Introduction to Clifford's Geometric Algebra

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Abstract. A brief application-oriented introduction to W.K. Clifford's geometric algebras, including conformal geometric algebra (CGA).

Keywords. Clifford's geometric algebra, conformal geometric algebra.

1. Introduction

Geometric algebra has become popularly used in applications dealing with geometry. This framework allows to reformulate and redefine problems involving geometry in a more intuitive and general way. We aim to give a brief introduction to geometric algebra as a geometric object formalism. Geometric algebra was defined thanks to the work of W. K. Clifford [1] to unify and generalize Grassmann algebra [6] and W.R. Hamilton's quaternions [7] into a universal algebraic framework by adding the inner product to H. G. Grassmann's outer product. Geometric algebra is, in a similar manner as Gibbs vectors, a vector formalism. However, the main product called the geometric product is defined for any dimension and is associative in contrast to the cross product which is only defined in three dimensions and lacks associativity[1]. Since geometric algebra is based on Grassmann algebra, we will start this introduction to geometric algebra with the main product of Grassmann algebra, namely the outer product.

1.1. Outer product

The outer product, also called exterior product or wedge product and denoted as $\wedge$ was defined by H. G. Grassmann [6]. Briefly, the outer products of vectors create new algebraic objects that have geometric meaning. The outer product of two vectors $\mathbf{a}, \mathbf{b}$, creates the oriented area spanned by the two vectors. The

[1]The reason is that *only* in three dimensions does there exist a unique orthogonal direction to which a vector with the length of the area of the parallelogram, spanned by the two vector factors, can be assigned.

outer product of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, represents the oriented volume spanned by the three vectors, as illustrated in Figure 1.

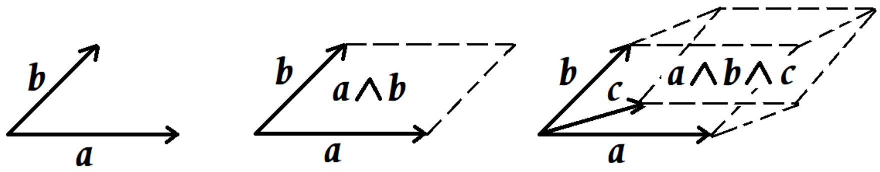


FIGURE 1. Geometric meaning of the outer product between vectors. In the middle, the outer products of the two vectors computes the oriented parallelogram formed by **a** and **b**. On the right, the outer product represents the oriented volume spanned by three vectors.

Let us consider the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^d$ defined in a d-dimensional space and a scalar $\alpha \in \mathbb{R}$. The outer product fulfills the properties:

$$
\begin{array}{lrl}
\text{scalar product:} & \alpha \wedge \mathbf{a} &= \alpha\mathbf{a}, \\
\text{associativity:} & (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} &= \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}), \\
\text{anti-symmetry:} & \mathbf{a} \wedge \mathbf{b} &= -\mathbf{b} \wedge \mathbf{a}\ , \\
 & \mathbf{a} \wedge \mathbf{a} &= 0, \\
\text{distributivity:} & \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} \wedge \mathbf{b}) + (\mathbf{a} \wedge \mathbf{c}), \\
\text{scaling:} & \mathbf{a} \wedge (\alpha\mathbf{b}) &= \alpha(\mathbf{a} \wedge \mathbf{b}).
\end{array}
$$

Furthermore, all these properties have a geometric interpretation. For instance, the oriented area formed by **a** with itself is obviously of 0 magnitude.

The outer product of pairs of orthonormal basis vectors $\mathbf{e}_i, \mathbf{e}_j \in \mathbb{R}^d$, $i \neq j$, denoted as $\mathbf{e}_i \wedge \mathbf{e}_j$, or as $\mathbf{e}_{ij}$ for brevity, yields a new subspace of basis bivectors. Note that since the outer product is anti-symmetric, $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$ if $i \neq j$ and $\mathbf{e}_{ij} = 0$ otherwise. Let us develop an example of the outer product of two general vectors in two dimensions defined by their components along the basis vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$.

$$
\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= (\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2) \ \wedge \ (\beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2) \\
&= \cancel{(\alpha_1\beta_1)(\mathbf{e}_1 \wedge \mathbf{e}_1)} + \ (\alpha_1\beta_2)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&\quad + (\alpha_2\beta_1)(\mathbf{e}_2 \wedge \mathbf{e}_1) + \cancel{(\alpha_2\beta_2)(\mathbf{e}_2 \wedge \mathbf{e}_2)} \\
&= (\alpha_1\beta_2 - \alpha_2\beta_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&= (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_{12}.
\end{aligned}
\tag{1.1}
$$

More geometrically, the outer product between two vectors **a** and **b** equals

$$
\mathbf{a} \wedge \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\alpha)\mathbf{e}_{12}, \tag{1.2}
$$

where α denotes the angle between the two vectors, and $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, their lengths (Euclidean norm).

Let us now consider the outer product of more than two vectors. We remark that if the three vectors are in $\mathbb{R}^2$ then the outer product results in 0. However in $\mathbb{R}^3$, it is possible to accommodate higher dimensional subspaces.

The outer product of any three orthonormal basis vectors $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \in \mathbb{R}^d, d \geq 3$, results in a three-dimensional subspace element, part of the trivector basis. An example of a computation of the outer product of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ resulting in a trivector is now developed. Since the outer product is associative, we start by the computation of the outer product[2] $\mathbf{a} \wedge \mathbf{b}$ and then compute the outer product between this result and $\mathbf{c}$ as follows:

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= (\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3) \ \wedge \ (\beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3) \\ &= (\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_{12} + (\alpha_1\beta_3 - \alpha_3\beta_1)\mathbf{e}_{13} + (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_{23}, \end{aligned} \tag{1.3}$$

and the resulting trivector is computed as

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = &\Big((\alpha_1\beta_2 - \alpha_2\beta_1)\mathbf{e}_{12} + (\alpha_1\beta_3 - \alpha_3\beta_1)\mathbf{e}_{13} + (\alpha_2\beta_3 - \alpha_3\beta_2)\mathbf{e}_{23}\Big) \\ &\wedge \Big(\gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2 + \gamma_3 \mathbf{e}_3\Big) \\ = &\Big(\gamma_3(\alpha_1\beta_2 - \alpha_2\beta_1) + \gamma_2(\alpha_1\beta_3 - \alpha_3\beta_1) + \gamma_1(\alpha_2\beta_3 - \alpha_3\beta_2)\Big)\ \mathbf{e}_{123}, \end{aligned} \tag{1.4}$$

where the factor in the third line equals the determinant of the vector components, i.e. the oriented volume spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

To summarize, the subspace elements obtained so far in three dimensions are

$$\Big(\underbrace{1}_{\text{scalars}}, \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\text{vector space}}, \underbrace{\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}}_{\text{bivector space}}, \underbrace{\mathbf{e}_{123}}_{\text{trivector space}} \Big) \tag{1.5}$$

Extending this for any k-dimensional subspace element, or k-blade of a d-dimensional vector space, it is possible to compute the number of elements in a k-vector space basis as:

$$\binom{d}{k} = \frac{d!}{(d-k)!d!}. \tag{1.6}$$

Finally, using the binomial theorem, the total number of basis blades is computed as:

$$\sum_{k=0}^{d} \binom{d}{k} = 2^d. \tag{1.7}$$

The number of basis vector factors that form a blade is called the grade of the blade. The grade of a k-blade is simply k. Note that for a d-dimensional space, the subspace grades range from 0 to d. The highest grade d-dimensional blade of a d-dimensional vector space is called the pseudoscalar, often denoted as I.

It is possible to form a linear combination of subspace elements in the total vector space of all blades whose dimension is 2^d. This new high dimensional entity is called a multivector, denoted here with a capital letter. For

[2]Note that special to three dimensions we have the cross product to be $\mathbf{a} \times \mathbf{b} = -\mathbf{e}_{123}(\mathbf{a} \wedge \mathbf{b})$, $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

example, a multivector A of the GA of a three-dimensional vector space can be written as:

$$A = \alpha + \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 + \alpha_{12} \mathbf{e}_{12} + \alpha_{13} \mathbf{e}_{13} + \alpha_{23} \mathbf{e}_{23} + \alpha_{123} \mathbf{e}_{123}, \quad (1.8)$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{123} \in \mathbb{R}$.

In general, $\langle A \rangle_k$ denotes the grade k part of the multivector A. In the above example, $\langle A \rangle_2 = \alpha_{12}\mathbf{e}_{12} + \alpha_{13}\mathbf{e}_{13} + \alpha_{23}\mathbf{e}_{23}$ corresponds to the grade two (bivector) part of A.

1.2. Inner product

So far, we gave an overview of the outer product and Grassmann algebra. Geometric algebra also has an inner product of two multivectors. This inner product between vectors, denoted as $\mathbf{a} \cdot \mathbf{b}$, is symmetric, distributive $(\mathbf{a}+\mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c}$, and fulfills $(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b})$, $\alpha \in \mathbb{R}$. The inner product of two orthogonal Euclidean basis vectors is $\mathbf{e}_i \cdot \mathbf{e}_j = 0$, when $i \neq j$. Furthermore, this inner product allows to define the signature of the vector space, as the triplet p, q, r such that

$$\mathbf{e}_i \cdot \mathbf{e}_i = \begin{cases} +1 & \text{for } i = 1, \cdots, p \\ -1 & \text{for } i = p+1, \cdots, p+q \\ 0 & \text{for } i = p+q+1, \cdots, p+q+r. \end{cases} \quad (1.9)$$

We use the common notation $\mathbb{R}^{p,q,r}$ for a vector space with signature (p, q, r).

More geometrically, the inner product between two vectors $\mathbf{a}$ and $\mathbf{b}$ yields[3]

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\alpha), \quad (1.10)$$

where α is simply the angle between the two vectors. The inner product of vectors can also be defined through the metric matrix $\mathbf{e}_i \cdot \mathbf{e}_j$, $\forall (i,j) \in [1,d]^2$. The simplest metric matrix is Euclidean, an identity matrix whose rank is the dimension d of the vector space.

The generalization of the definition of the inner product for basis elements of higher grades is now presented. The rule to multiply a vector $\mathbf{a}$ with a grade l vector is defined as:

$$\begin{aligned} \mathbf{a} \cdot B &= \mathbf{a} \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l) \\ &= (\mathbf{a} \cdot \mathbf{b}_1)\mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l - (\mathbf{a} \cdot \mathbf{b}_2)\mathbf{b}_1 \wedge \mathbf{b}_3 \cdots \wedge \mathbf{b}_l + \cdots \\ &\quad +(-1)^{l+1}(\mathbf{a} \cdot \mathbf{b}_l)\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_{l-2} \wedge \mathbf{b}_{l-1}. \end{aligned} \quad (1.11)$$

[3] In the following equation a Euclidean space $\mathbb{R}^{n,0}$ is assumed. It also works for anti-Euclidean spaces $\mathbb{R}^{0,n}$, $\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\| \, \|\mathbf{b}\| \cos(\alpha)$. In case of mixed signature, like a Minkowski plane $\mathbb{R}^{1,1}$, the angle function $\cos(\alpha)$ would have to be replaced by $\cosh(\alpha)$.

The inner product[4] between any grade $k > 1$ element A and grade $l > 1$ element called B is defined recursively as follows

$$\begin{aligned} A \cdot B &= (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_m) \\ &= \Bigg(\mathbf{a}_1 \cdot \Big(\mathbf{a}_2 \cdot \Big(\cdots (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l))\Big)\Big)\Bigg). \end{aligned} \tag{1.12}$$

The grade of the result of $A \cdot B$ is then $|l - k|$.

Thanks to the inner product, it is possible to define the norm $\|A\|$ of a Euclidean multivector A. To actually define it, we also need to introduce the *reverse* product order applied to a k-vector A, denoted as $\widetilde{A}$ and simply defined as

$$\widetilde{A} = (\mathbf{a}_1 \wedge \mathbf{a}_2 \cdots \wedge \mathbf{a}_k)^\sim = (\mathbf{a}_k \wedge \mathbf{a}_{k-1} \cdots \wedge \mathbf{a}_1). \tag{1.13}$$

Note that the reverse product is an involution. Then, the square norm of a k-vector A is simply defined as

$$\|A\|^2 = A \cdot \widetilde{A}. \tag{1.14}$$

1.3. Geometric product

The main product of geometric algebra is called *geometric product*[5]. It was defined by W. K. Clifford in [1] and it combines the outer and inner products to one invertible product. For vectors, it is simply defined as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \tag{1.15}$$

The geometric product is associative, distributive over the addition and linear but generally not commutative. The inverse $\mathbf{a}^{-1}$ of a vector $\mathbf{a}$ with non zero square is

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{aa}}, \tag{1.16}$$

and we can easily check that $\mathbf{a}^{-1}\mathbf{a} = \dfrac{\mathbf{aa}}{\mathbf{aa}} = 1$. The definition of the geometric product depends on the grades of the factors. The geometric product between a bivector A and a k-vector $(k > 1)$ B is:

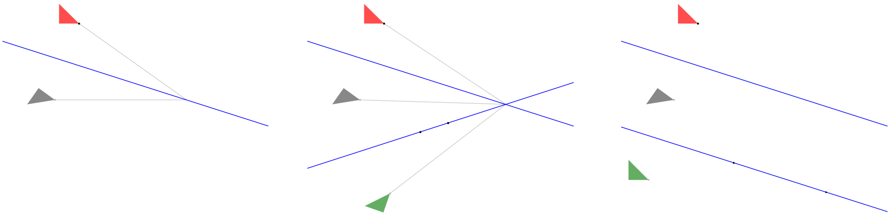
$$AB = A \cdot B + A \wedge B + A \times B, \tag{1.17}$$

where $A \times B = (AB - BA)/2$ denotes the commutator product between A and B, see e.g. [4]. Given a signature (p, q, r) and the geometric product, we use the common notation $\mathbb{G}^{p,q,r} = Cl(p, q, r)$ for the resulting multivector space that includes all elements of geometric algebra.

Using the previous definitions, the geometric product also allows to compute the *dual* of a multivector. Briefly, the dual maps a blade of grade k

[4] A more sophisticated version of the inner product between blades are the left and right contractions, frequently used in software, because they eliminate grade specific exceptions [3,9].

[5] At the end of this section we give a full mathematical definition of Clifford's geometric algebra, while this section mainly focusses on geometric aspects and some frequent application relevant computations in Clifford algebra.



(A) (B) (C)

Figure 2. (A) represents the reflection of the red triangle called $\mathbf{x}$ with respect to the blue line, whose normal vector is $\mathbf{m}$. The computation of the gray triangle is $\mathbf{x}' = -\mathbf{m}\mathbf{x}\mathbf{m}^{-1}$. (B) A reflection is applied to $\mathbf{x}'$ with respect to $\mathbf{n}$. The resulting transformation is a rotation for (B) since the two lines intersect, whereas it is a translation for (C) expressed in conformal GA. The software used is Ganja, see [2].

to a blade of grade $d-k$, if d is the vector space dimension. Algebraically, it is defined as

$$A^* = \frac{A\widetilde{I}}{I \cdot \widetilde{I}}. \tag{1.18}$$

The geometric product allows to define geometric transformations. First, the most basic geometric transformation in geometric algebra is reflection (at a hyperplane). Given a normal vector $\mathbf{m}$ to a line, the reflection of an entity $\mathbf{x}$ (e.g. a vector, point, etc.) with respect to the line is simply the difference between the projection on the line (hyperplane $\perp$ $\mathbf{m}$) and the projection on $\mathbf{m}$, and can be written as:

$$\mathbf{x}' = -\mathbf{m}\mathbf{x}\mathbf{m}^{-1} = \mathbf{x}_{\perp\mathbf{m}} - \mathbf{x}_{||\mathbf{m}}. \tag{1.19}$$

Note that the reflection is defined in the same way for the reflection of any multivector A.

Then, any rigid body motion (rotation or translation) can be expressed as the composition of two reflections in geometric algebra. If the first reflection whose normal vector is $\mathbf{m}$, is followed by a second reflection with normal vector $\mathbf{n}$, applied to a point $\mathbf{x} \in \mathbb{R}^d$, we obtain the point $\mathbf{x}'$ computed as

$$\mathbf{x}' \quad = -\mathbf{n}(-\mathbf{m}\mathbf{x}\mathbf{m}^{-1})\mathbf{n}^{-1} = (\mathbf{n}\mathbf{m})\mathbf{x}(\mathbf{n}\mathbf{m})^{-1}. \tag{1.20}$$

The transfomation corresponds to a rotation (around the intersection of the lines perpendicular to $\mathbf{n}$ and $\mathbf{m}$) when the two reflection lines intersect, and to a translation if the two reflection lines are parallel. This situation is illustrated in Fig. 2.

Indeed, assuming $\mathbf{n}$ and $\mathbf{m}$ are both normalized we have for rotations

$$\mathbf{x}' = \big(\cos(\phi) + \sin(\phi)\mathbf{i}_{\mathbf{m}\wedge\mathbf{n}}\big)\mathbf{x}\big(\cos(\phi) - \sin(\phi)\mathbf{i}_{\mathbf{m}\wedge\mathbf{n}}\big),$$

$$\mathbf{i}_{\mathbf{m}\wedge\mathbf{n}} = \frac{\mathbf{m}\wedge\mathbf{n}}{\|\mathbf{m}\wedge\mathbf{n}\|}, \qquad \cos\phi = \mathbf{m}\cdot\mathbf{n}, \qquad \sin\phi = \|\mathbf{m}\wedge\mathbf{n}\|, \tag{1.21}$$

where ϕ is the angle between $\mathbf{m}$ and $\mathbf{n}$ in the rotation plane captured by $\mathbf{m}\wedge\mathbf{n}$. Note that the angle of this rotation corresponds to 2ϕ. When the two reflection lines are parallel, if $\mathbf{t}$ is the normal distance vector between the two parallel lines, then the resulting translation, expressed in conformal GA, is by $2\mathbf{t}$.

Generally, a rotation V by angle θ in the rotation plane with bivector[6] $\mathbf{i}$ is defined as

$$V = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{i} = e^{\frac{\theta}{2}\mathbf{i}}. \tag{1.22}$$

Then, a point $\mathbf{x}$ is rotated to $\mathbf{x}'$ as follows:

$$\mathbf{x}' = V\mathbf{x}\widetilde{V}, \tag{1.23}$$

where the reverse $\widetilde{V} = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{i})^{\sim} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\widetilde{\mathbf{i}} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{i}$. So, geometric algebra includes the quaternions of Hamilton as well, as we can see in V. Furthermore, the combination of rotations is an outermorphism meaning that

$$V(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\mathbf{a}_k)\widetilde{V} = V\mathbf{a}_1\widetilde{V}\wedge V\mathbf{a}_2\widetilde{V}\wedge\cdots V\mathbf{a}_k\widetilde{V}, \tag{1.24}$$

which is very useful since geometric algebra can represent a wide range of geometric objects with the outer product of points.

Finally we state a mathematically rigorous definition of Clifford's geometric algebra.

Definition 1.1 (Clifford's geometric algebra [5, 9]). Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n\}$, with $n = p + q$, $e_k^2 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1, \ldots, p$, $\varepsilon_k = -1$ for $k = p+1, \ldots, n$, be an *orthonormal base* of the inner product vector space $\mathbb{R}^{p,q}$ with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k\delta_{k,l}, \qquad k, l = 1, \ldots n, \tag{1.25}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This non-commutative product and the additional axiom of *associativity* generate the 2^n-dimensional Clifford geometric algebra $Cl(p,q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathcal{G}_{p,q} = \mathbb{R}_{p,q}$ over $\mathbb{R}$. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{h_1}e_{h_2}\ldots e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$, $e_\emptyset = 1$, forms a graded (blade) basis of $Cl(p,q)$. The grades k range from 0 for scalars, 1 for vectors, 2 for bivectors, s for s-vectors, up to n for pseudoscalars. The vector space $\mathbb{R}^{p,q}$ is included in $Cl(p,q)$ as the subset of 1-vectors. The general elements of $Cl(p,q)$ are real linear combinations of basis blades e_A, called Clifford numbers, multivectors or hypercomplex numbers.

[6]The notation $\mathbf{i}$ for the bivector of the plane of rotation is intentional in order to show the relationship with rotations in two dimensions by complex numbers and in three dimensions by quaternions.

1.4. Conformal Geometric Algebra (CGA)

Up to now, we defined the main operations of geometric algebra (GA). This section gives an example of how GA represents geometric objects in an intuitive way.

Conformal GA (CGA) of $\mathbb{R}^3$ is $\mathbb{G}_{4,1} = Cl(4,1)$, based on an embedding in the five-dimensional vector space $\mathbb{R}^{3+1,0+1}$. The basis vectors of the space are divided into three groups: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ corresponding to the Euclidean vectors in $\mathbb{R}^3$, as well as $\{\mathbf{e}_0\}$, and $\{\mathbf{e}_\infty\}$, for origin and infinity, respectively. The inner products between them are defined in Table 1.

TABLE 1. Inner products between CGA basis vectors.

	$\mathbf{e_0}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{e_\infty}$
$\mathbf{e_0}$	0	0	0	0	−1
$\mathbf{e_1}$	0	1	0	0	0
$\mathbf{e_2}$	0	0	1	0	0
$\mathbf{e_3}$	0	0	0	1	0
$\mathbf{e_\infty}$	−1	0	0	0	0

An alternative basis of the CGA vector space $\mathbb{R}^{4,1}$ can be the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ along with the basis vectors $\mathbf{e}_+$ squaring to $+1$, and $\mathbf{e}_-$ squaring to -1. This basis corresponds to a diagonal metric matrix of inner products. The transformation between the two bases can be defined as follows[7]:

$$\mathbf{e}_\infty = \mathbf{e}_+ + \mathbf{e}_-, \qquad \mathbf{e}_0 = \frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+). \tag{1.26}$$

In this space, it is possible to define a (conformal) point whose Euclidean position vector is $\mathbf{x}_\epsilon = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ as:

$$\mathbf{x} = \mathbf{e}_0 + \mathbf{x}_\epsilon + \frac{1}{2}\left\|\mathbf{x}_\epsilon\right\|^2 \mathbf{e}_\infty. \tag{1.27}$$

The major property of the inner product between two conformal points $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = -\frac{1}{2}\left\|\mathbf{x}_{\epsilon 2} - \mathbf{x}_{\epsilon 1}\right\|^2. \tag{1.28}$$

This corresponds to the squared Euclidean distance between the two points, very useful for defining geometric objects.

Firstly, the very intuitively defined objects in CGA include, a.o., circles, lines, spheres and planes. A circle is easily constructed as the outer product of any three points (control points) $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, on it:

$$C = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{p}_3. \tag{1.29}$$

[7] Alternatively, one can use the more symmetric transformation $\mathbf{e}_\infty = (\mathbf{e}_+ + \mathbf{e}_-)/\sqrt{2}$, $\mathbf{e}_0 = (\mathbf{e}_- - \mathbf{e}_+)/\sqrt{2}$, see e.g. [8], that is of advantage in certain application contexts.

Secondly, moving one of the circle points to infinity opens and flattens the circle yielding a line L, i.e. one of the points on C is replaced by the point at infinity $\mathbf{e}_\infty$, namely:

$$L = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{e}_\infty. \tag{1.30}$$

Thirdly, by increasing the dimensions of objects by one, a sphere S can be obtained by the outer product of any four points on it:

$$S = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{p}_3 \wedge \mathbf{p}_4. \tag{1.31}$$

Finally, sending one of the points to infinity will open and flatten the sphere and results in a plane π defined as:

$$\pi = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{p}_3 \wedge \mathbf{e}_\infty. \tag{1.32}$$

This represents an efficient and intuitive way to define geometric objects using geometric algebra. Fig. 3 illustrates the definition of some geometric objects from control points in CGA.

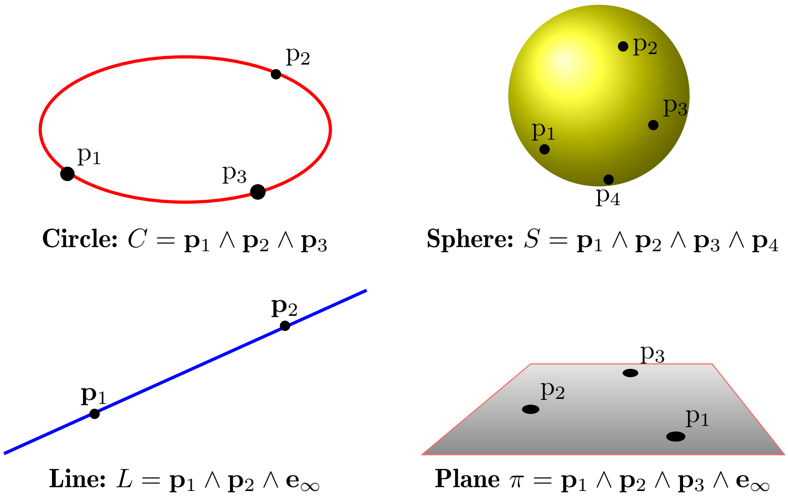


FIGURE 3. Definition of some geometric primitives from control points in CGA.

2. Conclusion

This paper has given a brief practical introduction to W.K. Clifford's geometric algebras, including conformal geometric algebra.

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