, An ecr-pcr rule for fusion of evidences defined on a non-exclusive framework of discernment, Chinese Journal of Aeronautics (2021).

[14] L. Zhou, H. Cui, C. Huang, B. Kang, J. Zhang, Counter deception in belief functions using shapley value methodology, International Journal of Fuzzy Systems (2021) 1–15.

[15] X. Mi, B. Kang, On the belief universal gravitation (bug), Computers & Industrial Engineering 148 (2020) 106685.

[16] T. Denoeux, S. Sriboonchitta, O. Kanjanatarakul, Evidential clustering of large dissimilarity data, Knowledge-Based Systems 106 (2016) 179–195.

[17] L. G. Polpitiya, K. Premaratne, M. N. Murthi, D. Sarkar, A framework for efficient computation of belief theoretic operations, in: 2016 19th International Conference on Information Fusion (FUSION), IEEE, 2016, pp. 1570–1577.

[18] J. Yang, H.-Z. Huang, L.-P. He, S.-P. Zhu, D. Wen, Risk evaluation in failure mode and effects analysis of aircraft turbine rotor blades using dempster–shafer evidence theory under uncertainty, Engineering Failure Analysis 18 (8) (2011) 2084–2092.

[19] Z.-G. Liu, Q. Pan, J. Dezert, A. Martin, Combination of classifiers with optimal weight based on evidential reasoning, IEEE Transactions on Fuzzy Systems 26 (3) (2017) 1217–1230.

[20] J. N. Heendeni, K. Premaratne, M. N. Murthi, J. Uscinski, M. Scheutz, A generalization of bayesian inference in the dempster-shafer belief theoretic framework, in: 2016 19th International Conference on Information Fusion (FUSION), IEEE, 2016, pp. 798–804.

[21] X. Deng, Y. Cui, An improved belief structure satisfaction to uncertain target values by considering the overlapping degree between events, Information Sciences 580 (2021) 398–407.

[22] L. A. Zadeh, A simple view of the dempster-shafer theory of evidence and its implication for the rule of combination, AI magazine 7 (2) (1986) 85–85.

[23] A. Roquel, S. Le Hégarat-Mascle, I. Bloch, B. Vincke, Decomposition of conflict as a distribution on hypotheses in the framework on belief functions, International journal of approximate reasoning 55 (5) (2014) 1129–1146.

[24] K.-S. Chin, C. Fu, Weighted cautious conjunctive rule for belief functions combination, Information Sciences 325 (2015) 70–86.

[25] J.-B. Yang, D.-L. Xu, Evidential reasoning rule for evidence combination, Artificial Intelligence 205 (2013) 1–29.

[26] X. Deng, D. Han, J. Dezert, Y. Deng, Y. Shyr, Evidence combination from an evolutionary game theory perspective, IEEE transactions on cybernetics 46 (9) (2015) 2070–2082.

[27] A.-L. Jousselme, P. Maupin, Distances in evidence theory: Comprehensive survey and generalizations, International Journal of Approximate Reasoning 53 (2) (2012) 118–145.

[28] M. Loudahi, J. Klein, J.-M. Vannobel, O. Colot, Evidential matrix metrics as distances between meta-data dependent bodies of evidence, IEEE transactions on cybernetics 46 (1) (2015) 109–122.

[29] F. Xiao, Ced: A distance for complex mass functions, IEEE transactions on neural networks and learning systems 32 (4) (2020) 1525–1535.

[30] H. Zhang, Y. Deng, Entropy measure for orderable sets, Information Sciences 561 (2021) 141–151.

[31] Y. Li, Y. Deng, Generalized ordered propositions fusion based on belief entropy, International Journal of Computers Communications & Control 13 (5) (2018) 792–807.

[32] Z. Sunberg, J. Rogers, A belief function distance metric for orderable sets, Information Fusion 14 (4) (2013) 361–373.

[33] C. Cheng, F. Xiao, A distance for belief functions of orderable set, Pattern Recognition Letters 145 (2021) 165–170.

[34] A.-L. Jousselme, D. Grenier, É. Bossé, A new distance between two bodies of evidence, Information fusion 2 (2) (2001) 91–101.