# Semicircles in the arbelos with overhang and division by zero

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**Abstract.** We consider special semicircles, whose endpoints lie on a circle, for a generalized arbelos called the arbelos with overhang considered in [4] with division by zero.

## 1 Introduction

For a point $O$ on the segment $AB$ such that $|AO| = 2a$, $|BO| = 2b$, let $A_h$ (resp. $B_h$) be a point on the half line $OA$ (resp. $OB$) with initial point $O$ such that $|OA_h| = 2(a + h)$ (resp. $|OB_h| = 2(b + h)$) for a real number $h$ satisfying $\min(a, b) < h$. In [4] we have considered a generalized arbelos consisting of the three semicircles $\alpha$, $\beta$ and $\gamma$ of diameters $A_hO$, $B_hO$ and $AB$, respectively, constructed on the same side of $AB$. The figure is denoted by $(\alpha, \beta, \gamma)_h$ and is called the arbelos with overhang $h$ (see Figure 1). The ordinary arbelos is obtained from $(\alpha, \beta, \gamma)_h$ if $h = 0$, which is denoted by $(\alpha, \beta, \gamma)_0$.

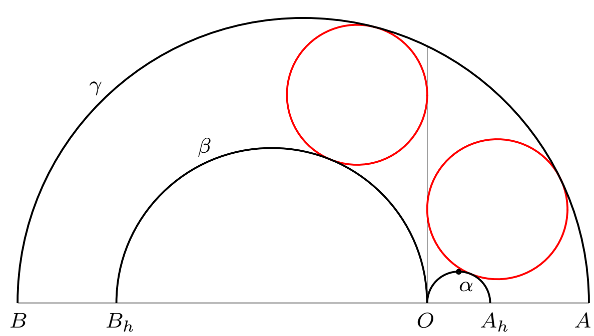

Figure 1: $(\alpha, \beta, \gamma)_h$, $-\min(a, b) < h < 0$.

Let $c = a + b$. The circle touching $\alpha$ (resp. $\beta$) externally, $\gamma$ internally, and the axis from the side opposite to $B$ (resp. $A$) has radius

$$r_{\mathrm{A}} = \frac{ab}{c + h}.$$

The two circles are called the twin circles of Archimedes of $(\alpha, \beta, \gamma)_h$. Circles of radius $r_{\mathrm{A}}$ are called Archimedean circles of $(\alpha, \beta, \gamma)_h$ or said to be Archimedean with respect to $(\alpha, \beta, \gamma)_h$.

In this article we consider special semicircles, which are counterpart to the incircle and Archimedean circles of $(\alpha, \beta, \gamma)_h$ using division by zero. At the last part of this paper we consider special case of $(\alpha, \beta, \gamma)_h$ considered by Aida [1]. We consider using a rectangular coordinate system with origin $O$ such that the farthest point on $\alpha$ have coordinates $(a + h, a + h)$ (see Figure 1). The radical axis of $\alpha$ and $\beta$ is called the axis.

## 2 Incircle and insemicircle

In this section we consider the incircle of $(\alpha, \beta, \gamma)_h$ and an inscribed semicircle in $(\alpha, \beta, \gamma)_h$. If a circle touches $\alpha$ and $\beta$ externally and $\gamma$ internally, we call the circle the incircle of $(\alpha, \beta, \gamma)_h$ (see Figure 2). If the endpoints of a semicircles lie on a circle, we say that the semicircle touches the circle at the endpoints. If a semicircle touches $\alpha$ and $\beta$, and $\gamma$ at the endpoints, we say that the semicircle is inscribed in $(\alpha, \beta, \gamma)_h$. We have considered such a semicircle in [2] for $(\alpha, \beta, \gamma)_0$. We use the next proposition.

**Proposition 1.** *A semicircle of radius $s$ touches a circle of radius $r$ at the endpoints if and only if $d^2 + s^2 = r^2$, where $d$ is the distance between the centers of the semicircle and the circle.*

Let $v = \sqrt{(c+h)^2 - 2ab + h^2}$.

**Theorem 1.** *The following statements hold.*
(i) *The incircle of $(\alpha, \beta, \gamma)_h$ has radius*

$$i_c = \frac{ab(c+2h)}{(c+h)^2 - ab}. \tag{1}$$

(ii) *If a semicircle is inscribed in $(\alpha, \beta, \gamma)_h$, then it has radius*

$$i_s = \frac{-v^2 + \sqrt{8ab(c+2h)^2 + v^4}}{2(c+2h)}. \tag{2}$$

*Proof.* We prove (ii). Let $(x, y)$ and $i_s$ be the coordinates of the center and the radius of the semicircle inscribed in $(\alpha, \beta, \gamma)_h$. Then we get $(x - (a+h))^2 + y^2 = ((a+h) + i_s)^2$, $(x + (b+h))^2 + y^2 = ((b+h) + i_s)^2$ and $(x - (a-b))^2 + y^2 + i_s^2 = c^2$ by Proposition 1. Eliminating $x$ and $y$ from the three equations and solving the resulting equation for $i_s$, we get (2). The part (i) is proved similarly. □

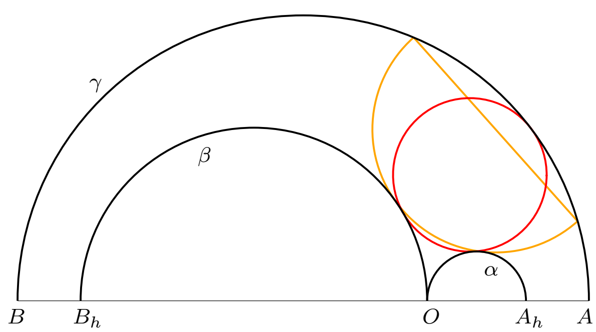


Figure 2.

The theorem shows that an inscribed semicircle in $(\alpha, \beta, \gamma)_h$ is determined uniquely. Hence we can call it the insemicircle of $(\alpha, \beta, \gamma)_h$.

We consider a condition where a semicircle of radius $i_s$ touches $\gamma$. If one of the endpoints of a semicircle $S_1$ lies on a semicircle $S_2$ and the other endpoints of $S_1$ lies on the reflection of $S_2$ in its diameter, we still say that $S_1$ touches $S_2$ at the endpoints. The circle of center of coordinates $((a+h)m, 0)$ (resp. $(-(b+h)n, 0)$ and passing through $O$ is denoted by $\alpha_m$ (resp. $\beta_n$) for a real number $m$ (resp. $n$) (see Figure 3). For points $P$ and $Q$ on a semicircle $\delta$, we say that $P$, $Q$ and the endpoints of $\delta$ lie counterclockwise if $P$, $Q$ and one of the endpoints of $\delta$ lie counterclockwise. If a circle touches $\alpha_m$, $\beta_n$ and $\gamma$ internally so that the points of tangency of this circle and each of $\beta_m$, $\alpha_n$ and $\gamma$ lie counterclockwise, we say that the circle touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately. Also if a semicircle touches $\alpha_m$ and $\beta_n$, and $\gamma$ at the endpoints so that the points of tangency of the semicircle and each of $\beta_n$, $\alpha_m$, and the endpoints lie counterclockwise, then we say that the semicircle touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately.

**Theorem 2.** *If $m \neq 0$ and $n \neq 0$, the following three statements are equivalent.*
(i) *A circle of radius $i_c$ touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately.*
(ii) *A semicircle of radius $i_s$ touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately.*
(iii) $c + 2h = \dfrac{a+h}{m} + \dfrac{b+h}{n}$.

*Proof.* Assume that (i) and $(x, y)$ are the coordinates of the center of the circle in (i). Then we have $(x - m(a+h))^2 + y^2 = (m(a+h)+i_c)^2$, $(x+n(b+h))^2+y^2 = (n(b+h)+i_c)^2$ and $(x-(a-b))^2+y^2 = (c-i_c)^2$. Eliminating $x$ and $y$ from the three equations with (1), we get (iii). Conversely we assume (iii), and a circle of radius $i_c$ touches $\alpha_m$, $\beta_{n'}$ and $\gamma$ appropriately for a real number $n'$. Then we have $a + b + 2h = (a + h)/m + (b + h)/n'$ just as we have shown, i.e., $n = n'$. Hence $\beta_n = \beta_{n'}$, i.e., (iii) implies (i). Therefore (i) and (iii) are equivalent. The equivalence of (ii) and (iii) is proved similarly. □

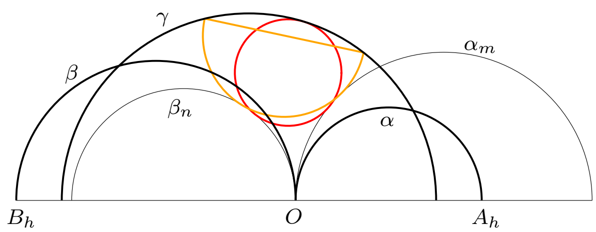


Figure 3: $1 < m$ and $0 < n < 1$.

Theorem 2 does not consider the case in which $\alpha_m$ or $\beta_n$ coincides with the axis. We consider the case in the next theorem (see Figure 4).

**Theorem 3.** *The following statements hold.*
(i) *A circle of radius $i_c$ touches $\alpha_m$ $(m > 0)$ externally, $\gamma$ internally and the axis if and only if*

$$m = m_0 = \frac{a+h}{c+2h}. \tag{3}$$

(ii) *A semicircle of radius $i_s$ touches $\alpha_m$ $(m > 0)$ and the axis, and $\gamma$ at the endpoints if and only if* (3) *holds.*
(iii) *A circle of radius $i_c$ touches $\beta_n$ $(n > 0)$ externally, $\gamma$ internally and the axis if and only if*

$$n = n_0 = \frac{b+h}{c+2h}. \tag{4}$$

(iv) *A semicircle of radius $i_s$ touches $\beta_n$ $(n > 0)$ and the axis, and $\gamma$ at the endpoints if and only if* (4) *holds.*

*Proof.* We prove (i). Let $(x, y)$ be the coordinates of the center of the circle of radius $i_c$ in (i). Then we have $x = i_c$, $(x - m(a + h))^2 + y^2 = (m(a + h) + i_c)^2$ and $(x - (a - b))^2 + y^2 = (a + b - i_c)^2$. Eliminating $x$ and $y$ from the three equations with (1), and solving the resulting equation for $m$, we get (3). Conversely, we assume that (3) and a circle of radius $i_c$ touches $\alpha_{m'}$ $(m' > 0)$ externally, $\gamma$ internally and the axis for a real number $m'$. Then we have $m' = m_0 = m$ as just we have proved. Therefore $\alpha_{m'} = \alpha_m$ and the converse is true. The rest of the theorem is proved similarly. □

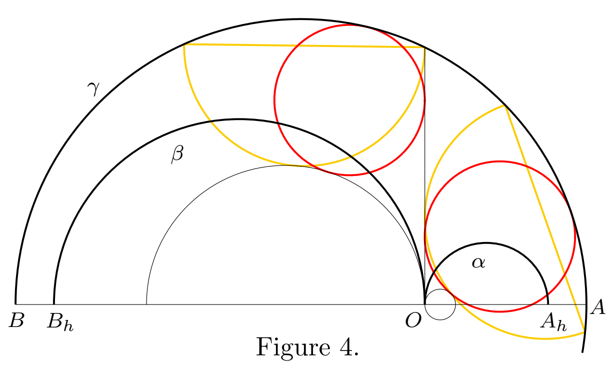


Figure 4.

If $m = m_0$, then $(a+h)/m = c + 2h$. Therefore if $(b+h)/n_x = 0$, and $\beta_{n_x}$ coincides with the axis, then we can consider that Theorem 2 is true in the case $(m, n) = (m_0, n_x)$. Similarly if $n = n_0$ and $(a+h)/m_x = 0$ and $\alpha_{m_x}$ coincides with the axis, we can also consider that Theorem 2 holds in the case $(m, n) = (m_x, n_0)$. Therefore Theorems 2 and 3 can be unified in this case. We consider about this in section 4.

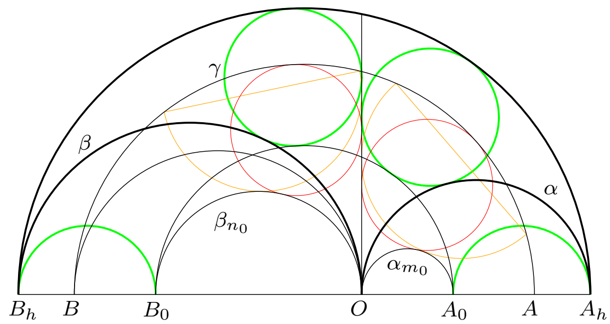


Figure 5.

**Theorem 4.** *If $A_0O$ and $B_0O$ are the diameters of the circles $\alpha_{m_0}$ and $\beta_{n_0}$, respectively, then the circles of diameters $A_0A_h$ and $B_0B_h$ are Archimedean circles of the arbelos made by $\alpha$, $\beta$ and the semicircle of diameter $A_hB_h$ constructed on the same side of $AB$ as $\gamma$. Therefore the circle of diameter $A_0B_0$ is concentric to $\gamma$ and touches the twin circles of Archimedes of the arbelos.*

*Proof.* Since the radius of the circle $\alpha_{m_0}$ equals $(a+h)m_0 = (a+h)^2/(c+2h)$ by (3), the circle of diameter $A_0A_h$ has radius

$$(a+h) - \frac{(a+h)^2}{c+2h} = \frac{(a+h)(b+h)}{c+2h},$$

which equals the radius of Archimedean circles of the arbelos made by $\alpha$, $\beta$ and the semicircle of diameter $A_hB_h$ (see Figure 5). Since the radius of the circle is symmetric in $a$ and $b$, the other circle also has the same radius. □

## 3 Archimedean semicircles

In this section we consider another kind of semicircles touching $\gamma$ at the endpoints.

**Theorem 5.** *The semicircle touching $\alpha$ and the axis and $\gamma$ at the endpoints is congruent to the semicircle touching $\beta$ and the axis and $\gamma$ at the endpoints. The common radius equals*

$$s_{\mathrm{A}} = \frac{1}{2}(\sqrt{(c+2h)^2 + 8ab} - c - 2h). \tag{5}$$

*Proof.* Let $(s, y)$ be the coordinates of the center of the semicircle touching $\alpha$ and the axis, and $\gamma$ at the endpoints. Then $s$ equals the radius of the semicircle, and we have $(s - (a-b))^2 + y^2 + s^2 = c^2$ by Proposition 1 and $(s - (a+h))^2 + y^2 = ((a+h)+s)^2$. Eliminating $y$ from the two equations and solving the resulting equation for $s$, we have $s = s_{\mathrm{A}}$. Since $s$ is symmetric in $a$ and $b$, the other semicircle also has the same radius. □

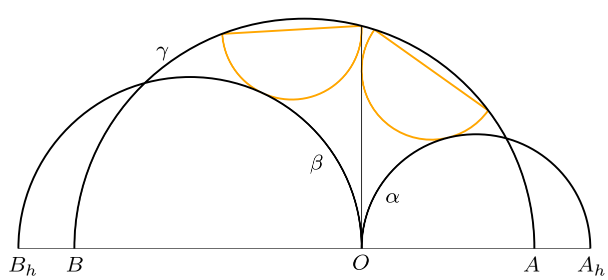


Figure 6.

The two congruent semicircles in Theorem 5 may be called *the twin semicircles of Archimedes* (see Figure 6). A semicircle of radius $s_{\rm A}$ is called an *Archimedean semicircle* of $(\alpha, \beta, \gamma)_h$ or said to be *Archimedean* with respect to $(\alpha, \beta, \gamma)_h$. Let $w_k = \sqrt{a^2 + kmn + b^2}$. Theorem 5 shows that $(\alpha, \beta, \gamma)_0$ has Archimedean semicircles of radius $(w_{10} - c)/2$.

**Theorem 6.** *Assume that $(m, n) \neq (1, 0), (0, 1)$ and a semicircle touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately. Then the semicircle is Archimedean with respect to $(\alpha, \beta, \gamma)_h$ if and only if*

$$\frac{1}{m} + \frac{1}{n} = 1. \tag{6}$$

*Proof.* Assume that a semicircle of radius $s_{\rm A}$ touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately and $(x, y)$ are the coordinates of its center. Then we get $(x - m(a + h))^2 + y^2 = (m(a + h) + s_{\rm A})^2$, $(x + n(b + h))^2 + y^2 = (n(b + h) + s_{\rm A})^2$, and $(x - (a - b))^2 + y^2 + {s_{\rm A}}^2 = c^2$. Eliminating $x$ and $y$ from the three equations, we have (6). Conversely we assume (6) and assume that a semicircle of radius $s_{\rm A}$ touches $\alpha_m$, $\beta_{n'}$ and $\gamma$ appropriately. Then we have $1/m + 1/n' = 1$. Hence we get $n = n'$, i.e., $\beta_n = \beta_{n'}$. Hence the converse holds. □

While we have obtained the next theorem in [4].

**Theorem 7.** *If $(m, n) \neq (1, 0), (0, 1)$ and a circle touches $\alpha_m$, $\beta_n$ and $\gamma$ appropriately, then the circle is Archimedean with respect to $(\alpha, \beta, \gamma)_h$ if and only if* (6) *holds.*

By Theorems 6 and 7 we have the next theorem.

**Theorem 8.** *If $(m, n) \neq (1, 0), (0, 1)$, the following statements are equivalent.*
(i) *The circle touching $\alpha_m$, $\beta_n$, and $\gamma$ appropriately is Archimedean with respect to $(\alpha, \beta, \gamma)_h$.*
(ii) *The semicircle touching $\alpha_m$, $\beta_n$, and $\gamma$ appropriately is Archimedean with respect to $(\alpha, \beta, \gamma)_h$.*
(iii) (6) *holds.*

It is commonly considered that the circles $\alpha_0$ and $\beta_0$ are point circles and coincide with the origin $O$. This implies that Theorem 8 is not true in the cases $(m, n) = (1, 0), (0, 1)$. Therefore Theorems 8 does not consider the case of the twin circles of Archimedean and the case of the twin semicircles of Archimedes. We consider the case in the next section.

## 4 Division by zero

In this section we show that we can consider that the circles $\alpha_0$ and $\beta_0$ coincide with the axis using recently made definition of division by zero [5].

For a field $F$ we consider the following bijection $\psi : F \to F$:

$$\psi(a) = \begin{cases} a^{-1} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

It is a custom to denote $z\psi(a)$ by $z/a$ if $a \neq 0$, i.e., $z\psi(a) = a/z$ for $a \neq 0$. Following to this, we write

$$z \cdot \psi(0) = \frac{z}{0} \quad for\ \forall z \in F. \tag{7}$$

Then we have

$$z \cdot \psi(a) = \frac{z}{a} \quad for\ \forall a, z \in F. \tag{8}$$

Especially we have

$$\frac{z}{0} = z \cdot 0 = 0 \quad for\ \forall z \in F. \tag{9}$$

Notice that the concept of the reduction to common denominator can not be used for $z/0$, i.e., we have the following relation in general in the case $b = 0$ or $d = 0$:

$$\frac{a}{b} + \frac{c}{d} \neq \frac{ad + bc}{bd}.$$

We consider the circle $\alpha_m$ in the case $m = 0$. The circle $\alpha_m$ has an equation $(x - m(a+h))^2 + y^2 = m^2(a+h)^2$, or

$$-2m(a+h)x + (x^2 + y^2) = 0. \tag{10}$$

This implies $x^2 + y^2 = 0$ if $m = 0$. Hence $\alpha_0$ coincides with the origin in this case. On the other hand, (10) can be written as

$$-2(a+h)x + \frac{x^2 + y^2}{m} = 0. \tag{11}$$

Therefore we get $-2(a+h)x = 0$, i.e., $x = 0$ if $m = 0$ by (9), i.e., $\alpha_0$ coincides with the axis in this case. Now we can consider that $\alpha_0$ is the origin or the axis, or the axis as the union of them. Similarly $\beta_0$ can be considered as the origin or the axis.

We can now consider that $\alpha_0$ and $\beta_0$ coincide with the axis. Then Theorem 2 holds in the case $(m, n) = (m_0, 0), (0, n_0)$ by (9). Also Theorem 8 holds in the case $(m, n) = (1, 0), (0, 1)$. Our current mathematics avoids to consider (9). But our above observation shows that (9) is useful.

Division by zero was founded by Saburou Saitoh in 2014. He has been making a list of successful example applying division by zero and its generalization called division by zero calculus, and there are more than 1200 evidences. It shows that a new world of mathematics can be opened if we admit them. For an extensive reference of division by zero and division by zero calculus including those evidences, see [5].

## 5 Aida arbelos

Aida (1747-1817) considered a figure consisting of two touching semicircles at their midpoints and the circle passing through the endpoints of the semicircles [1] (see Figure 7). He gave several notable properties of this figure, which are summarized in [3]. We conclude this paper by considering special circles and special semicircles for this figure.

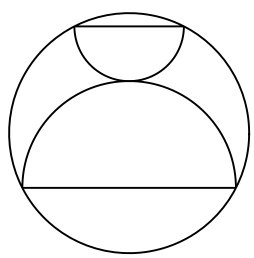

Figure 7: Aida's figure.

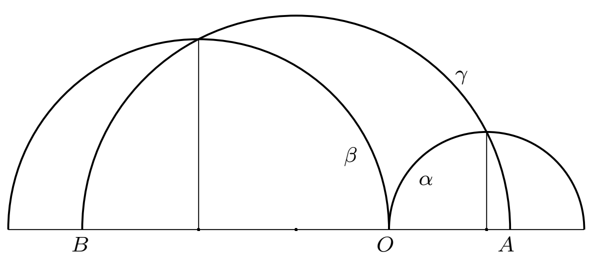


Figure 8: Aida arbelos.

Aida's figure is obtained from $(\alpha, \beta, \gamma)_h$, when $h = r_{\mathrm{A}}$ [3], or

$$h = \frac{ab}{c+h}. \tag{12}$$

Because (12) is equivalent to

$$r_{\mathrm{A}} = h = \frac{1}{2}(w_6 - c), \tag{13}$$

and (13) implies that the farthest points on $\alpha$ and $\beta$ from $AB$ lie on $\gamma$, where recall $w_k = \sqrt{a^2+kab+b^2}$. In this case we call $(\alpha,\beta,\gamma)_h$ an Aida arbelos (see Figure 8). Replacing $h$ in the denominator of the right side of (12) by the right side of (12) repeatedly, we get a continued fraction expansion of $r_{\rm A}$ for the Aida arbelos:

$$r_{\rm A} = \frac{ab}{c+h} = \frac{ab}{c+\dfrac{ab}{c+h}} = \frac{ab}{c+\dfrac{ab}{c+\dfrac{ab}{c+\ddots}}}.$$

We assume $h \geq 0$. Let $\overline{\alpha}$ and $\overline{\beta}$ be the semicircles of diameters $AO$ and $BO$, respectively, constructed on the same side of $AB$ as $\gamma$, i.e., $\overline{\alpha}$, $\overline{\beta}$ and $\gamma$ form $(\alpha,\beta,\gamma)_0$. The incircle of the curvilinear triangle made by $\alpha$, $\overline{\alpha}$ (resp. $\beta$, $\overline{\beta}$) and the radical axis of $\alpha$ (resp. $\beta$) and $\gamma$ has radius $(1/r_{\rm A}+1/h)^{-1}$ for $(\alpha,\beta,\gamma)_h$ [4]. Therefore the radius equals $r_{\rm A}/2$ for the Aida arbelos. The circles are denoted by green in Figure 9. The circle touching $\alpha$ or $\beta$ externally, $\gamma$ externally and the axis has radius $ab/h$ for $(\alpha,\beta,\gamma)_h$ [4]. Hence the radius equals $ab/r_{\rm A} = c+r_{\rm A}$ for the Aida arbelos by (12). The circles are denoted by magenta in Figure 9.

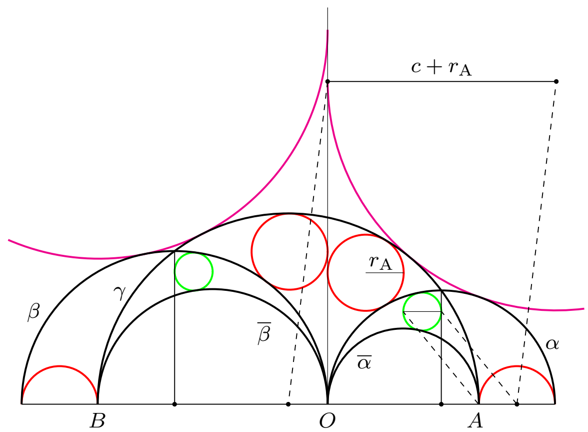


Figure 9: The green circles have radius $r_{\rm A}/2$.

Substituting (13) in (5), we get that the radius of Archimedean semicircles of the Aida arbelos equals

$$s_{\rm A} = \frac{1}{2}(w_{14}-w_6).$$

Since $i_c = w_6h/c$ for the Aida arbelos [3], we get that the inradius of the Aida arbelos equals

$$i_c = \frac{w_6(w_6-c)}{2c}$$

by (13). Therefore we have

$$i_c + r_{\rm A} = \frac{2ab}{c}.$$

Hence the sum of $i_c$ and $r_{\rm A}$ for the Aida arbelos equals the diameter of the Archimedean circle of $(\alpha,\beta,\gamma)_0$. Let $u = (w_6^4+16a^2b^2)^{1/4}$.

**Theorem 9.** *If the insemicircle of the Aida arbelos has center of coordinates $(x_s, y_s)$, we have*

$$i_s = \frac{u^2-c^2}{2w_6}, \tag{14}$$

$$(x_s, y_s) = \left(\frac{(b-a)i_s}{w_6}, \frac{4ab\sqrt{4ab+u^2}}{w_6^2}\right). \tag{15}$$

*Proof.* By (2) and (13), we get (14). Solving the equations $(x_s - (a+h))^2 + y_s^2 = ((a+h) + i_s)^2$ and $(x_s + (b+h))^2 + y_s^2 = ((b+h) + i_s)^2$ with (14), we get (15). □

The next theorem shows that the result for the insemicircle of $(\alpha, \beta, \gamma)_0$ obtained in [2] also holds for the Aida arbelos (see Figure 10).

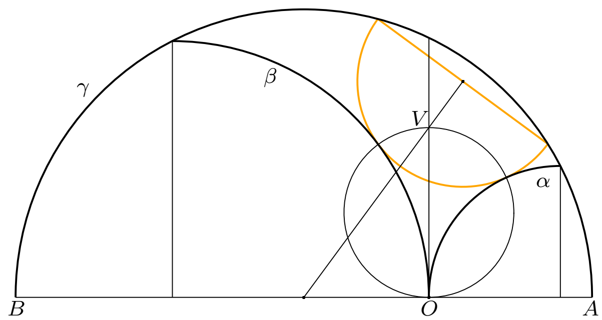

Figure 10.

**Theorem 10.** *If the line joining the centers of* $\gamma$ *and the insemicircle of the Aida arbelos meets the axis in a point* $V$*, then the circle of diameter* $OV$ *is orthogonal to the insemicircle. Hence the circle passes through the points of tangency of two of* $\alpha$*,* $\beta$ *and the insemicircle.*

*Proof.* From (13) and (15), the circle of diameter $OV$ has radius

$$r_v = \frac{4ab\sqrt{4ab+u^2}}{w_{10}^2 + u^2}$$

and the center of coordinates $(0, y_v) = (0, r_v)$. Then we have $(x_s - 0)^2 + (y_s - y_v)^2 = r_v^2 + i_s^2$. □

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