HOW HARD IS THE TENSOR RANK?

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Abstract. We build a combinatorial technique to solve several long-standing problems on the complexity of tensor decompositions. These include the polynomial time equivalence between the problem of computing the tensor rank over an integral domain $\mathcal{R}$ and the solvability of a system of polynomial equations over $\mathcal{R}$. In particular, the tensor rank is undecidable over $\mathbb{Z}$, which answers a question posed by Gonzalez and Ja'Ja' in 1980, and another special case $\mathcal{R} = \mathbb{Q}$ answers a question of Bläser. Also, we determine the algorithmic complexity of the symmetric rank, which confirms the NP-hardness conjecture of Hillar and Lim. As a byproduct of our approach, we answer a question discussed by Buss, Frandsen, Shallit in 1999 and determine the algorithmic complexity of the *minimal rank matrix completion*, and we solve two problems of Grossmann and Woerdeman on the *fractional minimal rank*.

1. Introduction

The objective of this study is to understand the algorithmic complexity of *tensor rank*, which is the function sending a tensor $T \in U \otimes V \otimes W$ to the smallest integer r for which one can write

$$T = \lambda_1 T_1 + \ldots + \lambda_r T_r$$

with $T_i = u_i \otimes v_i \otimes w_i$ and $\lambda_i \in \mathcal{F}$. Here, one usually takes $\mathcal{F}$ to be a field, but it is convenient for us to allow it to be an arbitrary commutative ring. As one can see from this definition, we restrict our attention to three-way tensor products because, as we show in this paper, this setting is already sufficient to understand the algorithmic complexity of rank decompositions. The rank of a three-way tensor corresponds to the invariant known as the *multiplicative complexity* for bilinear programs [22, 24, 59], which appears in the famous problem on the complexity of matrix multiplication [16, 35, 58]. The general problem of tensor rank decompositions was introduced eighty years ago [26] and, apart from the above mentioned application in computational complexity theory, it appears as a fundamental tool in statistics [48], signal processing [36], psychology [13], linguistics [57], chemometrics [15] and many other contexts, see a more detailed survey in [32].

The first step towards understanding the computational complexity of tensor rank was made by Håstad [24], who showed that this problem is NP-hard over $\mathbb{Q}$ and NP-complete over finite fields. A recent paper of Hillar and Lim [25] shows that Håstad's approach works well enough to prove the NP-hardness over $\mathbb{R}$ and $\mathbb{C}$ as well. In this paper, we show that, for any integral domain $\mathcal{R}$, the computation of the tensor rank over $\mathcal{R}$ is polynomial time equivalent to the general problem of the solvability of a system of polynomial equations over $\mathcal{R}$. We note a recent

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paper of Schaefer and Štefankovič [47], which proves the same result but restricted to the case when $\mathcal{R}$ is a field. Our result is more general, and, in particular, it shows that the tensor rank over $\mathbb{Z}$ is undecidable, which answers the question asked by Gonzalez and Ja'Ja' in 1980. We note that our result is original even in the above mentioned situation of $\mathcal{R}$ being a field because, although the paper [47] has already went through the peer review, the first version of the current paper [50] appeared on arXiv earlier than the first version of [47]. In particular, this covers the case of $\mathcal{R} = \mathbb{Q}$ and answers a question of Bläser [6]. As a byproduct of the work described in this paragraph, we determine the algorithmic complexity of *minimal rank matrix completion* and answer a question discussed by Buss, Frandsen, and Shallit in [9] and by Laurent in [34], and also we solve two recent problems of Grossmann, Woerdeman on the so-called *fractional minimal rank* [23].

The *symmetric tensor rank* $\operatorname{srk}(T)$ appears if we assume $U = V = W$ and $u_i = v_i = w_i$ in the above definition of rank, and this invariant is also relevant in pure mathematics and engineering [14, 31, 41, 56]. Practical applications motivate a search of algorithms computing this invariant as well as the study of its computational complexity. Many authors expected that the symmetric rank is computationally difficult, and this problem was discussed in the foundational paper [14]. Several subsequent papers were devoted specifically to the problem of computing the symmetric rank [3, 8] and did not prove its intractability. In 2013, Hillar and Lim posed the NP-hardness of the symmetric rank as a conjecture in a further highly cited paper [25], and this question was reiterated in subsequent studies [21]. Several other papers [41, 45] stated the NP-hardness as a fact, but it remained an open problem before the publication of our work. We prove that the symmetric rank admits the same description of the complexity as the one given for the tensor rank in the above paragraph, provided that $\mathcal{R}$ is a field with $|\mathcal{R}| \geqslant 4$.

2. Tensor decompositions

We consider tensors of order three over a commutative ring $\mathcal{R}$. From the combinatorial point of view, a *tensor* is a three-dimensional array T with elements $T(i|j|k)$ in $\mathcal{R}$, where i, j, k run over corresponding *indexing sets* I, J, K. We write

$$T \in \mathcal{R}^{I\times J\times K}$$

and say that T is an $I\times J\times K$ tensor over $\mathcal{R}$. The *size* of T is defined as $|I|\times|J|\times|K|$. A tensor T is called *symmetric* if $I = J = K$ and $T(i|j|k) = T(i'|j'|k')$ whenever (i, j, k) is a permutation of (i', j', k'). Given three vectors

$$a \in \mathcal{R}^I,\ b \in \mathcal{R}^J,\ c \in \mathcal{R}^K,$$

we define the $I \times J \times K$ tensor $a \otimes b \otimes c$ by setting its (i, j, k)th entry to be $a_i b_j c_k$. Tensors arising in this way are called *decomposable* or *simple* with respect to $\mathcal{R}$. We note that, if we allow the vectors a, b, c to contain elements not from $\mathcal{R}$ but rather from some extension $\mathcal{S}$, we may possibly get a different set of simple tensors.

Definition 2.1. Let $\mathcal{R} \subseteq \mathcal{S}$ be commutative rings, and let T be a tensor over $\mathcal{R}$. The *rank* of T *with respect to* $\mathcal{S}$ is the smallest integer r such that T can be written as a sum of r tensors decomposable over $\mathcal{S}$. This quantity is denoted by $\operatorname{rk}_{\mathcal{S}} T$.

It is well known that the rank of a tensor with entries in $\mathcal{R}$ may depend on $\mathcal{S}$ even if $\mathcal{R}$ is a field [2, 18]. In the setting of integral domains, the rank may depend on the extension even for matrices, which we think of as $m \times n \times 1$ tensors.

Example 2.2. (Example 17 in [49].) *The rank of the matrix*

$$\begin{pmatrix} x & -z & 0 \\ 0 & y & x \\ y & 0 & z \end{pmatrix}$$

is three over the ring $\mathbb{R}[x,y,z]$ *and two over the field* $\mathbb{R}(x,y,z)$.

In order to discuss the computational complexity, we need to assume that the elements of $\mathcal{R}$ can be encoded by strings in some finite alphabet so that the addition and multiplication in $\mathcal{R}$ can be performed by polynomial time algorithms. We do not impose this assumption on the extension $\mathcal{S}$ as in Definition 2.1. In particular, our considerations are valid for the real ranks of rational tensors, which corresponds to $\mathcal{R} = \mathbb{Q}$, $\mathcal{S} = \mathbb{R}$. We are ready to formulate one of the main results.

Theorem 2.3. *Let* $\mathcal{R} \subseteq \mathcal{S}$ *be integral domains, and let* $f_1, \ldots, f_p$ *be polynomials with coefficients in* $\mathcal{R}$. *There is a polynomial time algorithm that constructs an order-three tensor* $\mathcal{T}$ *over* $\mathcal{R}$ *and an integer* r *such that the following are equivalent:*
(1) the equations $f_1 = 0, \ldots, f_p = 0$ *have a simultaneous solution in* $\mathcal{S}$*;*
(2) the rank of $\mathcal{T}$ *with respect to* $\mathcal{S}$ *does not exceed* r.

Moreover, these $\mathcal{T}$ *and* r *do not depend on the choice of* $\mathcal{S}$.

On the other hand, a straightforward formulation of Definition 2.1 gives a system of polynomial equations with coefficients in $\mathcal{R}$, and the inequality $\operatorname{rk}_{\mathcal{S}} T \leqslant r$ holds if and only if this system has a solution in $\mathcal{S}$. Therefore, Theorem 2.3 gives a complete description of the algorithmic complexity of the tensor rank.

Theorem 2.4. *Let* $\mathcal{R} \subseteq \mathcal{S}$ *be integral domains. Given a tensor* T *over* $\mathcal{R}$ *and* $r \in \mathbb{Z}$*, checking the inequality* $\operatorname{rk}_{\mathcal{S}} T \leqslant r$ *is polynomial time equivalent to deciding if a given system of polynomial equations with coefficients in* $\mathcal{R}$ *has a solution in* $\mathcal{S}$.

A particularly important special case $\mathcal{R} = \mathbb{Q}$, $\mathcal{S} = \mathbb{R}$ shows that the real tensor rank is what is called an $\exists\mathbb{R}$-*complete* problem [38]. In other words, the real tensor rank is polynomial time equivalent to many classical problems in geometry, which include oriented matroids [40], polytope realizability [44], Nash equilibria [17], graph drawings [4], art galleries [1], linkages [46]. A similar characterization is known for other problems on rank decompositions in linear algebra, including the nonnegative rank [51] and positive semidefinite rank of matrices [52].

Theorem 2.4 with $\mathcal{R} = \mathcal{S} = \mathbb{Q}$ shows that the rational tensor rank is polynomial time equivalent to deciding if a given *Diophantine equation* has a rational solution, which answers the question of Bläser, see Open Problem 2 on page 119 in [6]. We recall that the rational Diophantine solvability is a famous problem that is believed to be undecidable, but its complexity status remains open despite extensive research [29, 30, 37, 39, 43]. Therefore, Theorem 2.3 can be seen as a conditional proof of the undecidability of the rational tensor rank, which would confirm Conjecture 13.3 in the paper [25] by Hillar and Lim. We remark that the solvability of Diophantine equations over $\mathbb{Z}$ was the content of *Hilbert's tenth problem*, and this question was proved to be undecidable a half century ago [37]. Using Theorem 2.3 together with this undecidability result, we get the following.

Corollary 2.5. *Tensor rank over* $\mathbb{Z}$ *is undecidable.*

Corollary 2.5 answers the question by Gonzalez and Ja'Ja' dating back to 1980, see page 77 of [22]. Finally, we note that the solvability of Diophantine equations is

NP-hard over any integral domain [29], so we have another corollary of Theorem 2.3, generalizing the results of Håstad [24] and Hillar and Lim [25], who stated the NP-hardness of the tensor rank over $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and over finite fields.

Corollary 2.6. *Tensor rank is NP-hard over any integral domain.*

Now we switch to the symmetric case. As explained in the introduction, the *symmetric rank* of a symmetric tensor T *with respect to* a field $\mathcal{S}$ is the smallest number of simple symmetric tensors over $\mathcal{S}$ whose linear span contains T. We can prove the symmetric counterpart of Theorem 2.3.

Theorem 2.7. *Let $\mathcal{F} \subseteq \mathcal{K}$ be fields with $|\mathcal{K}| \geqslant 4$. The problem of checking if*

$$\operatorname{srk}_{\mathcal{K}} T \leqslant r$$

for a symmetric tensor T over $\mathcal{F}$ and $r \in \mathbb{Z}$ is polynomial time equivalent to deciding if a given family of polynomials with coefficients in $\mathcal{F}$ have a common zero over $\mathcal{K}$.

In particular, the computation of the symmetric rank is NP-hard over any field with at least four elements, which proves the conjecture of Hillar and Lim [25] discussed in the introduction. The most well studied cases are $\mathbb{R}$ and $\mathbb{C}$, see [14, 25, 41], so our additional assumption on the cardinality seems to be quite mild. In particular, the symmetric ranks are often studied in terms of the Waring rank function of homogeneous polynomials, which is not well defined over fields of characteristic 2 or 3, see [7]. Our technique could cover the case of cardinality three as well, but, as said above, this case is not especially relevant and leads to significant technical difficulties, so we decided to omit it. Also, we did not work with the case $|\mathcal{K}| = 2$ because it is quite pathological, which can be seen, in particular, from the fact that the symmetric tensor

$$e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1$$

does not admit any symmetric decomposition over $\mathbb{Z}/2\mathbb{Z}$.

3. Substitutions and matrix completions

In comparison to many recent studies that approach the general and symmetric tensor decomposition problems from the point of view of algebraic geometry [3, 12, 32], our methods involve a more combinatorial background. In particular, one of the ideas of our approach is to combine the standard *substitution method* of tensor rank computation with a classical problem of linear algebra, known as the *minimal rank matrix completion*, which we discuss later in this section.

Definition 3.1. If T is an $I \times J \times K$ tensor, then we define the kth 3-*slice* of T as an $I \times J$ matrix whose (i, j) entry equals $T(i|j|k)$. For all $i \in I$, $j \in J$, we can define the ith 1-slice of T and the jth 2-slice of T similarly.

The substitution method rests on the following easy lemma. We refer the reader to [33] for a recent account on this method and to [53, 54] for further developments. The paper [27] gives an earlier appearance of a related result.

Lemma 3.2. *Let $\mathcal{F}$ be a field, and let T be a tensor in $\mathcal{F}^{I\times J\times K}$ with $K = \{1, \ldots, k\} \cup \{1', \ldots, \tau'\}$. Let S_i be the ith 3-slice of T and assume that $S_{1'}, \ldots, S_{\tau'}$ are linearly independent and rank-one. Then $\operatorname{rank}_{\mathcal{F}} T$ is equal to*

$$\tau' + \min \operatorname{rank}_{\mathcal{F}} \mathcal{T}(V_1, \ldots, V_k),$$

where $\mathcal{T}(V_1, \ldots, V_k)$ *is the tensor formed by the slices* $S_1 - V_1, \ldots, S_k - V_k$*, and the matrices* $V_1, \ldots, V_k$ *are taken in the* $\mathcal{F}$*-linear span of* $S_{1'}, \ldots, S_{\tau'}$*.*

Another tool important for our paper is the *minimal rank matrix completion* problem. If $*$ is a new placeholder symbol, we say that a matrix M with entries in $\mathcal{R} \cup \{*\}$ is an *incomplete matrix* over $\mathcal{R}$; any matrix obtained by replacing the $*$'s with elements in $\mathcal{S}$ is called a *completion* of M over $\mathcal{S}$. What is the smallest value that the rank of a completion of a given incomplete matrix may take? The reduction of the problem of computing the rank of a tensor over a field to the minimal rank matrix completion is straightforward by Lemma 3.2, see Section 5 for details. The opposite reduction was given by Derksen in [20], so he showed that the minimal rank completion and tensor rank are polynomial time equivalent problems in the case of fields. However, the algorithmic complexity of both problems remained open, so we need to prove the following result on the way to Theorem 2.4.

Theorem 3.3. *Let* $\mathcal{R} \subseteq \mathcal{S}$ *be commutative rings. The problem of deciding if a given incomplete matrix with entries in* $\mathcal{R} \cup \{*\}$ *has a completion of rank three with respect to* $\mathcal{S}$ *is polynomial time equivalent to the problem of deciding if a given system of polynomial equations with coefficients in* $\mathcal{R}$ *has a solution over* $\mathcal{S}$*.*

The author believes that this result is new even in the case of fields. Numerous related problems are shown to be NP-hard, and, as noted by Derksen in [19], the NP-hardness of the problem being discussed in the case of fields follows from the earlier paper by Peeters [42]. There are several related problems whose complexity is described completely, which include a result similar to Theorem 3.3 but for the version of the minimal rank problem in which some of the $*$ entries may be required to take the same value [9]. However, the complexity of our version was discussed by Buss, Frandsen, Shallit without any progress on lower bounds, see page 575 in [9]. Also, Laurent writes that the minimal rank completion seems to be a difficult task, but again she does not mention any particular result on the complexity of this problem, see page 1972 in [34]. As said above, our Theorem 3.3 does not only prove the NP-hardness, but fully determines the complexity of the problem over any commutative ring. Other related completion problems whose complexity has been known are the *Euclidean distance completion* [34], *minimal rank sign pattern completion* [5], and the problem of minimizing the rank of matrices over a finite field that *fit* a given graph [42]. We note that the approximate version of the minimal rank completion problem naturally arises in applied mathematics [10, 28], and a version with random positions of the $*$'s is particularly important [11].

Our paper is structured as follows. The forthcoming Section 4 is devoted to the proof of Theorem 3.3, which is obtained by a method that can be seen as a variation of a recent investigation of the complexity of the *positive semidefinite rank* of a matrix [52]. As a byproduct of our approach, we get the solutions of two problems on the *fractional minimal rank* posed by Grossmann and Woerdeman [23]. In Section 5, we employ the construction recently used by Derksen [19] and deduce Theorem 2.3 from Theorem 3.3. As said above, an immediate consequence of this construction is that Theorem 3.3 implies Theorem 2.3 over a field, and we adapt this method to any integral domain. This completes the proofs of all the results announced above except the complexity of the symmetric rank. In Section 6, we switch to this situation and complete the proof of Theorem 2.7.

4. The complexity of minimal rank matrix completion

The main goal of this section is to prove Theorem 3.3 and hence determine the algorithmic complexity of the minimal rank matrix completion problem. To this end, it would be sufficient to assume that $\mathcal{R}$ and $\mathcal{S}$ are commmutative rings satisfying $\mathcal{R} \subseteq \mathcal{S}$, but the above mentioned application to the problems of Grossmann and Woerdeman [23] requires us to consider a slightly more general setting.

Remark 4.1. In this section, we assume that $\mathcal{R}$ and $\mathcal{S}$ are rings satisfying $\mathcal{R} \subseteq \mathcal{S}$, and, additionally, we assume that the implication

$$(C'C = I) \to (CC' = I) \tag{4.1}$$

holds for the 3×3 matrices C, C' over $\mathcal{S}$. In particular, this happens when $\mathcal{S}$ is commutative or when $\mathcal{S}$ itself is a matrix ring over a field.

We proceed with several notational conventions needed to reflect the algorithmic context of the problem. In this section, we represent polynomials with variables $x_1, \ldots, x_n$ and coefficients in $\mathcal{R}$ as elements of the free ring $\mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n \rangle$, which are non-commutative polynomials with integer coefficients, where apart from the variables $x_1, \ldots, x_n$ we have one additional variable for every element in $\mathcal{R}$. In other words, the elements of $\mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n \rangle$ are the sums

$$p_1 + \ldots + p_s \tag{4.2}$$

in which every p_i is a monomial with either $p_i = w_i$ or $p_i = -w_i$, where

$$w_i = \pi_{i1}\pi_{i2}\ldots\pi_{iq_i} \tag{4.3}$$

is a word in the alphabet $\mathcal{R} \cup \{x_1, \ldots, x_n\}$. The product of monomials is defined as the concatenation of the corresponding words, and the sign of the product is the product of the signs of the multipliers. Two elements of the form (4.2) are equal if they can be brought to the same form by appropriate permutations of the monomials and cancellations of the pairs of identical words appearing with opposite signs. Every element f in $\mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n \rangle$ naturally defines a function from $\mathcal{S}^n$ to $\mathcal{S}$, where the value $f(\xi_1, \ldots, \xi_n)$ is computed by replacing every occurrence of x_i with ξ_i in f with a subsequent evaluation of the obtained expression over $\mathcal{S}$.

Remark 4.2. We write $\mathbb{O}$ to denote the empty sum as in (4.2), and we define $\mathbb{I}$ as the empty word of the form (4.3) taken with the positive sign. The elements $\mathbb{O}$ and $\mathbb{I}$ are the zero and one of the ring $\mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n \rangle$, respectively.

Example 4.3. For $a \in \mathcal{R}$, the polynomial $xy - yx + a$ belongs to $\mathbb{Z}\langle \mathcal{R}, x, y \rangle$. If $\mathcal{S}$ is commutative, it represents the constant function always equal to a.

We proceed with a reduction needed for the proof of Theorem 3.3. For any monomial $p_i = +w_i$ or $p_i = -w_i$ with w_i as in (4.3), we define

$$\sigma(p_i) = \{\pm\mathbb{I},\, \pm\pi_{i1},\, \pm\pi_{i1}\pi_{i2},\, \pm\pi_{i1}\pi_{i2}\pi_{i3},\, \ldots,\, \pm p_i\} \cup \{\pm\pi_{i1},\, \pm\pi_{i2},\, \ldots,\, \pm\pi_{iq_i}\}.$$

For a general input polynomial f represented as $p_1 + \ldots + p_s$, we set

$$\sigma(f) = \sigma(p_1) \cup \ldots \cup \sigma(p_s) \cup \{\mathbb{O}, \pm p_1, \pm(p_1 + p_2), \ldots, \pm f\},$$

and for a finite set $F = \{f_1, \ldots, f_t\}$, we take $\sigma(F) = \sigma(f_1) \cup \ldots \cup \sigma(f_t)$. Clearly, the construction of the set $\sigma(F)$ can be done in polynomial time.

Example 4.4. If $F = \{xy - yx + a\}$ with $a \in \mathcal{R}$, then

$$\sigma(F) = \{\mathbb{O},\, \pm\mathbb{I},\, \pm x,\, \pm y,\, \pm xy,\, \pm yx,\, \pm a,\, \pm(xy - yx),\, \pm(xy - yx + a)\}. \tag{4.4}$$

Now we fix a finite set F of input polynomials, each of which is represented as a sum of monomials in $\mathbb{Z}\langle\mathcal{R}, x_1, \ldots, x_n\rangle$. In the rest of this section, we simply write σ instead of $\sigma(F)$, and σ^3 stands for the set of all triples of elements in σ.

Definition 4.5. We denote by $\mathcal{H} = \mathcal{H}(F)$ the set of those vectors in σ^3 that have one of the coordinates equal to $\mathbb{I}$ or $-\mathbb{I}$. We define the matrix $\mathcal{U} = \mathcal{U}(x_1, \ldots, x_n)$ whose columns are vectors in $\mathcal{H}$, and we define $\mathcal{W}(x_1, \ldots, x_n) = \mathcal{U}^\top \mathcal{U}$.

Example 4.6. If $F = \{xy - yx + a\}$ with $a \in \mathcal{R}$, then $\mathcal{H}$ is the set of all triples of the elements in (4.4) which have at least one of the coordinates equal to either $\mathbb{I}$ or $-\mathbb{I}$. Since there are 15 elements in (4.4) different from $\pm\mathbb{I}$, we have a total of

$$2^3 + 3 \cdot 2^2 \cdot 15 + 3 \cdot 2 \cdot 15^2$$

or 1538 elements in $\mathcal{H}$ in this case.

In what follows, we label the columns of $\mathcal{U}$ by the corresponding elements of $\mathcal{H}$. In particular, the (u, v) entry of $\mathcal{W}$ with $u, v \in \mathcal{H}$ equals the dot product $u \cdot v$.

Definition 4.7. For all $u, v \in \mathcal{H}$, the polynomial $\delta(u, v) \in \mathbb{Z}\langle\mathcal{R}, x_1, \ldots, x_n\rangle$ is defined as $u \cdot v$, or, equivalently, by the formula $\delta(u, v) = \mathcal{W}(u|v)$. We define the $\mathcal{H} \times \mathcal{H}$ matrix $\mathcal{B} = \mathcal{B}(F)$ with entries in $\mathcal{R} \cup \{*\}$ as follows:

(B1) if $\delta(u, v) = \rho$ with $\rho \in \mathbb{Z}\langle\mathcal{R}\rangle$, then we define $\mathcal{B}(u|v) \in \mathcal{R}$ as the value of ρ,
(B2) if $\delta(u, v) = f$ with $f \in F$, then we take $\mathcal{B}(u|v) = 0$,
(B3) in the remaining cases, we set $\mathcal{B}(u|v) = *$.

Example 4.8. We note that $\mathcal{B}(xy - yx + a)$ is an incomplete matrix with known entries in $\mathbb{Z}[a] \subseteq \mathcal{R}$. According to Example 4.6, its size is 1538×1538.

It is clear that the matrix $\mathcal{W}(\xi_1, \ldots, \xi_n)$ is a rank-three completion of $\mathcal{B}$ provided that $(\xi_1, \ldots, \xi_n) \in \mathcal{S}^n$ is a simultaneous solution of the equations $f_1 = 0, \ldots, f_t = 0$. We are going to show that all rank-three completions arise in this way up to the natural action of the group of invertible 3×3 matrices.

Lemma 4.9. *Let P be an $\mathcal{H} \times 3$ matrix over $\mathcal{S}$, and let L be a $3 \times \mathcal{H}$ matrix over $\mathcal{S}$ such that the product PL is a completion of $\mathcal{B}$. Let C be the matrix obtained by taking the columns of L with indexes in $E = \{(\mathbb{I}, \mathbb{O}, \mathbb{O}), (\mathbb{O}, \mathbb{I}, \mathbb{O}), (\mathbb{O}, \mathbb{O}, \mathbb{I})\}$. Then*

$$PC = \mathcal{U}(\xi_1, \ldots, \xi_n)^\top \quad \text{and} \quad C^{-1}L = \mathcal{U}(\xi_1, \ldots, \xi_n), \tag{4.5}$$

where $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution of the equations $f_1 = 0, \ldots, f_t = 0$.

Proof. Step 1. Since the 3×3 submatrix of $\mathcal{B}$ with row and column indexes in E is the unity matrix, we get that the submatrix C' of P formed by the rows with indexes in E satisfies $C'C = I$, where I is the 3×3 identity matrix. Since the condition (4.1) in Remark 4.1 applies, we also have $CC' = I$, and the transformation $(P, L) \to (PC, C'L)$ cannot change the property of PL to be a completion of $\mathcal{B}$. Therefore, we can assume without loss of generality that C is the unity matrix, which means that the rows of P with indexes in E and the columns of L with indexes in E already satisfy the desired conclusion as in the equalities (4.5).

Step 2. For any $u \in \mathcal{H}$, we denote the uth row of P by $p(u)$, and we denote the uth column of L by $l(u)$. The assumption of the lemma states that the product PL is a completion of $\mathcal{B}$, which means that

$$p(u) \cdot l(v) = \mathcal{B}(u|v) \quad \text{whenever } \mathcal{B}(u|v) \neq *. \tag{4.6}$$

Using this language, we can rewrite the result of Step 1 as

$$p(\mathbb{I},\mathbb{O},\mathbb{O})=(1,0,0),\;\; p(\mathbb{O},\mathbb{I},\mathbb{O})=(0,1,0),\;\; p(\mathbb{O},\mathbb{O},\mathbb{I})=(0,0,1), \tag{4.7}$$

$$l(\mathbb{I},\mathbb{O},\mathbb{O})=(1,0,0),\;\; l(\mathbb{O},\mathbb{I},\mathbb{O})=(0,1,0),\;\; l(\mathbb{O},\mathbb{O},\mathbb{I})=(0,0,1). \tag{4.8}$$

Now let $u\in\mathcal{H}$ be a vector whose j-th coordinate belongs to $\mathbb{Z}\langle\mathcal{R}\rangle$, that is, this coordinate does not depend on $x_1,\dots,x_n$, so, as a function, it identically equals to some $u_j\in\mathcal{R}$. Straightforwardly, if we now write e_j for the length three vector with $\mathbb{I}$ at the j-th position and with $\mathbb{O}$'s everywhere else, the product $e_j\cdot u$ equals the j-th coordinate of u, and we get

$$\mathcal{B}(e_j|u)=u_j \tag{4.9}$$

by the item (B1) of Definition 4.7, and hence

$$L(j|u)=p(e_j)\cdot l(u)=u_j, \tag{4.10}$$

where the first equality comes from (4.7), and the second equality is deduced from (4.9) by the condition (4.6). From (4.10) we get that $L(j|u)$ equals the value of $\mathcal{U}(j|u)$ whenever $\mathcal{U}(j|u)\in\mathbb{Z}\langle\mathcal{R}\rangle$, and, either by the symmetry of the construction or with a similar argument using (4.8) instead of (4.7), we get that $P(u|j)$ equals the result of the evaluation of $\mathcal{U}(j|u)$, again provided that $\mathcal{U}(j|u)\in\mathbb{Z}\langle\mathcal{R}\rangle$.

Step 3. Using Step 2, we get that, for any variable x_i, there is $y_i\in\mathcal{S}$ such that

$$l(\mathbb{I},\mathbb{O},x_i)=(1,0,y_i). \tag{4.11}$$

In what follows, we denote by y_i the element of $\mathcal{S}$ such that the formula (4.11) is satisfied. We also write $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$.

Step 4. We say that the label $u=(a,b,c)$ is *p-good* if

$$p(u)=(a(y),b(y),c(y)),$$

where $a(y)$ is the result of the evaluation of the polynomial $a\in\mathbb{Z}\langle\mathcal{R},x_1,\dots,x_n\rangle$ at the point $y=(y_1,\dots,y_n)$. Similarly, we say that u is *l-good* if

$$l(u)=(a(y),b(y),c(y)).$$

By Step 2, the labels consisting of elements in $\mathbb{Z}\langle\mathcal{R}\rangle$ are both necessarily p-good and l-good. In order to complete the proof, we need to check that

(i) every label in $\mathcal{H}$ is both p-good and l-good, and

(ii) $f(y)=0$ for all $f\in F$.

Step 5. Now let us see what happens if a vector $(g,\mathbb{O},h)$ is l-good.

Step 5.1. Since either $g=\pm\mathbb{I}$ or $h=\pm\mathbb{I}$ by Definition 4.5, we have $gh=hg$ and

$$(-h,g,g)\cdot(g,\mathbb{O},h)=(-h,g,g)\cdot(\mathbb{O},-\mathbb{I},\mathbb{I})=\mathbb{O}, \tag{4.12}$$

so we can use the item (B1) of Definition 4.7 to get

$$\mathcal{B}(-h,g,g|g,\mathbb{O},h)=\mathcal{B}(-h,g,g|\mathbb{O},-\mathbb{I},\mathbb{I})=0, \tag{4.13}$$

and then the application of (4.6) to (4.13) gives

$$p(-h,g,g)\cdot l(g,\mathbb{O},h)=p(-h,g,g)\cdot l(\mathbb{O},-\mathbb{I},\mathbb{I})=0. \tag{4.14}$$

Since the vector $(g,\mathbb{O},h)$ is l-good by the assumption of Step 5, and the vector $(\mathbb{O},-\mathbb{I},\mathbb{I})$ is l-good by the result of Step 2, the equalities (4.14) imply

$$p(-h,g,g)\cdot(g(y),0,h(y))=p(-h,g,g)\cdot(0,-1,1)=0$$

or that $p(-h, g, g) = (\pi_1, \pi_2, \pi_3) \in \mathcal{S}^3$ with

$$\pi_1 g(y) + \pi_3 h(y) = 0 \text{ and } \pi_2 = \pi_3. \tag{4.15}$$

Again, since either $g = \pm\mathbb{I}$ or $h = \pm\mathbb{I}$, the equalities (4.15) can be used to check that $(-h, g, g)$ is a p-good vector[1]. A similar consideration starting from the equalities

$$(-h, g, g) \cdot (g, h, \mathbb{O}) = (\mathbb{O}, \mathbb{O}, \mathbb{I}) \cdot (g, h, \mathbb{O}) = \mathbb{O},$$

taken instead of (4.12), shows that the vector $(g, h, \mathbb{O})$ is l-good.

Step 5.2. Still assuming that $(g, \mathbb{O}, h)$ is l-good, we deduce that $(-h, \mathbb{O}, g)$ is p-good by the argument as in Step 5.1 starting from

$$(-h, \mathbb{O}, g) \cdot (g, \mathbb{O}, h) = (-h, \mathbb{O}, g) \cdot (\mathbb{O}, \mathbb{I}, \mathbb{O}) = \mathbb{O}.$$

Similarly, the condition that $(-h, \mathbb{O}, g)$ is p-good in turn implies that $(g, \mathbb{O}, h)$ is l-good, again by the same argument but starting from

$$(-h, \mathbb{O}, g) \cdot (g, \mathbb{O}, h) = (\mathbb{O}, \mathbb{I}, \mathbb{O}) \cdot (g, \mathbb{O}, h) = \mathbb{O}.$$

Step 5.3. Now we can get the main conclusion of Step 5, using the symmetry and results of Steps 5.1 and 5.2. Namely, a vector $(g, \mathbb{O}, h)$ is l-good if and only if any permutation of $(g, \mathbb{O}, h)$ is l-good, which in turn happens if and only if any permutation of $(-h, \mathbb{O}, g)$ is p-good.

Step 6. Now we assume that $(\mathbb{I}, \mathbb{O}, \alpha)$, $(\mathbb{I}, \mathbb{O}, \beta)$ are l-good vectors.

Step 6.1. Further, we assume that $\alpha + \beta$ is in the set σ from Definition 4.5. Using the argument outlined in Step 5.1 but with

$$(\mathbb{I}, \mathbb{I}, \mathbb{O}) \cdot (-\mathbb{I}, \mathbb{I}, \alpha) = (\mathbb{O}, -\alpha, \mathbb{I}) \cdot (-\mathbb{I}, \mathbb{I}, \alpha) = \mathbb{O}$$

instead of (4.12), we conclude that $(-\mathbb{I}, \mathbb{I}, \alpha)$ is an l-good vector[2]. Similarly, the vector $(-\beta, -\alpha - \beta, \mathbb{I})$ can be shown to be p-good starting from the equalities

$$(-\beta, -\alpha - \beta, \mathbb{I}) \cdot (-\mathbb{I}, \mathbb{I}, \alpha) = (-\beta, -\alpha - \beta, \mathbb{I}) \cdot (\mathbb{I}, \mathbb{O}, \beta) = \mathbb{O}$$

involving the l-good vectors $(-\mathbb{I}, \mathbb{I}, \alpha)$ and $(\mathbb{I}, \mathbb{O}, \beta)$. Finally, the fact that the vector $(\mathbb{O}, \mathbb{I}, \alpha + \beta)$ is l-good can be shown from the equalities

$$(\mathbb{I}, \mathbb{O}, \mathbb{O}) \cdot (\mathbb{O}, \mathbb{I}, \alpha + \beta) = (-\beta, -\alpha - \beta, \mathbb{I}) \cdot (\mathbb{O}, \mathbb{I}, \alpha + \beta) = \mathbb{O}$$

involving the p-good vectors $(-\beta, -\alpha - \beta, \mathbb{I})$ and $(\mathbb{I}, \mathbb{O}, \mathbb{O})$. Therefore, we can conclude that the vector $(\mathbb{I}, \mathbb{O}, \alpha + \beta)$ is l-good by the result of Step 5.3.

Step 6.2. Now we switch to the case when $\beta\alpha$ belongs to σ. Then, the vector $(\beta\alpha, \mathbb{I}, \alpha)$ is l-good by the argument similar to that in Step 5.1 but starting from

$$(\mathbb{O}, -\alpha, \mathbb{I}) \cdot (\beta\alpha, \mathbb{I}, \alpha) = (\mathbb{I}, \mathbb{O}, -\beta) \cdot (\beta\alpha, \mathbb{I}, \alpha) = \mathbb{O}$$

instead of (4.12), where the vectors $(\mathbb{O}, -\alpha, \mathbb{I})$ and $(\mathbb{I}, \mathbb{O}, -\beta)$ are p-good by Step 5. Similarly, the vector $(\mathbb{I}, -\beta\alpha, \mathbb{O})$ is p-good by the same argument starting with

$$(\mathbb{I}, -\beta\alpha, \mathbb{O}) \cdot (\mathbb{O}, \mathbb{O}, \mathbb{I}) = (\mathbb{I}, -\beta\alpha, \mathbb{O}) \cdot (\beta\alpha, \mathbb{I}, \alpha) = \mathbb{O}.$$

Therefore, the vector $(\mathbb{I}, \mathbb{O}, \beta\alpha)$ is l-good by Step 5.3.

Step 7. The results of Step 6 show that the vector $(\mathbb{I}, \mathbb{O}, s)$ is l-good for all $s \in \sigma$.

---

[1] For instance, if $g = \mathbb{I}$, then $\pi_2 = \pi_3 = 1$ by the result of Step 2, and we get $\pi_1 = -h(y)$ from the first equality in (4.15). The cases when $g = -\mathbb{I}$ or $h = \pm\mathbb{I}$ can be treated in a similar fashion.

[2] This is possible because $(\mathbb{I}, \mathbb{I}, \mathbb{O})$ is p-good by Step 2 and $(\mathbb{O}, -\alpha, \mathbb{I})$ is p-good by Step 5.

Step 8. In order to check the condition (i) as in Step 4, because of the symmetry, it is sufficient to check that the vector $(\mathbb{I}, u, v)$ is l-good whenever u, v are taken in σ. Again, this follows from the argument similar to Step 5.1 applied to the equalities

$$(-u, \mathbb{I}, \mathbb{O}) \cdot (\mathbb{I}, u, v) = (-v, \mathbb{O}, \mathbb{I}) \cdot (\mathbb{I}, u, v) = \mathbb{O}$$

involving the vectors $(-u, \mathbb{I}, \mathbb{O})$ and $(-v, \mathbb{O}, \mathbb{I})$ that are p-good by Steps 5 and 7.

Step 9. In order to check the condition (ii) as in Step 4, we consider an arbitrary polynomial f in F. Since we have

$$(\mathbb{O}, \mathbb{O}, \mathbb{I}) \cdot (\mathbb{I}, \mathbb{O}, f) = f,$$

we get $\mathcal{B}(\mathbb{O}, \mathbb{O}, \mathbb{I}|\mathbb{I}, \mathbb{O}, f) = 0$ by the item (B2) of Definition 4.7. This implies

$$p(\mathbb{O}, \mathbb{O}, \mathbb{I}) \cdot l(\mathbb{I}, \mathbb{O}, f) = 0$$

by the condition (4.6), and hence

$$(0, 0, 1) \cdot (1, 0, f(y)) = 0 \tag{4.16}$$

because, according to Step 8, the vectors $(\mathbb{O}, \mathbb{O}, \mathbb{I})$, $(\mathbb{I}, \mathbb{O}, f)$ are both p-good and l-good. Since the equality (4.16) means that $f(y) = 0$, the proof is complete. □

The following corollary is immediate from Lemma 4.9.

Corollary 4.10. *The matrix $\mathcal{B}(F)$ admits a completion of rank three with respect to $\mathcal{S}$ if and only if the equations $f_1 = 0, \ldots, f_t = 0$ have a common solution in $\mathcal{S}$.*

Proof. As said above, the matrix $\mathcal{W}(\xi_1, \ldots, \xi_n)$ is a rank-three completion of $\mathcal{B}$, provided that $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution of the polynomial equations $f_1 = 0, \ldots, f_t = 0$. Conversely, if there is no such a solution over $\mathcal{S}$, then by Lemma 4.9 the matrix $\mathcal{B}$ admits no completion of rank three with respect to $\mathcal{S}$. □

Since the reduction $F \to \mathcal{B}(F)$ is polynomial time, we get Theorem 3.3 from Corollary 4.10. Now we proceed with the application of our technique to the problems of Grossmann and Woerdeman [23]. Namely, for a positive integer b and an incomplete $m \times n$ matrix A with known entries in a field $\mathcal{F}$, they define $A \otimes I_b$ to be the $mb \times nb$ incomplete matrix seen as the $m \times n$ block matrix in which

(1) if $A(i|j) = *$, then the (i, j) block is the $b \times b$ block of $*$'s,
(2) if $A(i|j) \neq *$, then the (i, j) block is the scalar matrix $A(i|j)I_b$.

The authors of [23] define the *fractional minimal rank* of A as

$$\operatorname{fmr}(A) = \inf_b \frac{\min \operatorname{rk} A \otimes I_b}{b} \tag{4.17}$$

with the numerator being the minimal rank of any completion of $A \otimes I_b$ over $\mathcal{F}$. Question 4 in Section 5 of [23] asked, is the infimum in (4.17) necessarily attained? We give a negative resolution of this question. We decided to work with $\mathcal{F} = \mathbb{Q}$, but the following construction is vaild over any field of characteristic zero.

Example 4.11. Let $\mathcal{R} = \mathcal{F} = \mathbb{Q}$ and consider the matrix $B = \mathcal{B}(xy - yx + 1)$ as in Example 4.8. Then $\operatorname{fmr}(B) = 3$ and $\min \operatorname{rk} B \otimes I_b \geqslant 3b + 1$ for all b.

Proof. For $b > 1$, we can think of $B \otimes I_b$ as the matrix $\mathcal{B}(xy - yx + 1)$ but constructed with respect to the ring $\mathcal{R} = \operatorname{Mat}_b(\mathbb{Q})$ instead of $\mathcal{R} = \mathbb{Q}$. We note that the polynomial $xy - yx + 1$ cannot be vanished by matrices over a field of characteristic zero because the traces of xy and yx are equal. We apply Lemma 4.9, which is valid in this setting because of Remark 4.1, and we conclude that no completion of $B \otimes I_b$ can be represented as the product PL of rational matrices unless P has more than $3b$ columns. In other words, we have $\min\operatorname{rk} B \otimes I_b \geqslant 3b+1$ for all b.

Now let D_b be the $b \times b$ matrix with numbers $1, \ldots, 1, 1-b$ on the main diagonal and zeros everywhere else. A standard result of matrix theory [55] shows that every trace-zero matrix over $\mathbb{Q}$ is a commutator, so, if we considered the matrix B'_b defined as $\mathcal{B}(xy - yx + D_b)$ with respect to the ring $\mathcal{S} = \operatorname{Mat}_b(\mathbb{Q})$, we would have

$$\min\operatorname{rk} B'_b \leqslant 3b.$$

As we see from the discussion in Example 4.8, the matrix $B \otimes I_b$ can be obtained from B'_b by altering a fixed number of its entries, which implies

$$|\min\operatorname{rk} B \otimes I_b - \min\operatorname{rk} B'_b| = O(1) \text{ as } b \to \infty$$

and thus $\operatorname{fmr}(B) = 3$. □

Our technique allows one to solve another problem in [23]. The formulation of this problem requires the notion of the *triangular minimum rank* tmr of a partial matrix, but, since the corresponding definition is relatively complicated, we decided not to reproduce it here. The relevant properties are that

(T1) $\operatorname{tmr}(A) \leqslant \operatorname{fmr}(A) \leqslant \min\operatorname{rk} A$ for any incomplete matrix A, and
(T2) $\operatorname{tmr}(A) \geqslant \operatorname{rk} A'$ for any complete submatrix A' of A.

Conjecture 4.12 (Problem 3 in Section 5 of [23])**.** For all incomplete matrices A, we have either $\operatorname{tmr}(A) < \operatorname{fmr}(A) < \min\operatorname{rk} A$ or $\operatorname{tmr}(A) = \operatorname{fmr}(A) = \min\operatorname{rk} A$.

Using the matrix B in Example 4.11, we can disprove this conjecture.

Example 4.13. We have $\operatorname{tmr}(B) = \operatorname{fmr}(B) = 3$ and $\min\operatorname{rk} B > 3$.

Proof. The equality $\operatorname{fmr}(B) = 3$ is immediate from Example 4.11. Further, we have $\min\operatorname{rk} B > 3$ by Corollary 4.10 because the equation $xy - yx + 1 = 0$ has no solutions over $\mathbb{Q}$. Also, we have $\operatorname{tmr}(B) \geqslant 3$ from the condition (T2) because B contains a unit 3×3 submatrix (namely, this is the submatrix with the row and column indexes in the set E as in Lemma 4.9). Finally, we have $\operatorname{tmr}(B) \leqslant 3$ from the condition (T1) because we already know that $\operatorname{fmr}(B) = 3$. □

5. A proof of Theorem 2.3

We switch to the setting of Theorem 2.3, so many results of this section require that the rings $\mathcal{R} \subseteq \mathcal{S}$ are integral domains, that is, they are commutative and have no zero divisors. In several lemmas, we will need to refer to the construction of the matrix $\mathcal{B}(F)$ and the corresponding family of polynomials $F = \{f_1, \ldots, f_t\}$ as in the previous section. The matrix $\mathcal{B}(F)$ is simply denoted as $\mathcal{B}$.

Lemma 5.1. *Let $\mathcal{F}$ be a field containing an integral domain $\mathcal{S}$. Assume that W_1, W_2, W_3 are rank-one matrices over $\mathcal{F}$ such that $W_1 + W_2 + W_3$ is a completion of $\mathcal{B}$. Assume that, for some $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{F}$, some rank-one matrix W_0 coincides with $\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3$ everywhere except possibly several entries that are $*$'s in $\mathcal{B}$. Then there is an element $\mu \in \mathcal{F}$ such that W_0 is one of $\mu W_1, \mu W_2, \mu W_3$.*

Proof. We define the *E-submatrix* of $\mathcal{B}$ as the one formed by the rows and columns with indexes in $E = \{(\mathbb{I}, \mathbb{O}, \mathbb{O}), (\mathbb{O}, \mathbb{I}, \mathbb{O}), (\mathbb{O}, \mathbb{O}, \mathbb{I})\}$. We note that this submatrix is the unity matrix and, in particular, it does not contain the $*$'s. The corresponding E-submatrices of W_1, W_2, W_3 are rank-one and sum to a rank-three matrix, so the rank of the E-submatrix of $\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3$ equals the number of those λ_i's that are nonzero. Therefore, it suffices to consider the case when W_0 coincides with W_3 everywhere except possibly several entries that are $*$'s in $\mathcal{B}$. We are going to show that $W_0 = W_3$; for $j = 0, 1, 2, 3$, we write

$$W_j = a_j b_j^\top \text{ with } a_j, b_j \in \mathcal{F}^{\mathcal{H}}.$$

Using an appropriate scaling, we assume that the coordinates of a_0 and b_0 with indexes in E are equal to the corresponding coordinates of a_3 and b_3.

We define P as the matrix formed by the columns a_1, a_2, a_3, and we take L to be the matrix formed by the rows $b_1^\top, b_2^\top, b_3^\top$. The matrix $PL = W_1 + W_2 + W_3$ is a completion of $\mathcal{B}$, and Lemma 4.9 implies that

$$C^{-1}L = \mathcal{U}(\xi_1, \ldots, \xi_n),$$

where $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution of $f_1 = 0, \ldots, f_t = 0$, and C is the 3×3 matrix formed by the columns of L with indexes in E. Similarly, we define Q as the matrix formed by the rows $b_1^\top, b_2^\top, b_0^\top$, and we get that

$$C^{-1}Q = \mathcal{U}(\psi_1, \ldots, \psi_n),$$

where $(\psi_1, \ldots, \psi_n)$ is another simultaneous solution of $f_1 = 0, \ldots, f_t = 0$. The entries of $Q - L$ are all zero except possibly those in the third row, so the matrix

$$\mathcal{U}(\psi_1, \ldots, \psi_n) - \mathcal{U}(\xi_1, \ldots, \xi_n) = C^{-1}(Q - L) \tag{5.1}$$

has rank at most one. By its definition, the set $\mathcal{H}$ consists of vectors one of whose coordinates is constant, so that the matrix (5.1) has a zero in every column. Since the rank of (5.1) is at most one, it has a zero row, which suffices to conclude that $(\psi_1, \ldots, \psi_n) = (\xi_1, \ldots, \xi_n)$. Therefore, the matrix (5.1) is zero, which means that $Q = L$ or $b_0 = b_3$. Using the symmetry, we get $a_0 = a_3$ and $W_0 = W_3$. $\square$

We are going to construct a reduction from the matrix completion problem to tensor rank, and we use the construction that previously appeared in [19].

Definition 5.2. Let $\mathcal{B}$ be the matrix as in Section 4; we recall that the rows and columns of $\mathcal{B}$ have indexes in the set $\mathcal{H}$. We enumerate by

$$k_1 = (i_1, j_1), \ \ldots, \ k_\tau = (i_\tau, j_\tau)$$

the entries which correspond to the $*$'s in $\mathcal{B}$, so τ is the number of such entries. We set $K = \{0, 1, \ldots, \tau\}$, and we define the $\mathcal{H} \times \mathcal{H} \times K$ tensor $\mathcal{A} = \mathcal{A}(\mathcal{B})$ as follows:

(1) $\mathcal{A}(u|v|t) = \mathcal{B}(u|v)$ if $t = 0$ and $\mathcal{B}(u|v) \neq *$,
(2) $\mathcal{A}(u|v|t) = 1$ if $k_t = (u, v)$,
(3) $\mathcal{A}(u|v|t) = 0$ in the remaining cases.

In other words, we begin by taking the matrix $\mathcal{B}$ as in Section 4, and we substitute its $*$'s with zeros to obtain the matrix which we further call A. Then we get $\mathcal{A}$ by the addition to A of the τ new 3-slices equal to the matrix units corresponding to the positions of the $*$ entries of $\mathcal{B}$. Derksen [19] showed that

$$\operatorname{rk}_{\mathcal{S}} \mathcal{A}(\mathcal{B}) = \tau + \min \operatorname{rk}_{\mathcal{S}} \mathcal{B} \tag{5.2}$$

in the case when $\mathcal{S}$ is a field; this result comes from Lemma 3.2 as well. We are going to adapt the substitution technique and prove an appropriate analogue of (5.2) for any integral domain $\mathcal{S}$. More precisely, we prove that the inequality $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \leqslant \tau + 3$ is valid if and only if $\mathcal{B}$ admits a completion of rank three with respect to $\mathcal{S}$.

Lemma 5.3. *If $\mathcal{S}$ is an integral domain, then* $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \geqslant \tau + 3$.

Proof. Since $\mathcal{S}$ is an integral domain, there is a field containing $\mathcal{S}$, and the assertion follows from Lemma 3.2 or from the above mentioned result by Derksen. □

Lemma 5.4. *Let $\mathcal{S}$ be an integral domain. If $\mathcal{B}$ admits a completion of rank three with respect to $\mathcal{S}$, then* $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \leqslant \tau + 3$.

Proof. If B is such a completion, then we can get a tensor B_0 of rank three over $\mathcal{S}$ by setting $B_0(u|v|t) = B(u|v)$ if $t = 0$ and $B_0(u|v|t) = 0$ otherwise. Further, we define a rank-one tensor S_t whose entries are all zeros except $S_t(i_t|j_t|0) = -B(i_t|j_t)$ and $S_t(i_t|j_t|t) = 1$. We get $\mathcal{A} = B_0 + S_1 + \ldots + S_\tau$, so the result follows. □

Lemma 5.5. *If $\mathcal{S}$ is an integral domain and* $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \leqslant \tau + 3$*, then $\mathcal{B}$ admits a completion of rank three with respect to $\mathcal{S}$.*

Proof. Let $\mathcal{F}$ be a field containing $\mathcal{S}$, and let

$$\mathcal{A} = S_1 + \ldots + S_{\tau+3}$$

be a decomposition of $\mathcal{A}$ into the sum of tensors that are rank-one with respect to $\mathcal{S}$. Let V be the $\mathcal{F}$-linear space spanned by the zeroth 3-slices of $S_1, \ldots, S_{\tau+3}$ with the coordinates $k_1, \ldots, k_\tau$ removed. Since the 3-slices of $\mathcal{A}$ with indexes $1, \ldots, \tau$ are linearly independent and have zeros outside $k_1, \ldots, k_\tau$, we get $\dim V \leqslant 3$. We say that a 3-slice is *non-trivial* if it has a non-zero element somewhere except $k_1, \ldots, k_\tau$.

Therefore, if there were at least four S_i's with non-trivial zeroth 3-slices, then these 3-slices would become linearly dependent after the removal of the $*$ positions. Using Lemma 5.1, we would get that there are two S_i's whose zeroth 3-slices are non-zero and coincide up to scalings by nonzero elements of $\mathcal{F}$. The sum of these two S_i's would still be a simple tensor with respect to $\mathcal{F}$, which would imply $\operatorname{rk}_{\mathcal{F}} \mathcal{A} \leqslant \tau + 2$ and contradict to Lemma 5.3. Therefore, there are at most three S_i's with non-trivial zeroth 3-slices, and the sum of these 3-slices is a desired completion of $\mathcal{B}$. □

Lemmas 5.4 and 5.5 prove that $\operatorname{rk}_{\mathcal{S}} \mathcal{A}(\mathcal{B}) \leqslant \tau + 3$ if and only if $\mathcal{B}$ admits a completion of rank three with respect to an integral domain $\mathcal{S}$. By Corollary 4.10, such a completion exists if and only if the polynomials in the family F as in Section 4 have a common zero over $\mathcal{S}$. Since the reduction $F \to \mathcal{A}(\mathcal{B}(F))$ can be computed in polynomial time, we complete the proof of Theorem 2.3.

6. Symmetric tensors

The goal of this section is to prove Theorem 2.7 and, more generally, the analogue of Theorem 2.3 for the symmetric rank of tensors over a field. Our argument is a reduction of the standard tensor rank problem to the symmetric version.

Definition 6.1. We say that a tensor T_0 is obtained from an $I \times J \times K$ tensor T by *adjoining* an $I \times J$ matrix A as a 3-slice if the 3-slices of T_0 are precisely those of T and A. We use similar definitions for adjoining 1-slices and 2-slices.

Definition 6.2. For all $p, q \in I \cap J$, we define the (p,q)*-unit* as the $I \times J$ matrix M such that $M(i|j) = 1$ if $i, j \in \{p, q\}$ and $M(i|j) = 0$ otherwise. In particular, such a matrix becomes a conventional matrix unit whenever $p = q$.

Now we are ready to present the main tool of this section.

Definition 6.3. Let I, J, K be disjoint indexing sets, and let T be an $I \times J \times K$ tensor over a field $\mathcal{F}$. We define the new indexing set $H = I \cup J \cup K$ and the $H \times H \times H$ tensor $S = S(T)$ as follows. For all $\alpha, \beta, \gamma \in H$, we take

(S1) $S(\alpha|\beta|\gamma) = T(i|j|k)$ if (α, β, γ) is a permutation of $(i, j, k) \in I \times J \times K$,

(S2) $S(\alpha|\beta|\gamma) = 0$ otherwise.

Definition 6.4. Let I, J, K, H be the indexing sets as in Definition 6.3, and let S be an $H \times H \times H$ tensor over $\mathcal{F}$. We define I^2 as the set of all $\{p, q\}$ with p, q in I. The sets J^2, K^2 are defined similarly, and we denote $\mathcal{H} = H \cup I^2 \cup J^2 \cup K^2$. We define the $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ tensor $\mathcal{T} = \mathcal{T}(S)$ by adjoining of the π-unit 1-slices to S, and then the subsequent adjoining of the π-unit 2-slices and the π-unit 3-slices to the resulting tensors. Here, an index π runs over the set $I^2 \cup J^2 \cup K^2$.

Remark 6.5. In Definition 6.4 and in what follows, we use $\pi \in I^2 \cup J^2 \cup K^2$ as the label of the slice of $\mathcal{T}(S(T))$ corresponding to the adjoined π-unit matrix.

Remark 6.6. The resulting tensor $\mathcal{T}(S(T))$ is symmetric.

In what follows, we assume $|I| = |J| = |K| = n$, and we are going to prove that

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(S(T)) = \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2 + n) \tag{6.1}$$

whenever $|\mathcal{F}| \geqslant 4$. This would show that $T \to \mathcal{T}(S(T))$ is a polynomial time many-one reduction from the standard rank problem to the symmetric one; we note that the assumption $|I| = |J| = |K|$ does not cause a loss of generality because the rank of a tensor remains unchanged if one adjoins several zero slices to it. In particular, the equality (6.1) implies Theorem 2.7, which is the main goal of this section. Therefore, the rest of our paper is devoted to the confirmation of (6.1).

Lemma 6.7. *Let $S(T)$ be the tensor as in Definition 6.3, and $\mathcal{T}(S(T))$ be the tensor as in Definition 6.4. Then* $\operatorname{rk}_{\mathcal{F}} \mathcal{T}(S(T)) \geqslant \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2 + n)$.

Proof. Let M_3 be a linear combination of the 3-slices of $\mathcal{T}$ with indexes in $I^2 \cup J^2 \cup K^2$. By Definition 6.4, all non-zero entries of these slices belong to the blocks $(I|I)$, $(J|J)$, $(K|K)$, and the same conclusion holds for M_3. Therefore, the addition of M_3 to any slice of $\mathcal{T}$ does not change its $(I|J|K)$ block. Similarly, the addition of any linear combination of the 2-slices (and 1-slices, afterwards) with indexes in $I^2 \cup J^2 \cup K^2$ to any 2-slice (or 1-slice, respectively) of the resulting tensor does not affect its $(I|J|K)$ block.

We apply Lemma 3.2 to the 1-slices with indexes in $I^2 \cup J^2 \cup K^2$, then to the 2-slices with these indexes, and then to the 3-slices. We get the desired inequality because the total number of adjoined linearly independent slices is

$$3(|I^2| + |J^2| + |K^2|) = 4.5(n^2 + n)$$

and because the $(I|J|K)$ block of $\mathcal{T}$ is T. $\square$

Since the rank cannot exceed the symmetric rank, Lemma 6.7 implies

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(S(T)) \geqslant \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2 + n),$$

which gives one direction of the equality (6.1). Our proof of the opposite direction is more technical, and we need some more notation.

Definition 6.8. Let T be an $I \times I \times I$ symmetric tensor over a field $\mathcal{F}$. Let ρ be a permutation of I, and let (f_i) be a family of non-zero elements of $\mathcal{F}$, where the index i runs over I. We say that the tensor whose $(i|j|k)$ entry equals

$$f_i\, f_j\, f_k\, T(\rho_i|\rho_j|\rho_k)$$

is obtained from T by a *monomial transformation*. Indexes $i, \hat{\imath} \in I$ are called *twins* for T if the $\hat{\imath}$th 1-, 2-, and 3-slices are equal to the corresponding ith slices. The *removal* of an index i is the operation of restricting T to the indexing set $I \setminus \{i\}$.

Observation 6.9. *A monomial transformation and the removal of a twin do not change the symmetric ranks of a given tensor.*

Lemma 6.10. *Let $\mathcal{F}$ be a field with $|\mathcal{F}| \geqslant 4$, and let x be a scalar in $\mathcal{F}$. We define the $2 \times 2 \times 2$ symmetric tensor A such that*

$$A(1|1|1) = x,\ A(1|1|2) = 1,\ A(1|2|2) = A(2|2|2) = 0.$$

Then $\operatorname{srank}_{\mathcal{F}} A \leqslant 3$.

Proof. We define the values

$$q = \frac{p}{px-1},\ s_1 = \frac{1}{p(2-px)},\ s_2 = \frac{(px-1)^2}{p(px-2)},\ s_3 = \frac{p^2}{px-1}$$

depending on a parameter p, and we check that

$$A = s_1(1,p)^{\otimes 3} + s_2(1,q)^{\otimes 3} + s_3(0,1)^{\otimes 3}$$

provided that we can choose a value of p satisfying $0 \notin \{p, 1-px, 2-px\}$. This is possible unless we have both $x=0$ and $2=0$ at the same time, but then

$$A = \sigma_1(1,1)^{\otimes 3} + \sigma_2(1,q)^{\otimes 3} + \sigma_3\left(1, \frac{q}{q+1}\right)^{\otimes 3}$$

with

$$\sigma_1 = \frac{q^2}{q+1},\ \sigma_2 = \frac{1}{q^3+q^2},\ \sigma_3 = \frac{(q+1)^3}{q^2}$$

and $q \notin \{0,1\}$. $\square$

Further, let us define an (i,j)th 3-*transversal* of a tensor T as the set of entries in which the first two coordinates are equal to i and j, respectively. The notions of 1- and 2-transversals are defined in a similar way.

Lemma 6.11. *Let I, J, K, H be the indexing sets as in Definition 6.3, and let U be a symmetric $H \times H \times H$ tensor over a field $\mathcal{F}$ with $|\mathcal{F}| \geqslant 4$. If $U(i|j|k) = 0$ whenever $i \in I$, $j \in J$, $k \in K$, then*

$$\operatorname{srk}_{\mathcal{F}}\ \mathcal{T}(U) \leqslant 4.5(n^2+n),$$

where $\mathcal{T}(U)$ is the tensor as in Definition 6.4.

Proof. We take an arbitrary total order $\succeq$ on $I \cup J \cup K$, and we write $p \succ q$ if $p \succeq q$ and $p \neq q$. For any $X \in \{I, J, K\}$ and $\pi = \{p, q\} \in X^2$ with $p \succ q$, we define the $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ tensor $\mathcal{L}_\pi$ as follows. For all $r, s \in \{p, q\}$, $h \in H$, $x, y, z \in \mathcal{H}$, we set

(L1) $\mathcal{L}_\pi(r|s|h) = U(p|q|h)$ if either $h \notin X$ or $h \succ p$,

(L2) $\mathcal{L}_\pi(r|s|\pi) = 1$,

(L3) $\mathcal{L}_\pi(z|x|y) = \mathcal{L}_\pi(y|z|x) = \mathcal{L}_\pi(x|y|z)$ if at least one of these is already defined,

(L4) the entries which are not yet defined are zero.

Every $\mathcal{L}_\pi$ can be reduced to the tensor as in Lemma 6.10 by the transformations as in Observation 6.9. We have

$$\operatorname{srk}_{\mathcal{F}} \mathcal{L}_\pi \leqslant 3,$$

so the result of the lemma would follow if we check that the tensor

$$\Phi = \mathcal{T}(U) - \sum \mathcal{L}_\pi$$

has symmetric rank at most $9n$ with respect to $\mathcal{F}$, where the summation goes over all possibilities of π for which we defined $\mathcal{L}_\pi$ above. We can check that all the non-zero entries of Φ are covered by the union of the (u,u)-th 1-, 2-, 3-transversals over all $u \in H$. We get

$$\Phi = \sum_{u \in H} \mathcal{M}_u,$$

where $\mathcal{M}_u$ is defined as, for all $p, q, r \in \mathcal{H}$,

(M1) $\mathcal{M}_u(p|q|r) = \Phi(p|q|r)$ if u appears at least twice among p, q, r,

(M2) $\mathcal{M}_u(p|q|r) = 0$ otherwise.

This implies $\operatorname{srk}_{\mathcal{F}} \Phi \leqslant 9n$ because each of the $3n$ tensors $\mathcal{M}_u$ has symmetric rank at most three again by Observation 6.9 and Lemma 6.10. □

Lemma 6.12. *Let $\mathcal{T}(S(T))$ be the tensor as in Definition 6.4. If $|\mathcal{F}| \geqslant 4$, then*

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(S(T)) \leqslant \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2 + n).$$

Proof. Assuming that $\operatorname{rk}_{\mathcal{F}} T = r$, we consider an appropriate decomposition

$$T = \sum_{t=1}^{r} a_t \otimes b_t \otimes c_t,$$

where each of (a_t), (b_t), (c_t) is a family of r vectors with indexing sets I, J, K, respectively. We construct the family (w_t) of vectors indexed with $\mathcal{H}$ by setting the I part of w_t equal to a_t, the J part equal to b_t, the K part equal to c_t, and setting all the other entries equal to zero, which means that the $I^2 \cup J^2 \cup K^2$ part of w_t is zero. Now we see that the tensor

$$\mathcal{T}(S(T)) - \sum_{t=1}^{r} w_t \otimes w_t \otimes w_t$$

satisfies the assumptions imposed on the tensor $\mathcal{T}(U)$ as in Lemma 6.11, and the application of this lemma completes the proof. □

Lemmas 6.7 and 6.12 prove the equality (6.1) for any field $\mathcal{F}$ of cardinality at least four. This shows that $T \to \mathcal{T}(S(T))$ is a desired reduction of the tensor rank problem to the symmetric rank, which completes the proof of Theorem 2.7.

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