ON THE GENERAL NO-THREE-IN-LINE PROBLEM

T. AGAMA

ABSTRACT. In this paper we show that the number of points that can be placed in the grid $n \times n \times \cdots \times n$ (d *times*) $= n^d$ for all $d \in \mathbb{N}$ with $d \geq 2$ such that no three points are collinear satisfies the lower bound

$$\gg n^{d-1} \sqrt[2d]{d}.$$

This pretty much extends the result of the no-three-in-line problem to all dimension $d \geq 3$.

1. **Introduction**

The no-three-in-line problem is a well-known problem in discrete geometry that seeks for the maximum number of points that can be placed in an $n \times n$ grid in such a way that no three of the points are collinear. The problem was posed by the then English mathematician Henry Dudeney in 1917. The problem is apparently trivial for all $n \leq 46$, so the only version of the problem still open is for all sufficiently large values of n. Quite a number of progress has been made in the context of obtaining upper and lower lower bounds in the plane and the three dimensional euclidean space. An argument of Erdős (see [3]) yields the lower bound

$$\gg (1 - \epsilon)n$$

for the any $\epsilon > 0$ and n sufficiently large as the number of points that can be placed in the $n \times n$ grid so that no three are collinear. This was improved (see [4]) to

$$\gg \left(\frac{3}{2} - \epsilon\right)n$$

in the grid $n \times n$ with no three collinear. Various upper bound to the problem had also been conjectured. For instance it is conjectured that (see [5]) the number of points that can be placed in an $n \times n$ grid so that no three are collinear has the optimal solution cn with

$$c = \frac{\pi}{\sqrt{3}} \approx 1.814.$$

A generalized version of the problem has also been studied in (see [1]). There it is shown that the number of points that can be placed in an $n \times n \times n$ grid such that no three of them are collinear is $\Theta(n^2)$.

In the current paper we generalize the problem to dimensions $d \geq 2$ under the requirement that our configuration has no three collinear points. By applying the

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method of compression (see [2]), we obtain a lower bound for the number of such points as

$$\gg n^{d-1}\sqrt[2d]{d}.$$

What follows are the lower bound for the grid $n \times n$ and $n \times n \times n$.

In the sequel the notation $f(n) \gg g(n)$ for any $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ would mean there exists some constant $c > 0$ such that $f(n) \geq cg(n)$. In the case the constant depends on some variable, say s then we write simply $f(n) \gg_s g(n)$. We write $f(n) \sim g(n)$ to mean $\lim\limits_{n \longrightarrow \infty} \frac{f(n)}{g(n)} = 1$.

2. **Preliminary results**

Definition 2.1. By the compression of scale $1 \geq m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$, we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of rescaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin. Intuitively, compression induces some kind of motion on points in the Euclidean space $\mathbb{R}^n$ for $n \geq 2$.

Proposition 2.1. *A compression of scale $1 \geq m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. **The mass of compression.**
In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $1 \geq m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1\leq i,j\leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ is such that $x_i\neq x_j$ for $1\leq i,j\leq n$.

Lemma 2.4. *We have*

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $1\geq m>0$.

Proposition 2.2. *Let* $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ *with* $x_i\neq 0$ *for each* $1\leq i\leq n$ *and* $x_i\neq x_j$ *for* $i\neq j$*, then we have*

$$m\log\left(1-\frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \ll m\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n\geq 2$.

Proof. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$ with $x_j\neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

$\square$

Definition 2.6. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale $m > 0$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

3. The ball induced by compression

In this section we introduce the notion of the ball induced by a point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of a given scale. We launch more formally the following language.

Definition 3.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and $x_i \neq 0$ for all $1 \leq i \leq n$. Then by the ball induced by $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of scale $1 \geq m > 0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| < \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ if it satisfies the inequality.

Remark 3.2. Next we prove that smaller balls induced by points should essentially be covered by the bigger balls in which they are embedded. We state and prove this statement in the following result.

In the geometry of balls induced under compression of scale $m > 0$, we assume implicitly that

$$0 < m \leq 1.$$

For simplicity we will on occasion choose to write the ball induced by the point $\vec{x} = (x_1, x_2, \ldots, x_n)$ under compression as

$$\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}]}[\vec{x}].$$

We adopt this notation to save enough work space in many circumstances. We first prove a preparatory result in the following sequel. We find the following estimates for the compression gap useful.

Proposition 3.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

In particular, if $m = m(n) = o(1)$ as $n \longrightarrow \infty$, then we have the estimate

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

for $\vec{x} \in \mathbb{R}^n$ with $x_i \geq 1$ for each $1 \leq i \leq n$.

Proposition 3.1 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

with $m := m(n) = o(1)$ as $n \longrightarrow \infty$ if and only if $||\vec{x}|| \lesssim ||\vec{y}||$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $x_i \geq 1$ for all $1 \leq i \leq n$. This important transference principle will be mostly put to use in obtaining our results. In particular, we note that in the latter case, we can write the asymptotic

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \sim \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] = ||\vec{x}||^2.$$

Corollary 3.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq x_i$ for $j \neq i$ and $x_i, x_j \geq 1$ for each $1 \leq i, j \leq n$. If $m := m(n) = o(1)$ as $n \longrightarrow \infty$, then we have

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \geq n\mathrm{Inf}(x_j^2) - 2mn + O\left(m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \leq n\mathrm{sup}(x_j^2) - 2mn + O\left(m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

Lemma 3.3 (Compression estimate)**.** *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_i \geq 1$ for all $1 \leq i \leq n$ with $x_i \neq x_j$ $(i \neq j)$. If $m := m(n) = o(1)$ as $n \longrightarrow \infty$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\mathrm{sup}(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1} - 2mn.$$

Remark 3.4. It is important to note that the inequality in Corollary 3.1 implies the inequalities in Lemma 3.3. At any given moment, we will decide which of the versions of these inequalities to use. Indeed the inequalities in Corollary 3.1 are mostly applicable to various problems that the one in Lemma 3.3.

Theorem 3.5. *Let $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$ with $z_i \geq 1$ for all $1 \leq i \leq n$ and $m := m(n) = o(1)$ as $n \longrightarrow \infty$. Then $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}|| < ||\vec{y}||$ if and only if*

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

with $||\vec{y} - \vec{z}|| < \epsilon$ for some $\epsilon > 0$.

Proof. Let $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$ and $z_i \geq 1$ for all $1 \leq i \leq n$ such that $||\vec{y}|| > ||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}|| \lesssim ||\vec{z}||$, which is absurd. In this case, we can take $\epsilon := \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}]$. Conversely, suppose

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 3.1 that $||\vec{z}|| \lesssim ||\vec{y}||$. Under the requirement $||\vec{y} - \vec{z}|| < \epsilon$ for some $\epsilon > 0$, we obtain the inequality

$$\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| \leq \left|\left|\vec{y} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| + \epsilon$$
$$= \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}] + \epsilon$$

with $m = m(n) = o(1)$ as $n \longrightarrow \infty$. By choosing $\epsilon > 0$ sufficiently small, we deduce that $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

In the geometry of balls under compression, we will assume that n is sufficiently large for $\mathbb{R}^n$. In this regime, we will always take the scale of compression $m := m(n) = o(1)$ as $n \longrightarrow \infty$.

Theorem 3.6. *Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ with $x_i \geq 1$ for each $1 \leq i \leq n$. If $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}|| < ||\vec{x}||$ for $||\vec{y} - \vec{x}|| < \delta$ for $\delta > 0$ sufficiently small, then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for $m := m(n) = o(1)$ as $n \longrightarrow \infty$.

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}|| < ||\vec{x}||$ for $||\vec{y} - \vec{x}|| < \delta$, then it follows from Theorem 3.5 that $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] \gtrsim \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$ with $||\vec{y} - \vec{x}|| < \delta$ for $\delta > 0$ sufficiently small. Suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \not\subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}|| < ||\vec{y}||$ such that $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{z} - \vec{y}|| < \epsilon$ for $\epsilon > 0$ sufficiently small. It is not very difficult to see that this point does exist. Notice that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\mathbb{V}_m[\vec{y}]]}[\mathbb{V}_m[\vec{y}]]$$

so that under the regime where the two balls overlap then either $\vec{y} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ or $\mathbb{V}_m[\vec{y}] \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ since these points are symmetric to the center of ball. However in the latter case, we choose the point $\vec{z}$ such that to $||\mathbb{V}_m[\vec{y}]|| < ||\vec{z}||$. We can assume without loss of generality that $\vec{y} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ so that we choose the point $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}|| < ||\vec{y}||$ such that $||\vec{z} - \vec{y}|| < \epsilon$ for $\epsilon > 0$ sufficiently small, then $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 3.5 that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \gtrsim \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$$

with $||\vec{z} - \vec{x}|| < \epsilon + \delta$. It follows from Theorem 3.5 that $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ since ϵ, δ are taken to be sufficiently small. This is inconsistent with $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The case where the balls do not overlap is easier and can be treated in the same manner. This completes the proof. □

Remark 3.7. Theorem 3.6 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

3.1. **Interior points and the limit points of balls induced under compression.** In this section we launch the notion of an interior and the limit point of balls induced under compression. We study this notion in depth and explore some connections.

Definition 3.8. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then a point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ is an interior point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for most $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$. An interior point $\vec{z}$ is then said to be a limit point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for all $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$

Remark 3.9. Next we prove that there must exist an interior and limit point in any ball induced by points under compression of any scale in any dimension.

Theorem 3.10. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$ with $y_i\geq 1$ for all $1\leq i\leq n$. Then the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains an interior point and a limit point.*

Proof. Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$ with $x_i\geq 1$ for all $1\leq i\leq n$ and suppose on the contrary that $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains no limit point. Then pick

$$\vec{z}_1\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

with $||\vec{z}_1||<||\vec{x}||$ such that $||\vec{z}_1-\vec{x}||<\epsilon$ for $\epsilon>0$ sufficiently small. Then by Theorem 3.6 and Theorem 3.5, it follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]\lesssim\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Again pick $\vec{z}_2\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$ with $||\vec{z}_2||<||\vec{z}_1||$ such that $||\vec{z}_2-\vec{z}_1||<\delta$ for $\delta>0$ sufficiently small. Then by employing Theorem 3.6 and Theorem 3.5, we have

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]}[\vec{z}_2]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]\lesssim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]$. By continuing the argument in this manner we obtain the infinite descending sequence of the gap of compression

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]\gtrsim\cdots\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_n]\gtrsim\cdots$$

thereby ending the proof of the theorem. □

Proposition 3.2. *The point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$ is the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$ for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$.*

Proof. Applying the compression $\mathbb{V}_1:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ on the point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$, we obtain $\mathbb{V}_1[\vec{x}]=(1,1,\ldots,1)$ so that $\mathcal{G}\circ\mathbb{V}_1[\vec{x}]=0$ and the corresponding ball induced under compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ contains only the point $\vec{x}$. It follows by Definition 3.10 the point $\vec{x}$ must be the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$. It follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for all $1\leq i\leq n$. For if the contrary

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

holds for some $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$, then there must exists some point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$. Since $\vec{x}$ is the only point in the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$, it follows that

$$\vec{x}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

which is inconsistent with the fact that $\vec{x}$ is the limit point of the ball. $\square$

3.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 3.11. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 3.12. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 3.13. *Let $\vec{x}\in\mathbb{R}^n$ with $x_i\neq x_j$ $(i\neq j)$ such that $x_i\geq 1$ for all $1\leq i\leq n$ and set $m:=m(n)=o(1)$ as $n\longrightarrow\infty$. The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ such that $||\vec{y}-\vec{x}||<\epsilon$ for $\epsilon>0$ sufficiently small is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ such that $||\vec{y}-\vec{x}||<\epsilon$ for $\epsilon>0$ sufficiently small be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Without loss of generality, we can choose some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{z}||<||\vec{x}||$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

for $||\vec{z}-\vec{x}||<\delta$ for $\delta>0$ sufficiently small. Applying Theorem 3.5, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\lesssim\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

This already contradicts the equality $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. The latter equality of compression gaps follows from the requirement that the balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the two indistinguishable balls and so must satisfy the equality

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\\ &=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Proposition 3.3. *No three admissible points on the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ are collinear.*

4. Main result

In this section we prove the main result of this paper.

Theorem 4.1. *The number of points that can be placed in the grid $n\times n\times\cdots\times n$ (d times) $=n^d$ for all $d\in\mathbb{N}$ and with $d\geq 2$ such that no three points are collinear satisfies the lower bound*

$$\gg n^{d-1}\sqrt[2d]{d}.$$

Proof. Let $m:=m(d)=o(1)$ as $d\longrightarrow\infty$ and pick a point $\vec{x}\in\mathbb{R}^d$ such that $\mathcal{G}\circ\mathbb{V}_1[\vec{x}]\sim n^d$ for a fixed n. We note that such a point exist; that is, we choose $\vec{x}$ such that the largest coordinate $\sup(x_i)_{i=1}^d=n^d$ and the smallest coordinate $\inf(x_i)_{i=1}^d\geq 1$. Next we apply the compression $\mathbb{V}_m$ on $\vec{x}$ and construct the induced ball

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

By virtue of the restriction $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\sim n^d$ all admissible points $\vec{x_k}$ for $\vec{x_k}\neq\vec{x}$ with $||\vec{x}_k-\vec{x}||<\epsilon$ for $\epsilon>0$ small on the ball has the property that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x_k}]\sim n^d$$

with

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}_k]}[\vec{x}_k]$$

by virtue of Theorem 3.13. Again for the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}_k]}[\vec{x}_k]$, we pick an admissible point $\vec{x}_l$ such that $\vec{x}_l\neq\vec{x}_k$ with $||\vec{x}_l-\vec{x}_k||<\epsilon$ for the same choice of $\epsilon>0$ small but with $||\vec{x}_l-\vec{x}||\geq\epsilon$. Then we have

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}_k]\sim n^d$$

with

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}_l]}[\vec{x}_l]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}_k]}[\vec{x}_k]$$

by virtue of Theorem 3.13. This process can be iterated, and it is seen that for any admissible point $\vec{x}_j$ on the ball so constructed $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$, we must have

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]\sim n^d$$

Next, we construct the smallest d dimensional box that covers this ball. In this box, we construct the $n\times n\times\cdots\times n$ (d *times*) $=n^d$ grid. We only consider admissible points of the ball that are on the constructed grid. In the grid $n \times n \times \cdots \times n$ (d *times*) $= n^d$ for all $d \in \mathbb{N}$ with $d \geq 2$ the number of points that can be arranged in such a way that no three are collinear can be lower bounded by counting only the number of admissible points on the ball so constructed and on the grid constructed, by virtue of Proposition 3.3, so that we obtain the lower bound

$$\begin{aligned}
&\geq \sum_{\substack{\vec{x_j} \in n^d \\ \sqrt[d]{\mathcal{G} \circ \mathbb{V}_m[\vec{x_j}]} = n}} 1 \\
&= \sum_{\vec{x_j} \in n^d} \frac{\sqrt[d]{\mathcal{G} \circ \mathbb{V}_m[\vec{x_j}]}}{n} \\
&\gg \frac{1}{n} \sum_{\vec{x_j} \in n^d} \sqrt[2d]{d} \sqrt[2d]{(\mathrm{Inf}(x_{j_i})_{i=1}^d)^2} \\
&\geq \frac{\sqrt[2d]{d}}{n} \sum_{\vec{x_j} \in n^d} 1 \\
&= n^{d-1} \sqrt[2d]{d}
\end{aligned}$$

and the claimed lower bound follows as a consequence since $\inf(x_{j_i})_{j=1}^d \geq 1$. $\square$

Corollary 4.1. The number of points that can be placed in the grid $n \times n$ such that no three points are collinear satisfies the lower bound

$$\geq nC_1 \sqrt[4]{2}$$

for some $C_1 > 0$.

Corollary 4.2. The number of points that can be placed in the grid $n \times n \times n$ such that no three points are collinear satisfies the lower bound

$$\geq n^2 C_2 \sqrt[6]{3}$$

for some $C_2 > 0$.

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Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: `theophilus@aims.edu.gh/emperordagama@yahoo.com`

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