ON THE GENERAL NO-THREE-IN-LINE PROBLEM

T. AGAMA

Abstract. In this paper we show that the number of points that can be placed in the grid $n \times n \times \cdots \times n$ (d *times*) $= n^d$ for all $d \in \mathbb{N}$ with $d \geq 2$ such that no three points are collinear satisfies the lower bound

$$\gg n^{d-1}\sqrt{d}\min_{\vec{x}\in n^d}\mathrm{Inf}(x_j)_{j=1}^{d}.$$

This pretty much extends the result of the no-three-in-line problem to all dimension $d \geq 3$.

1. Introduction

The no-three-in-line problem is a well-known problem in discrete geometry that seeks for the maximum number of points that can be placed in an $n \times n$ grid in such a way that no three of the points are collinear. The problem was posed by the then English mathematician Henry Dudeney in 1917. The problem is apparently trivially true for all $n \leq 46$, so the only version of the problem still open is for all sufficiently large values of n. Quite a number of progress has been made in the context of upper and lower lower bounds. An argument of Erdős (see [3]) yields the lower bound

$$\gg (1-\epsilon)n$$

for the any $\epsilon > 0$ and n sufficiently large as the number of points that can be placed in the $n \times n$ grid so that no three are collinear. This was improved (see [4]) to

$$\gg \left(\frac{3}{2}-\epsilon\right)n$$

in the grid $n \times n$ with no three collinear. Various upper bound to the problem had also been conjectured. For instance it is conjectured that (see [5]) the number of points that can be placed in an $n \times n$ grid so that no three are collinear has the optimal solution cn with

$$c = \frac{\pi}{\sqrt{3}} \approx 1.814.$$

A generalized version of the problem has also been studied in (see [1]). There it is shown that the number of points that can be placed in an $n \times n \times n$ grid such that no three of them are collinear is $\Theta(n^2)$.
In the current paper we generalize the problem to dimensions $d \geq 2$ under the requirement that our configuration has no three collinear points. By applying the

Date: June 29, 2021.
2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. points; collinear.

method of compression (see [2]), we obtain a lower bound for the number of such points as
$$\gg n^{d-1}\sqrt{d}\min_{\vec{x}\in n^d}\mathrm{Inf}(x_j)_{j=1}^{d}.$$
What follows are the lower bound for the grid $n\times n$ and $n\times n\times n$.

2. Preliminary results

Definition 2.1. By the compression of scale $m>0$ ($m\in\mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V}:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that
$$\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)$$
for $n\geq 2$ and with $x_i\neq x_j$ for $i\neq j$ and $x_i\neq 0$ for all $i=1,\ldots,n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n\geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $m>0$ with $\mathbb{V}_m:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\mathbb{V}_m[(y_1,y_2,\ldots,y_n)]$, then it follows that
$$\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)=\left(\frac{m}{y_1},\frac{m}{y_2},\ldots,\frac{m}{y_n}\right).$$
It follows that $x_i=y_i$ for each $i=1,2,\ldots,n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. The mass of compression.

In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $m>0$ ($m\in\mathbb{R}$) fixed, we mean the map $\mathcal{M}:\mathbb{R}^n\longrightarrow\mathbb{R}$ such that
$$\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)])=\sum_{i=1}^{n}\frac{m}{x_i}.$$

It is important to notice that the condition $x_i\neq x_j$ for $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1=x_2=\cdots=x_n$, then it will follows that $\mathrm{Inf}(x_j)=\mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies
$$m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)-k}\leq\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)])\leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}$$
and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ must satisfy $x_i\neq x_j$ for all $1\leq i,j\leq n$. Hence in this paper

this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i, j \leq n$.

Lemma 2.4. *The estimate holds*
$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$
where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$.

Proposition 2.2. *Let* $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$*, then the estimates holds*
$$m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$
for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that
$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$
and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound
$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$
□

Definition 2.6. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale m $\mathbb{V}_m$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression
$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

Proposition 2.3. *Let* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *for* $n \geq 2$ *with* $x_j \neq 0$ *for* $j = 1, \ldots, n$*, then we have*
$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$
In particular, we have the estimate
$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$
for $\vec{x} \in \mathbb{N}^n$*, where* $m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]$ *is the error term in this case.*

Lemma 2.7 (Compression estimate)**.** *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\mathrm{sup}(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1} - 2mn.$$

3. **Compression lines**

In this section we study the notion of lines induced under compression of a given scale and the associated geometry. We first launch the following language.

Definition 3.1. Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \neq 0$ for $1 \leq i \leq n$. Then by the line $L_{\vec{x}, \mathbb{V}_m[\vec{x}]}$ produced under compression $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we mean the line joining the points $\vec{x}$ and $\mathbb{V}_m[\vec{x}]$ given by

$$\vec{r} = \vec{x} + \lambda(\vec{x} - \mathbb{V}_m[\vec{x}])$$

where $\lambda \in \mathbb{R}$.

Remark 3.2. In striving for the simplest possible notation and to save enough work space, we will choose instead to write the line produced under compression $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $L_{\mathbb{V}_m[\vec{x}]}$. Next we show that the lines produced under compression of two distinct points not on the same line of compression cannot intersect at the corresponding points and their images under compression.

Lemma 3.3. *Let $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with $\vec{a} \neq \vec{x}$ and $a_i, x_j \neq 0$ for $1 \leq i, j \leq n$. If the point $\vec{a}$ lies on the corresponding line $L_{\mathbb{V}_m[\vec{x}]}$, then $\mathbb{V}_m[\vec{a}]$ also lies on the same line.*

Proof. Pick arbitrarily a point $\vec{a}$ on the line $L_{\mathbb{V}_m[\vec{x}]}$ produced under compression for any $\vec{x} \in \mathbb{R}^n$. Suppose on the contrary that $\mathbb{V}_m[\vec{a}]$ cannot live on the same line as $\vec{a}$. Then $\mathbb{V}_m[\vec{a}]$ must be away from the line $L_{\mathbb{V}_m[\vec{x}]}$. Produce the compression line $L_{\mathbb{V}_m[\vec{a}]}$ by joining the point $\vec{a}$ to the point $\mathbb{V}_m[\vec{a}]$ by a straight line. Then It follows from Proposition 2.3

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}].$$

Again pick a point $\vec{c}$ on the line $L_{\mathbb{V}_m[\vec{a}]}$, then under the assumption it follows that the point $\mathbb{V}_m[\vec{c}]$ must be away from the line. Produce the compression line $L_{\mathbb{V}_m[\vec{c}]}$ by joining the points $\vec{c}$ to $\mathbb{V}_m[\vec{c}]$. Then by Proposition 2.3 we obtain the following decreasing sequence of lengths of distinct lines

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a}] > \mathcal{G} \circ \mathbb{V}_m[\vec{c}].$$

By repeating this argument, we obtain an infinite descending sequence of lengths of distinct lines

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] > \mathcal{G} \circ \mathbb{V}_m[\vec{a_1}] > \cdots > \mathcal{G} \circ \mathbb{V}_m[\vec{a_n}] > \cdots.$$

This proves the Lemma. $\square$

Definition 3.4. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$. Then by the ball induced by $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ under compression of scale m, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1, x_2, \ldots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1, x_2, \ldots, x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n = 2$.

Theorem 3.5. *Let $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$. Then $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$, then it follows that $||\vec{y}|| > ||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}|| < ||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}|| \leq ||\vec{y}||$ and $\sup(z_j) \leq \sup(y_j)$ by Lemma 2.7. It follows that

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &\leq \left|\left|\vec{y} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 3.6. *Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$. If $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \not\subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 3.5 that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G} \circ \mathbb{V}_m[\vec{y}] &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{z}] \\ &> \mathcal{G} \circ \mathbb{V}_m[\vec{x}] \\ &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 3.7. Theorem 3.6 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

3.1. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 3.8. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{N}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 3.9. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 3.10. *The point* $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ *is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 3.5, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]<\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}||<||\vec{y}||$ or $||\vec{y}||<||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point.. This contradicts the fact that the point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point. Now we notice that $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ certainly implies $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Conversely we notice as well that $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$, which certainly implies $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$ by Theorem 3.5. Thus the conclusion follows. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|&=\left|\left|\vec{z}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\\&\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}] = \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \leq \frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Proposition 3.1. *No three admissible points on the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ are collinear.*

Remark 3.11. Proposition 3.1 is a known result and balls refereed to in the context of our work for dimensions bigger than three is just a geometric sphere, to which the result can be applied.

4. Main result

In this section we prove the main result of this paper.

Theorem 4.1. *The number of points that can be placed in the grid $n \times n \times \cdots \times n$ (d times) $= n^d$ for all $d \in \mathbb{N}$ and with $d \geq 2$ such that no three points are collinear satisfies the lower bound*

$$\gg n^{d-1}\sqrt{d}\min_{\vec{x}\in n^d}\mathrm{Inf}(x_j)_{j=1}^{d}.$$

Proof. Let us pick a point $\vec{x} \in \mathbb{N}^d$ such that $\mathcal{G}\circ\mathbb{V}_1[\vec{x}] = n$ for a fixed n. Next we apply the compression $\mathbb{V}_1$ on $\vec{x}$ and construct the induced ball

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}].$$

By virtue of the restriction $\mathcal{G}\circ\mathbb{V}_1[\vec{x}] = n$ all admissible points $\vec{x_k}$ for $\vec{x_k} \neq \vec{x}$ on the ball has the property that

$$\mathcal{G}\circ\mathbb{V}_1[\vec{x}] = \mathcal{G}\circ\mathbb{V}_1[\vec{x_k}] = n$$

by virtue of Theorem 3.10 and so they all lie within the grid $n\times n\times\cdots\times n$ (d *times*) $=$ n^d for all $d \in \mathbb{N}$ with $d \geq 2$. In the grid $n \times n \times \cdots \times n$ (d *times*) $= n^d$ for all $d \in \mathbb{N}$ with $d \geq 2$ the number of points that can be arranged in such a way that no three are collinear can be lower bounded by counting the number of admissible points on the ball so constructed, by virtue of Proposition 3.1, so that we obtain the lower bound

$$\geq \sum_{\substack{\vec{x_j}\in n^d\\ \mathcal{G}\circ\mathbb{V}_1[\vec{x_j}]=n}} 1$$
$$= \sum_{\vec{x_j}\in n^d}\frac{\mathcal{G}\circ\mathbb{V}_1[\vec{x_j}]}{n}$$
$$\gg \frac{1}{n}\sum_{\vec{x_j}\in n^d}\sqrt{d}\mathrm{Inf}(x_{j_i})_{i=1}^{d}$$
$$= \frac{\sqrt{d}}{n}\sum_{\vec{x_j}\in n^d}\mathrm{Inf}(x_{j_i})_{i=1}^{d}$$
$$\geq \frac{\sqrt{d}}{n}n^d\mathrm{min}_{\vec{x_j}\in n^d}\mathrm{Inf}(x_{j_i})_{i=1}^{d}$$
$$= n^{d-1}\sqrt{d}\mathrm{min}_{\vec{x_j}\in n^d}\mathrm{Inf}(x_{j_i})_{j=1}^{d}$$

and the claimed lower bound follows as a consequence. □

Corollary 4.1. The number of points that can be placed in the grid $n \times n$ such that no three points are collinear satisfies the lower bound

$$\gg n\sqrt{2}\mathrm{min}_{\vec{x}\in n^2}\mathrm{Inf}(x_j)_{j=1}^{2}.$$

Corollary 4.2. The number of points that can be placed in the grid $n \times n \times n$ such that no three points are collinear satisfies the lower bound

$$\gg n^2\sqrt{3}\mathrm{min}_{\vec{x}\in n^3}\mathrm{Inf}(x_j)_{j=1}^{3}.$$

[1].

References

1. Pór, Attila and Wood, David R *No-three-in-line-in-3D*, Algorithmica, vol. 47:4, Springer, 2007, 481–488.
2. T. Agama *The Compression method and applications*, arXiv preprint arXiv:1912.08075, 2019.
3. Roth, Klaus F *On a problem of Heilbronn*, Journal of the London Mathematical Society, vol. 1:3, Wiley Online Library, 1951, pp 198–204 .
4. Hall, Richard R and Jackson, Terence H and Sudbery, Anthony and Wild, Ken *Some advances in the no-three-in-line problem*, Journal of Combinatorial Theory, Series A, vol. 18:3, Elsevier, 1975, pp 336–341.
5. Pegg Jr, Ed *Math Games: Chessboard Tasks*, MAA Online, April, vol. 11, 2005.

Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

[1]

.