A Problem on Binomial Coefficients

Suaib Lateef
lateefholuyinka@gmail.com

Abstract
In this paper, we present a problem concerning the sum of powers of Binomial coefficients. We prove two special cases of the problem using some simple identities involving Binomial coefficients, and list another two cases but without proof.

Keywords: Franel numbers, Binomial coefficients.

1 Introduction

The binomial coefficients are the positive integers that occur as coefficients in the Binomial theorem. A binomial coefficient is written as $\binom{n}{k}$, where $n \geq k \geq 0$. Many properties of the Binomial coefficients have been discovered over time. For instance,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{1}$$

as well as

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \tag{2}$$

are two well known identities. We could see that (1) and (2) have simple closed forms.

In 1894, Franel [1] showed that

$$(n+1)^2 f_{n+1} = (7n^2 + 7n + 2) f_n + 8n^2 f_{n-1},$$

where $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ and n is a positive integer. He proved that no first-order recurrence exists for f_n so, there is no simple closed form for f_n. Also, in 1895, Franel [2] showed that

$$(n+1)^3 \phi_{n+1} = 2(2n+1)(3n^2 + 3n + 1)\phi_n + 4(4n-1)(4n+1)\phi_{n-1},$$

where $\phi_n = \sum_{k=0}^{n} \binom{n}{k}^4$ and n is a positive integer.

In this paper, we generate a corresponding expression for

$$_pf_n = \sum_{k=0}^{n} \binom{n}{k}^p,$$

where n is any positive integer and p is any Complex number.

2 Problem

If $n \geq 1$ is an integer and p is any Complex number, then

$$\sum_{k=0}^{n} \binom{n}{k}^p = 6\sum_{k=0}^{n-1} \binom{n}{k+1}^{p-2} \binom{n-1}{k}^2 - 4\sum_{k=0}^{n-1} \binom{n}{k+1}^{p-3} \binom{n-1}{k}^3. \tag{3}$$

3 Proofs of Special Cases

CASE p=1
From (3), if $p = 1$, we have that

$$\sum_{k=0}^{n} \binom{n}{k} = 6\sum_{k=0}^{n-1} \binom{n}{k+1}^{-1} \binom{n-1}{k}^2 - 4\sum_{k=0}^{n-1} \binom{n}{k+1}^{-2} \binom{n-1}{k}^3.$$

Proof.

$$\sum_{k=0}^{n} \binom{n}{k} = 6\sum_{k=0}^{n-1} \binom{n}{k+1}^{-1} \binom{n-1}{k}^2 - 4\sum_{k=0}^{n-1} \binom{n}{k+1}^{-2} \binom{n-1}{k}^3,$$

$$\sum_{k=0}^{n} \binom{n}{k} = 6\sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{\binom{n}{k+1}} \binom{n-1}{k} - 4\sum_{k=0}^{n-1} \left(\frac{\binom{n-1}{k}}{\binom{n}{k+1}}\right)^2 \binom{n-1}{k}.$$

Note that

$$\frac{\binom{n-1}{k}}{\binom{n}{k+1}} = \frac{k+1}{n}.$$

So,

$$\sum_{k=0}^{n} \binom{n}{k} = 6\sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right) \binom{n-1}{k} - 4\sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^2 \binom{n-1}{k},$$

$$\sum_{k=0}^{n} \binom{n}{k} = \frac{6}{n}\sum_{k=0}^{n-1} \binom{n-1}{k}(k+1) - \frac{4}{n^2}\sum_{k=0}^{n-1} \binom{n-1}{k}(k+1)^2,$$

$$\sum_{k=0}^{n}\binom{n}{k}=\frac{6}{n}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}k+\sum_{k=0}^{n-1}\binom{n-1}{k}\right)-\frac{4}{n^2}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}k^2+2\sum_{k=0}^{n-1}\binom{n-1}{k}k+\sum_{k=0}^{n-1}\binom{n-1}{k}\right).$$

We know that

$$\sum_{k=0}^{n-1}\binom{n-1}{k}=\frac{2^n}{2},$$

and

$$\sum_{k=0}^{n-1}\binom{n-1}{k}k=\frac{(n-1)2^n}{4},$$

and

$$\sum_{k=0}^{n-1}\binom{n-1}{k}k^2=\frac{n(n-1)2^n}{8}.$$

So,

$$\sum_{k=0}^{n}\binom{n}{k}=\frac{6}{n}\left(\frac{(n-1)2^n}{4}+\frac{2^n}{2}\right)-\frac{4}{n^2}\left(\frac{n(n-1)2^n}{8}+2\frac{(n-1)2^n}{4}+\frac{2^n}{2}\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}=2^n\left(\frac{3}{n}\left(\frac{(n-1)}{2}+1\right)-\frac{2}{n^2}\left(\frac{n(n-1)}{4}+n\right)\right),$$

$$\sum_{k=0}^{n}\binom{n}{k}=2^n\left(\frac{3(n+1)}{2n}-\frac{n+3}{2n}\right)$$

$$\sum_{k=0}^{n}\binom{n}{k}=2^n.$$

$\square$

CASE p=2

From (3), if $p=2$, we have

$$4\sum_{k=0}^{n-1}\binom{n}{k+1}^{-1}\binom{n-1}{k}^{3}=6\sum_{k=0}^{n-1}\binom{n-1}{k}^{2}-\sum_{k=0}^{n}\binom{n}{k}^{2}.$$

Proof.

$$4\sum_{k=0}^{n-1}\binom{n}{k+1}^{-1}\binom{n-1}{k}^{3}=6\sum_{k=0}^{n-1}\binom{n-1}{k}^{2}-\sum_{k=0}^{n}\binom{n}{k}^{2},$$

$$\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}}{\binom{n}{k+1}}\binom{n-1}{k}^{2}=\frac{3}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}^{2}-\frac{1}{4}\sum_{k=0}^{n}\binom{n}{k}^{2},$$

$$\sum_{k=0}^{n-1} \frac{k+1}{n} \binom{n-1}{k}^2 = \frac{3}{2} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 - \frac{1}{4} \sum_{k=0}^{n} \binom{n}{k}^2,$$

$$\frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (k+1) = \frac{3}{2} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 - \frac{1}{4} \sum_{k=0}^{n} \binom{n}{k}^2.$$

We know that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{2(2n-1)(2n-2)!}{n(n-1)!^2},$$

and

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 = \frac{(2n-2)!}{(n-1)!^2}.$$

So,

$$\frac{1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (k+1) = \frac{3(2n-2)!}{2(n-1)!^2} - \frac{2(2n-1)(2n-2)!}{4n(n-1)!^2},$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 (k+1) = \frac{3n(2n-2)!}{2(n-1)!^2} - \frac{(2n-1)(2n-2)!}{2(n-1)!^2},$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 (k+1) = \frac{(2n-2)!}{2(n-1)!}(n+1),$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 k + \sum_{k=0}^{n-1} \binom{n-1}{k}^2 = \frac{(2n-2)!}{2(n-1)!}(n+1),$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 k + \frac{(2n-2)!}{(n-1)!} = \frac{(2n-2)!}{2(n-1)!}(n+1),$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 k = \frac{(2n-2)!}{2(n-1)!}(n+1) - \frac{(2n-2)!}{(n-1)!},$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 k = \frac{(2n-2)!}{2(n-1)!}(n-1). \tag{5}$$

From (5), let

$$S = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 k = \binom{n-1}{0}^2 0 + \binom{n-1}{1}^2 1 + \binom{n-2}{2}^2 2 + \cdots + \binom{n-1}{n-1}^2 (n-1).$$

So,

$$2S = \left[\binom{n-1}{0}^2 0 + \binom{n-1}{1}^2 1 + \binom{n-2}{2}^2 2 + \cdots + \binom{n-1}{n-1}^2 (n-1)\right]$$
$$+ \left[\binom{n-1}{n-1}^2 (n-1) + \binom{n-1}{n-2}^2 (n-2) + \cdots + \binom{n-1}{0}^2 0\right].$$

Since $\binom{n-1}{k}^2 = \binom{n-1}{n-k-1}^2$, we have

$$2S = \binom{n-1}{0}^2 (n-1) + \binom{n-1}{1}^2 (n-1) + \binom{n-2}{2}^2 (n-1) + \cdots + \binom{n-1}{n-1}^2 (n-1),$$

$$2S = (n-1)\left[\binom{n-1}{0}^2 + \binom{n-1}{1}^2 + \binom{n-2}{2}^2 + \cdots + \binom{n-1}{n-1}^2\right],$$

$$2S = (n-1)\frac{(2n-2)!}{(n-1)!},$$

$$S = (n-1)\frac{(2n-2)!}{2(n-1)!}.$$

□

4 Other Special Cases

We list some other special cases without proof here;

$$\sum_{k=0}^{n} \binom{n}{k}^4 = 5\sum_{k=0}^{n} \binom{n}{k+1}^2 \binom{n-1}{k}^2 - \frac{4n-1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k}^4,$$

$$\sum_{k=0}^{n} \binom{n}{k}^4 = 20\sum_{k=0}^{n} \binom{n}{k+1} \binom{n-1}{k}^3 - 6\frac{4n-1}{n} \sum_{k=0}^{n-1} \binom{n-1}{k}^4.$$

References

[1] J. Franel, *On a question of Laisant*, L'intermédiaire des Mathématiciens, **1** (1984), 45–47.

[2] J. Franel, *On a question of Laisant*, L'intermédiaire des Mathématiciens, **2** (1985), 33–35.