ON THE METHOD OF DYNAMICAL BALLS

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ABSTRACT. In this paper we introduce and develop the notion of dynamical systems induced by a fixed $a \in \mathbb{N}$ and their associated induced dynamical balls. We develop tools to study problems requiring to determine the convergence of certain sequences generated by iterating on a fixed integer.

1. Introduction

The classical problem of deciding on the convergence of a given sequence is generally in principle not a hard problem. However, the difficulty may arise from the how the terms in the sequence are generated. A typical example of a sequence whose convergence may be difficult to determine is the Collatz sequence

$$f(n), f^2(n), f^3(n), \ldots, f^k(n), \ldots$$

where $f^s = f^{s-1} \circ f = f \circ f^{s-1}$ and

$$f(n) := \begin{cases} \frac{n}{2} & \textbf{if} \quad n \equiv 0 \pmod 2 \\ 3n+1 & \textbf{if} \quad n \equiv 1 \pmod 2 \end{cases}.$$

Collatz conjecture [2] is the problem requiring to decide on the convergence of the system for all $n \in \mathbb{N}$. Another problem of possibly similar difficulty is the problem to determine the convergence of the juggler sequence introduced by Pickover [3]

$$g(n), g^2(n), g^3(n), \ldots, g^k(n), \ldots$$

with $g^s = g \circ g^{s-1} = g^{s-1} \circ g = g \circ g \cdots \circ g$ (s rimes) for all $n \in \mathbb{N}$, where

$$g(n) := \begin{cases} \lfloor n^{\frac{1}{2}} \rfloor & \textbf{if} \quad n \equiv 0 \pmod 2 \\ \lfloor n^{\frac{3}{2}} \rfloor & \textbf{if} \quad n \equiv 1 \pmod 2 \end{cases}.$$

These problems are widely believed to be difficult and forbidden, given that there are currently no viable tool for making ample progress [2], [1].
In this paper we generalize these problems by introducing the notion of dynamical systems and their corresponding dynamical balls. We develop some tools to study problems of the form above.

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2. **Dynamical systems induced by sequences**

Definition 2.1. Let $f:\mathbb{N}\longrightarrow \mathbb{N}$. Then by the first k **dynamical** system induced by f on $a\in \mathbb{N}$, we mean the sequences generated by the system of iterations

$$f(a),f^2(a),f^3(a),\ldots,f^k(a)$$

where $f^s(a)=f\circ f^{s-1}(a)$ with $f^0(a)=a$ and equipped with the self generative energy

$$\mathcal{E}_a(f,f^2,\ldots,f^k):=\prod_{i=1}^{k}f^i(a)$$

with corresponding sequence of balls

$$\mathcal{B}_{f(a)}(a),\mathcal{B}_{f^2(a)}(a),\ldots,\mathcal{B}_{f^k(a)}(a)$$

where $f^s(a)$ is the radius of the s^{th} ball in the sequence and a the center of each ball. We call each ball in the sequence generated in this manner a **dynamical** ball. In other words, we say f induces a k dynamical system on $a\in\mathbb{N}$. We call $f^s(a)$ for $1\leq s\leq k$ the s^{th} dynamical system and $\mathcal{B}_{f^s(a)}(a)$ the s^{th} dynamical ball. We say the s^{th} dynamical system has an upward **measure** relative to the $(s-1)^{th}$ dynamical system if $f^s(a)>f^{s-1}(a)$. On the other hand, we say it has a downward **measure** relative to the $(s-1)^{th}$ dynamical system if $f^{s-1}(a)>f^s(a)$. Similarly, we say the s^{th} dynamical ball $\mathcal{B}_{f^s(a)}(a)$ is **inflated** relative to the ball $\mathcal{B}_{f^{s-1}(a)}(a)$ if $f^s(a)>f^{s-1}(a)$. We say it is **deflated** relative to the dynamical ball $\mathcal{B}_{f^{s-1}(a)}(a)$ if $f^{s-1}(a)>f^s(a)$. In the situation where $f^s(a)=f^{s-1}(a)$ then we say the s^{th} dynamical system is **stable** relative to the $(s-1)^{th}$ dynamical system, and the s^{th} dynamical ball $\mathcal{B}_{f^s(a)}(a)$ is **stable** relative to the $(s-1)^{th}$ dynamical ball $\mathcal{B}_{f^{s-1}(a)}(a)$.

Proposition 2.1. *Let f and g induce k dynamical system on $a\in\mathbb{N}$. Then*

$$\mathcal{E}_a(f,f^2,\ldots,f^k)=\mathcal{E}_a(g,g^2,\ldots,g^k)$$

if and only if there exists some permutation $\sigma:[1,k]\longrightarrow[1,k]$ such that $g^i(a)=f^{\sigma(j)}(a)$ for any $1\leq i\leq j\leq k$.

Proposition 2.2. *Let f induce a dynamical system on $a\in\mathbb{N}$. If the s^{th} dynamical system is stable relative to the $(s-1)^{th}$ dynamical system, then*

$$f^s(a)=f^{s+1}(a)=\ldots=f^{s+l}(a)=\ldots$$

for all $l\geq 1$.

Proof. Suppose the s^{th} dynamical system is stable relative to the $(s-1)^{th}$ dynamical system, then $f^s(a)=f^{s-1}(a)$ so that we have the chain of equality

$$f^s(a)=f^{s+1}(a)=f^{s+2}(a)=\ldots=f^{s+3}(a)=\ldots$$

by iteration. ∎

3. **Analysis on dynamical balls**

In this section we develop some topology of dynamical balls and study their interaction with each other. We launch the following languages in the sequel.

Definition 3.1. Let $\mathcal{B}_{f^j(a)}(a)$ be the j-dynamical ball induced by the k-dynamical system $f(a), f^2(a), f^3(a), \ldots, f^k(a)$ with $1 \leq j \leq k$. Then we say $y_n \in \mathcal{B}_{f^j(a)}(a)$ if and only if the inequality holds

$$|y_n - a| < f^j(a).$$

In particular we write $y_n \hat{\in} \mathcal{B}_{f^j(a)}(a)$ if and only if $|y_n - a| = f^j(a)$.

Proposition 3.1. *Let $x_n \in \mathcal{B}_{f^j(a)}(a)$ and $y_n \in \mathcal{B}_{f^i(a)}(a)$ with $1 \leq i < j \leq k$. Then the following holds*

(i) $x_n - y_n \in \mathcal{B}_{f^i(af^s(a))}(a)$ *provided* $f^i(a) + f^i(f^s(a)) \leq f^i(af^s(a))$ *for* $s = j - i$
(ii) $x_n - y_n \in \mathcal{B}_{f^i(a+f^s(a))}(a)$ *provided* $f^i(a) + f^i(f^s(a)) \leq f^i(a + f^s(a))$ *for* $s = j - i$.

Proof. The following containment $x_n \in \mathcal{B}_{f^j(a)}(a)$ and $y_n \in \mathcal{B}_{f^i(a)}(a)$ implies that $|x_n - a| \leq f^j(a)$ and $|y_n - a| \leq f^i(a)$ so that with $1 \leq i < j \leq k$, we can write the inequality

$$|x_n - y_n - a| \leq f^j(a) + f^i(a) = f^i(f^s(a)) + f^i(a)$$

and (i) and (ii) follows under the specified requirements. □

Proposition 3.2. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then the following embedding holds

$$\mathcal{B}_{f^j(a)}(a) \subseteq \mathcal{B}_{f^{j+1}(a)}(a)$$

if and only $f^j(a) \leq f^{j+1}(a)$

This is an easy consequence of the interpretation of the sequence of dynamical balls all centered at a fixed $a \in \mathbb{N}$ and evolves according to the radius whose values are terms in the corresponding induced dynamical system induced on $a \in \mathbb{N}$ by $f : \mathbb{N} \longrightarrow \mathbb{N}$.

3.1. **The limit of dynamical balls.**

In this subsection we introduce and study the notion of the limit of dynamical balls.

Definition 3.2. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then we denote the limit of the dynamical balls as $\lim\limits_{k \longrightarrow \infty} \mathcal{B}_{f^k(a)}(a)$. We write

$$\lim_{k \longrightarrow \infty} \mathcal{B}_{f^k(a)}(a) = \mathcal{B}_b(a)$$

if and only if for any $\delta > 0$ and for any $x_n \hat{\in} \mathcal{B}_b(a)$ there exists some $K_o > 0$ such that for all $k \geq K_o$ there exists some $y_n \hat{\in} \mathcal{B}_{f^k(a)}(a)$ such that
$$|x_n - y_n| < \delta$$
and we say the sequence of k-dynamical balls **converges**. Otherwise we say it **diverges**.

Let
$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$
be a k-dynamical system with corresponding sequence of dynamical balls
$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$
Then the following statements are equivalent:
For any $\epsilon > 0$ there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that
$$|x_n - y_n| < \epsilon$$
for $1 \leq s \leq k$ as $k \longrightarrow \infty$ if and only if the corresponding infinite dynamical system
$$f(a), f^2(a), f^3(a), \ldots, f^k(a), f^{k+1}(a), \ldots$$
is a Cauchy sequence. Let us suppose the corresponding infinite dynamical system is a Cauchy sequence, then it follows that for any $\epsilon > 0$ there exists a $K_o > 0$ such that for all $s \geq K_o$ then
$$|f^s(a) - f^{s-1}(a)| = |a + f^s(a) - (a + f^{s-1}(a))| < \epsilon$$
so that by taking $x_n = a + f^s(a)$ and $y_n = a + f^{s-1}(a)$ we see that $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$. Conversely suppose that for any $\epsilon > 0$ there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that
$$|x_n - y_n| < \epsilon$$
for $1 \leq s \leq k$ as $k \longrightarrow \infty$. By taking $x_n = a + f^s(a) \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n = a + f^{s-1}(a) \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$, we see that
$$|f^s(a) - f^{s-1}(a)| < \epsilon$$
for $1 \leq s \leq k$ as $k \longrightarrow \infty$.

Proposition 3.3. *Let $\mathcal{B}_{f^j(a)}(a)$ be a dynamical ball induced by the k-dynamical system $f(a), f^2(a), f^3(a), \ldots, f^k(a)$ with $1 \leq j \leq k$. Then for any $\epsilon > 0$ there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that*
$$|x_n - y_n| < \epsilon$$
for $1 \leq s \leq k$ as $k \longrightarrow \infty$ if and only if $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists.

Proof. Let $\epsilon > 0$ and suppose there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that
$$|x_n - y_n| < \epsilon$$
for $1 \leq s \leq k$ as $k \longrightarrow \infty$. Then it implies that the infinite dynamical system
$$f(a), f^2(a), \ldots, f^k(a), \ldots,$$

must be a Cauchy sequence, so that there exists some $L \in \mathbb{R}^+$ such that

$$\lim_{j \longrightarrow \infty} f^j(a) = L.$$

It follows that $\lim_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists and

$$\lim_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a) = \mathcal{B}_L(a).$$

Conversely, suppose $\lim_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists and let

$$\lim_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a) = \mathcal{B}_L(a).$$

Then it follows that for any $\epsilon > 0$ and for any $b \hat{\in} \mathcal{B}_L(a)$ there exists some $K_o > 0$ such that for all $s \geq K_o$ then there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ such that $|x_n - b| < \frac{\epsilon}{2}$. It follows similarly that there exists some $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that $|y_n - b| < \frac{\epsilon}{2}$ so that for all $s \geq K_o$

$$|x_n - y_n| \leq |x_n - b| + |y_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

4. **Dynamical waves and amplitude of waves induced by dynamical balls**

In this section we introduce and study the notion of dynamical waves and their corresponding notion of amplitudes induced by the evolution of dynamical balls.

Definition 4.1. Let $\mathcal{B}_{f^j(a)}(a)$ be a dynamical ball induced by the k-dynamical system $f(a), f^2(a), f^3(a), \ldots, f^k(a)$ with $1 \leq j \leq k$. Then we call the sequence of discrepancy

$$(|f^{j+1}(a) - f^j(a)|)_{1 \leq j \leq k}$$

the dynamical **waves** induced by the evolution of the dynamical balls. We call each term of the sequence a **wavelet** of the dynamical system. We call $\sup_{1 \leq j \leq k}(|f^{j+1}(a) - f^j(a)|)$ the **amplitude** of the wave and we denote amplitude by $\mathbb{A}_f(a, k)$.

Definition 4.2. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

By the **frequency** of the dynamical **wave** induced, we mean the formal sum

$$\mathcal{W}_a(f, k) := \sum_{j=1}^{k} \frac{|f^{j+1}(a) - f^j(a)|}{j}.$$

We denote the **frequency** of the wave of the corresponding infinite dynamical system as

$$\mathcal{W}_a(f) = \lim_{k \longrightarrow \infty} \sum_{j=1}^{k} \frac{|f^{j+1}(a) - f^j(a)|}{j} = \sum_{j=1}^{\infty} \frac{|f^{j+1}(a) - f^j(a)|}{j}.$$

It turns out that for any dynamical wave induced by f on $a \in \mathbb{N}$, we can decompose the total dynamical waves into two pieces, namely as small piece and a large piece as follows

$$\begin{aligned}\mathcal{D}_f(a,k) := &\sum_{2\leq s\leq k} |f^s(a)-f^{s-1}(a)|\\ =&\sum_{\substack{2\leq s\leq k\\ |f^s(a)-f^{s-1}(a)|>|f^2(a)-f(a)|}} |f^s(a)-f^{s-1}(a)|\\ +&\sum_{\substack{2\leq s\leq k\\ |f^s(a)-f^{s-1}(a)|<|f^2(a)-f(a)|}} |f^s(a)-f^{s-1}(a)|\end{aligned}$$

and we call the second sum on the right-hand side the **regular** part and the first sum the **random** part. Symbolically, we rewrite the above decomposition into random and regular part as

$$\mathcal{D}_f(a,k) := \mathbb{R}ad_f(a,k) + \mathbb{R}eg_f(a,k).$$

It is easy to see that for any dynamical system, we can write

$$\mathcal{D}_f(a,k) := \sum_{\substack{2\leq s\leq k\\ |f^s(a)-f^{s-1}(a)|>|f^2(a)-f(a)|}} |f^s(a)-f^{s-1}(a)| + O_{f,a}(k).$$

The corresponding total wave of the infinite dynamical system is obtained by taking the limits

$$\begin{aligned}\mathcal{D}_f(a) &= \lim_{k\longrightarrow\infty} \mathcal{D}_f(a,k)\\ &= \sum_{s=2}^{\infty} |f^s(a)-f^{s-1}(a)|.\end{aligned}$$

Proposition 4.1. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\mathcal{W}_a(f) < \infty$ if and only if $\lim_{j\longrightarrow\infty} \mathcal{B}_{f^j(a)}(a)$ exists.

Proof. Suppose $\mathcal{W}_a(f) < \infty$, then it follows that

$$\sum_{j=1}^{\infty} \frac{|f^{j+1}(a)-f^j(a)|}{j} < \infty$$

so that $\lim_{j\longrightarrow\infty} |f^{j+1}(a)-f^j(a)| = 0$ and it implies that $\lim_{j\longrightarrow\infty} f^j(a)$ exists and so is $\lim_{j\longrightarrow\infty} \mathcal{B}_{f^j(a)}(a)$. Conversely suppose $\lim_{j\longrightarrow\infty} \mathcal{B}_{f^j(a)}(a)$ exists then so is $\lim_{j\longrightarrow\infty} f^j(a)$ so that $\lim_{j\longrightarrow\infty} |f^{j+1}(a)-f^j(a)| = 0$ and $\mathcal{W}_a(f) < \infty$. □

Theorem 4.3 (Restriction law)**.** *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a) - f^s(a)| \neq |f^{t+1}(a) - f^t(a)|$ for all $s, t \geq 1$ with $s \neq t$ and $|f^{s+1}(a) - f^s(a)|, |f^{t+1}(a) - f^t(a)| \neq 0$ with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\lim\limits_{k \longrightarrow \infty} \mathbb{R}eg_f(a,k) < \infty$.

Proof. Let us suppose on the contrary that $\lim\limits_{k \longrightarrow \infty} \mathbb{R}eg_f(a,k) = \infty$ so that

$$\lim_{k \longrightarrow \infty} \sum_{\substack{2 \leq s \leq k \\ |f^s(a) - f^{s-1}(a)| < |f^2(a) - f(a)|}} |f^s(a) - f^{s-1}(a)|$$

contains infinitely many terms. It follows from the condition $|f^s(a) - f^{s-1}(a)| < |f^2(a) - f(a)|$ and the pigeon hole principle that there are infinitely many coinciding wavelets. It follows that there must exist some $s \neq t$ such that $|f^{s+1}(a) - f^s(a)| = |f^{t+1}(a) - f^t(a)|$. This contradicts the requirements of the dynamical system induced. □

Remark 4.4. Theorem 4.3 though simple is ultimately useful for determining the convergence of dynamical systems. The bound on the regular part of the total wave of any infinite dynamical system reduces the problem of convergence to just the random part of the decomposition. These ideas are summarized in the following proposition.

Proposition 4.2. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a) - f^s(a)| \neq |f^{t+1}(a) - f^t(a)|$ for all $s, t \geq 1$ with $s \neq t$ and $|f^{s+1}(a) - f^s(a)|, |f^{t+1}(a) - f^t(a)| \neq 0$ with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ *exists if and only if* $\lim\limits_{k \longrightarrow \infty} \mathbb{R}ad_f(a,k) < \infty$.

Proof. Suppose that $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists then it follows that $\lim\limits_{j \longrightarrow \infty} |f^j(a) - f^{j-1}(a)| = 0$. It implies that

$$\lim_{k \longrightarrow \infty} \mathbb{R}ad_f(a,k) < \mathcal{D}_f(a) < \infty.$$

Conversely suppose $\lim\limits_{k \longrightarrow \infty} \mathbb{R}ad_f(a,k) < \infty$, then by appealing to Theorem 4.3 we can write

$$\lim_{k \longrightarrow \infty} \mathcal{D}_f(a,k) = \lim_{k \longrightarrow \infty} \mathbb{R}ad_f(a,k) + \lim_{k \longrightarrow \infty} \mathbb{R}eg_f(a,k) < \infty$$

so that

$$\sum_{s=1}^{\infty} |f^s(a) - f^{s-1}(a)| < \infty.$$

Since $f^j(a) - f^{j-1}(a) \in \mathbb{Z}$, it implies that $\lim\limits_{j \longrightarrow \infty} |f^j(a) - f^{j-1}(a)| = 0$ and that $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ also exists. □

4.1. **Dynamical waves estimate.** In this section we establish some new estimates relating the frequency, amplitude and total waves of any dynamical systems.

Theorem 4.5. *Let*
$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$
be a k-dynamical system such that $|f^{s+1}(a) - f^s(a)| \neq |f^{t+1}(a) - f^t(a)|$ for all $s, t \geq 1$ with $s \neq t$ and $|f^{s+1}(a) - f^s(a)|, |f^{t+1}(a) - f^t(a)| \neq 0$ with corresponding sequence of dynamical balls
$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$
Then

(i)
$$\mathcal{W}_f(a,k) \ll \mathbb{A}_f(a,k) \log k$$

(ii)
$$\int\limits_1^{k-1} \frac{f^t(a)}{t^2} dt \ll_{f,a} \mathcal{W}_f(a,k) - \frac{\mathcal{D}_f(a,k)}{k}$$

(iii)
$$\mathcal{W}_f(a,k) = \frac{\mathbb{R}ad_f(a,k)}{k} + \int\limits_1^{k-1} \frac{\mathbb{R}ad_f(a,t)}{t^2} dt + O(\frac{1}{k})$$

(iv)
$$\int\limits_1^{k-1} \frac{|f^t(a) - f(a)|}{t^2} dt \leq \int\limits_1^{k-1} \frac{\mathbb{R}ad_f(a,t)}{t^2} dt + O(\frac{1}{k})$$

Proof. (i) We can write
$$\mathcal{W}_f(a,k) = \sum_{j=1}^{k-1} \frac{|f^{j+1}(a) - f^j(a)|}{j}$$
$$\leq \mathbb{A}_f(a,k) \sum_{j=1}^{k-1} \frac{1}{j} \ll \mathbb{A}_f(a,k) \log k.$$

(ii) By an application of partial summation, we can write the frequency of the dynamical wave
$$\mathcal{W}_f(a,k) = \frac{1}{k}\mathcal{D}_f(a,k) + \int\limits_1^{k-1} \frac{\mathcal{D}_f(a,t)}{t^2} dt$$
so that by exploiting the lower bound
$$\mathcal{D}_f(a,t) := \sum_{1 \leq j \leq t} |f^{j+1}(a) - f^j(a)| \geq |f^t(a) - f(a)| \gg_{f,a} f^t(a)$$
the asserted estimate follows.

(iii) By applying the decomposition of the total dynamical waves into random and the regular part as $\mathcal{D}_f(a,k)=\mathbb{R}ad_f(a,k)+\mathbb{R}eg_f(a,k)$, we can further write the estimate for the frequency in (ii) as

$$\begin{aligned}\mathcal{W}_f(a,k)&=\frac{1}{k}\mathcal{D}_f(a,k)+\int\limits_{1}^{k-1}\frac{\mathcal{D}_f(a,t)}{t^2}dt\\&=\frac{1}{k}\mathbb{R}ad_f(a,k)+\frac{1}{k}\mathbb{R}eg_f(a,k)+\int\limits_{1}^{k-1}\frac{\mathbb{R}ad_f(a,t)}{t^2}dt+\int\limits_{1}^{k-1}\frac{\mathbb{R}eg_f(a,t)}{t^2}dt\\&=\frac{\mathbb{R}ad_f(a,k)}{k}+\int\limits_{1}^{k-1}\frac{\mathbb{R}ad_f(a,t)}{t^2}dt+O(\frac{1}{k})\end{aligned}$$

since

$$\mathbb{R}eg_f(a,k)\ll_{f,a}1$$

and

$$\int\limits_{1}^{k-1}\frac{\mathbb{R}eg_f(a,t)}{t^2}dt\leq\int\limits_{1}^{\infty}\frac{\mathbb{R}eg_f(a,t)}{t^2}dt\ll_{f,a}\int\limits_{1}^{\infty}\frac{1}{t^2}dt$$

.

(iv) By plugging the estimate in (iii) into the upper bound

$$\int\limits_{1}^{k-1}\frac{|f^t(a)-f(a)|}{t^2}dt\leq\mathcal{W}_f(a,k)-\frac{\mathcal{D}_f(a,k)}{k}$$

the claimed upper bound also follows.

□

The estimates established in Theorem 4.5 can be used in a unifying manner to study the convergence of any dynamical system. The estimate in (iii) seems to stand out among them and confirms Proposition 4.2. We confirm the observation again as an application of the estimate.

Corollary 4.1. Let

$$f(a),f^2(a),f^3(a),\ldots,f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a)-f^s(a)|\neq|f^{t+1}(a)-f^t(a)|$ for all $s,t\geq 1$ with $s\neq t$ and $|f^{s+1}(a)-f^s(a)|,|f^{t+1}(a)-f^t(a)|\neq 0$ with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a),\mathcal{B}_{f^2(a)}(a),\ldots,\mathcal{B}_{f^k(a)}(a).$$

Then $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ exists if and only if $\lim\limits_{k\longrightarrow\infty}\mathbb{R}ad_f(a,k)<\infty$.

Proof. The result follows from the estimate

$$\mathcal{W}_f(a,k)=\frac{\mathbb{R}ad_f(a,k)}{k}+\int\limits_{1}^{k-1}\frac{\mathbb{R}ad_f(a,t)}{t^2}dt+O(\frac{1}{k})$$

and an appeal to Proposition 4.1. □

5. **Translation and dilation of dynamical balls**

In this section we introduce the notion of translation and dilation of dynamical balls. This would allow the movement of dynamical balls for the purposes of our work.

Definition 5.1. Let
$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$
be a k-dynamical system with corresponding sequence of dynamical balls
$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$
We call the map
$$\mathbb{T}_b : \mathcal{B}_{f^j(a)}(a) \longmapsto \mathcal{B}_{f^j(a+b)}(a+b) := \mathcal{B}_{f^j(\mathbb{T}_b(a))}(\mathbb{T}_b(a))$$
the **translation** of the dynamical ball $\mathcal{B}_{f^j(a)}(a)$ by a a scale factor b.

Definition 5.2. Let
$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$
be a k-dynamical system with corresponding sequence of dynamical balls
$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$
We call the map
$$\mathbb{D}_m : \mathcal{B}_{f^j(a)}(a) \longmapsto \mathcal{B}_{f^j(ma)}(ma) := \mathcal{B}_{f^j(\mathbb{D}_m(a))}(\mathbb{D}_m(a))$$
the **dilation** of the dynamical ball $\mathcal{B}_{f^j(a)}(a)$ by a scale factor m.

Proposition 5.1. *Let*
$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$
be a k-dynamical system with corresponding sequence of dynamical balls
$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$
Suppose $\lim\limits_{j\longrightarrow\infty} \mathcal{B}_{f^j(b)}(b)$ *exists. If* $\lim\limits_{j\longrightarrow\infty} \mathcal{B}_{f^j(a)}(a)$ *exists then* $\lim\limits_{j\longrightarrow\infty} \mathcal{B}_{f^j(a+b)}(a+b)$ *exists provided* $f^s(a+b) \leq f^s(a) + f^s(b)$ *whenever* $f^{s-1}(a+b) \geq f^{s-1}(a) + f^{s-1}(b)$ *for all* $s \geq 2$.

Proof. It suffices to show that for any $\epsilon > 0$ there exists some $N_o > 0$ such that for all $s \geq N_o$ then $|f^s(a+b) - f^{s-1}(a+b)| < \epsilon$.
Under the assumption $\lim\limits_{j\longrightarrow\infty} \mathcal{B}_{f^j(b)}(b)$ and $\lim\limits_{j\longrightarrow\infty} \mathcal{B}_{f^j(a)}(a)$ exist, then for any $\epsilon > 0$ there exist some $N_o, M_o > 0$ such that
$$|f^s(a) - f^{s-1}(a)| < \frac{\epsilon}{2}$$
for all $s \geq N_o$ and
$$|f^s(b) - f^{s-1}(b)| < \frac{\epsilon}{2}$$

for all $s \geq M_o$. By choosing $P = \max\{N_o, M_o\}$ and exploiting the condition $f^s(a+b) \leq f^s(a) + f^s(b)$ if $f^{s-1}(a+b) \geq f^{s-1}(a) + f^{s-1}(b)$ for all $s \geq 2$, it follows that

$$|f^s(a+b) - f^{s-1}(a+b)| \leq |f^s(a) - f^{s-1}(a)| + |f^s(b) - f^{s-1}(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $s \geq P = \max\{N_o, M_o\}$. This implies that $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a+b)}(a+b)$ exists since $\epsilon > 0$ can be chosen arbitrarily. □

Proposition 5.2. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

If $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ *exists then* $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(ma)}(ma)$ *exists provided* $f^s(ma) \leq mf^s(a)$ *whenever* $f^{s-1}(ma) \geq mf^{s-1}(a)$ *for all* $s \geq 2$ *and for a fixed* $m \in \mathbb{N}$.

Proof. Under the assumption $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists, then for any $\epsilon > 0$ there exists some $N_o > 0$ such that

$$|f^s(a) - f^{s-1}(a)| < \epsilon$$

for all $s \geq N_o$ so that under the conditions $f^s(ma) \leq mf^s(a)$ whenever $f^{s-1}(ma) \geq mf^{s-1}(a)$ for all $s \geq 2$ and for a fixed $m \in \mathbb{N}$, we can write by choosing $\epsilon = \frac{\delta}{m}$ for any $\delta > 0$

$$|f^s(ma) - f^{s-1}(ma)| \leq m|f^s(a) - f^{s-1}(a)| < m\epsilon = \delta$$

for all $s \geq N_o$. □

Remark 5.3. Proposition 5.1 and 5.2 provides a slick way of extending the convergence of an infinite dynamical system induced by a function f on any $a \in \mathbb{N}$ to some other numbers $z \in \mathbb{N}$ by translation.

[1].

References

1. Lagarias, Jeffrey C, *The ultimate challenge: The 3x+ 1 problem*, American Mathematical Soc., 2010.
2. Guy, Richard, *Unsolved problems in number theory*, Springer Science & Business Media, vol. 1, 2004.
3. Pickover, Clifford A, *Computers and the Imagination: Visual Adventures Beyond the Edge*, St. Martin's Press, Inc., 1991.

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