DIVISION BY ZERO CALCULUS IN FIGURES
– OUR NEW SPACE SINCE EUCLID –

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Abstract: We will show in this paper in a self contained way that our basic idea for our space is wrong since Euclid, simply and clearly by using many simple and interesting figures. The common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. We will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$ with many figures. In particular, we will introduce several fundamental concepts on Euclidean geometry which show new elementary concepts on our space. We will know that the division by zero is our elementary and fundamental mathematics.

Key Words: Division by zero, division by zero calculus, singularity, derivative, $0/0 = 1/0 = z/0 = 0$, $\tan(\pi/2) = 0$, infinity, discontinuous, point at infinity, gradient, Laurent expansion, Euclidean geometry, Wasan.

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1. Introduction - Simple History of the Division by Zero

By a **natural extension** of the fractions

$$\frac{b}{a} \tag{1}$$

for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{2}$$

incidentally in [23] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [7] for the case of real numbers. The result is a very special case for general fractional functions in [5].

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [21] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. In particular, note that Brahmagupta (598 -668 ?) established the four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brhmasphuasiddhnta. Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we will see that his definition is right and suitable.

Indeed, we will show typical examples for $0/0 = 0$. However, in this introduction, these examples are based on some natural feelings and are not mathematics, because we do still not give the definition of $0/0$. However, following our new mathematics, these examples and results may be accepted as natural ones later:

The conditional probability $P(A|B)$ for the probability of A under the condition that B happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and from the meaning, $P(A|B) = 0$ and so, $0/0 = 0$.

For the representation of inner product in vectors

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}},$$

if $\mathbf{A}$ or $\mathbf{B}$ is the zero vector, then we see that $0 = 0/0$. In general, the zero vector is orthogonal for any vector and then, $\cos\theta = 0$.

We have furthermore many and concrete examples as we will see in this paper.

However, we do not know the reason and motivation of the definition of $0/0 = 0$ by Brahmagupta, furthermore, for the important case $1/0$ we do not know any result there. – Indeed, we find many and many wrong logics on the division by zero, without the definition of the division by zero $z/0$. However, Sin-Ei Takahasi ([7]) discovered a simple and decisive interpretation (2) by analyzing the extensions of fractions and by showing the complete characterization for the property (2):

Proposition 1. *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b,a)F(c,d) = F(bc,ad)$$

for all

$$a,b,c,d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a,b \in \mathbf{C}, a \neq 0.$$

Then, we obtain, for any $b \in \mathbf{C}$

$$F(b,0) = 0.$$

In a long mysterious history of the division by zero, this proposition seems to be decisive.

Following the proposition, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and consider, for any complex number b, as (2); that is, for the mapping

$$W = \frac{1}{z}, \tag{3}$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [31, 32, 33]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([11, 14]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations in an undergraduate level and furthermore to our basic ideas for the space and universe.

Meanwhile, the division by zero (2) was derived from several independent approaches as in:

1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse to the fundamental equation $az = b$ that leads to the definition of the fraction $z = b/a$,

2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

3) by the unique extension of the fractions by S. Takahasi, as in the above, and

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$, – Here, we can consider also the above on the real numbers $\mathbf{R}$ for the function $y = 1/x$ –

Furthermore, in ([10]) we gave the results in order to show the reality of the division by zero:

A) a field structure containing the division by zero — the **Yamada field** $\mathbf{Y}$,

B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with its center at the origin on the complex z plane — the reflection point of zero is zero, (The classical result is wrong, see [14]),

and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

Furthermore, in ([11],[23]), we discussed many division by zero properties in the Euclidean plane - however, precisely, our new space is not the Euclidean space. More recently, we see the great impact to Euclidean geometry in connection with Wasan in ([15, 17, 18]). In ([8]), we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero.

The global results were published in the book [24].

2. Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is complete.

However for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results and for other reasons. See [11].

For example, for the typical linear mapping

$$y = \frac{x - 1}{x + 1},$$

it gives a mapping on $\{\mathbf{R} \setminus \{-1\}\}$ onto $\{\mathbf{R} \setminus \{1\}\}$ in one to one and from

$$y = 1 + \frac{-2}{x - (-1)},$$

we see that -1 corresponds to 1 and so the function maps the whole $\{\mathbf{R}\}$ onto $\{\mathbf{R}\}$ in one to one.

Meanwhile, note that for

$$y = (x-1) \cdot \frac{1}{x+1},$$

we should not enter $x = -1$ in the way

$$[(x-1)]_{x=-1} \cdot \left[\frac{1}{x+1}\right|_{x=-1} = (-2) \cdot 0 = 0.$$

However, in may cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We will refer to this delicate problem with many examples.

2.1. **Introduction of the division by zero calculus.** For any Laurent expansion around $x = a$,

$$f(x) = \sum_{n=-\infty}^{-1} C_n(x-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(x-a)^n \tag{4}$$

we define the division by zero calculus by the identity

$$f(a) = C_0. \tag{5}$$

For the correspondence (5) for the function $f(x)$, we will call it **the division by zero calculus**. By considering the derivatives, we can define any order derivatives of the function f at the singular point a as follows:

$$f^{(n)}(a) = n!C_n.$$

We can give the definition of the division by zero calculus for more general functions over analytic functions.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a).$$

In order to avoid any logical confusion in the division by zero, we would like to refer to the logical essence:

We define $1/0 = 0$ for the form; this precise meaning is that for the function $W = f(x) = 1/x$, we have $f(0) = 0$ following the division by zero calculus. In particular, from the function $f(x) \equiv 0$ we have $0/0 = 0$, similarly (see Figure 1).

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, **for the results by division by zero we should check the results, case by case.**

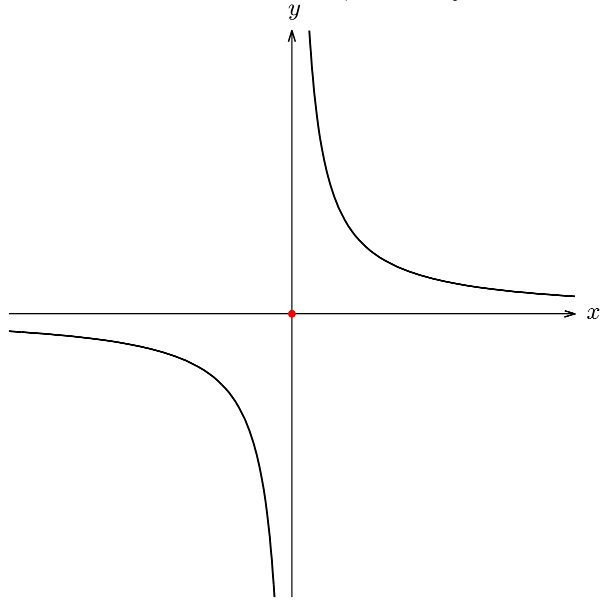

Figure 1.

For example, for the simple example for the line equation on the x, y plane

$$ax + by + c = 0$$

we have, formally

$$x + \frac{by + c}{a} = 0,$$

and so, by the division by zero, we have, for $a = 0$, the reasonable result

$$x = 0.$$

Indeed, for the equation $y = mx$, from

$$\frac{y}{m} = x,$$

we have, by the division by zero, $x = 0$ for $m = 0$. This gives the case $m = \pm\infty$ of the gradient of the line. – This will mean that the equation $y = mx$ represents the general line through the origin in this sense. – This method was applied in many cases, for example see [15, 17].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$

Here, we should consider that

$$\frac{ax + by}{c} + \frac{c}{c} = 0,$$

and for $c = 0$

$$\frac{c}{c} = 0,$$

we have the trivial identity.

Meanwhile, note that for the function $f(x) = x + \frac{1}{x}$, $f(0) = 0$, however, for the function

$$f^2(x) = x^2 + 2 + \frac{1}{x^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$

Furthermore, see many examples, [11].

2.2. **Ratio.** On the real $x-$ line, we fix two different point $P_1(x_1)$ and $P_2(x_2)$ and for $x = (x_1.x_2)$, we will consider the point, with a real number r

$$P(x;r) = \frac{x_1 + rx_2}{1+r}. \tag{6}$$

If $r = 1$, then the point $P(x;1)$ is the mid point of the two points P_1 and P_2 and for $r > 0$, the point P is on the interval (x_1, x_2). Meanwhile, for $-1 < r < 0$, the point P is on $(-\infty, x_1)$ and for $r < -1$, the point P is on $(x_2, +\infty)$. Of course, for $r = 0$, $P = P_1$. We see that r tends to $+\infty$ and $-\infty$, P tends to the point P_2. We see the pleasant fact that by the division by zero calculus, $P(x,-1) = P_2$. For this fact we see that for all real numbers r correspond to all real line numbers.

In particular, we see that in many text books on the undergraduate course the formula (6) is stated as a parameter representation of the line through the two pints P_1 and P_2. However, if we do not consider the case $r = -1$ by the division by zero calculus, the classical statement is not right, because the point P_2 may not be considered.

On this setting, we will consider another representation

$$P(x;m,n) = \frac{mx_2 - nx_1}{m-n}$$

for the exterior division point $P(x;m,n)$ in $m:n$ for the point P_1 and P_2. For $m = n$. we obtain, by the division by zero calculus, $P(x;m,m) = x_2$. Imagine the result that the point $P(x;m,m) = P_2$ and the point P_2 seems to be the point P_1. Such a strong discontinuity happens for many cases. See [11, 14].

By the division by zero, we can introduce the ratio for any complex numbers a, b, c, d as

$$\frac{AC}{CB} = \frac{c-a}{b-c}.$$

We will consider the **Appollonius circle** determined by the equation

$$\frac{AP}{PB} = \frac{|z-a|}{|b-z|} = \frac{m}{n} \tag{7}$$

for fixed $m, n \geq 0$. Then, we obtain the equation for the cirlce

$$\left| z - \frac{-n^2 a + m^2 b}{m^2 - n^2} \right|^2 = \frac{m^2 n^2}{(m^2-n^2)^2} \cdot |b-a|^2. \tag{8}$$

If $m = n \neq 0$, the circle is the line in (8). For $|m| + |n| \neq 0$, if $m = 0$, then $z = a$ and if $n = 0$, then $z = b$. If $m = n = 0$ then z is a or b.

The representation (7) is valid always, however, (8) is not reasonable for $m = n$. The property of the division by zero depends on the representations of formulas.

On the real line, the points $P(p), Q(1), R(r), S(-1)$ form a harmonic range of points if and only if

$$p = \frac{1}{r}.$$

If $r = 0$, then we have $p = 0$ that is now the representation of the point at infinity(see Figure 2).

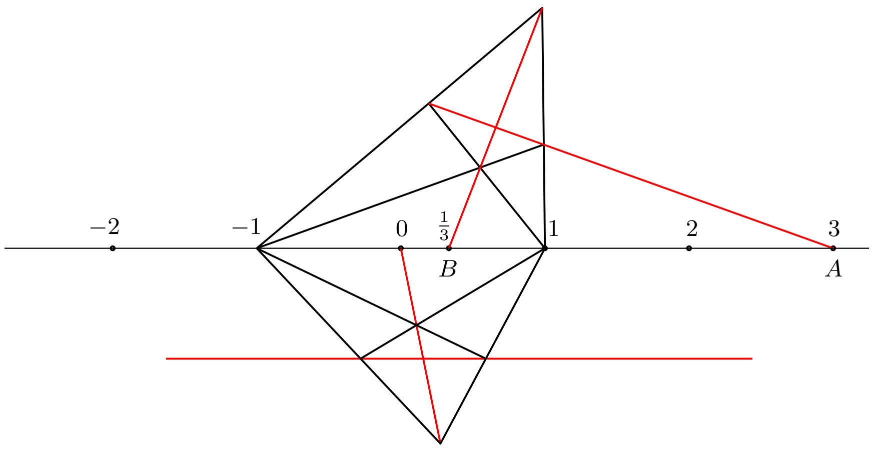

Figure 2.

3. Derivatives of a function

On derivatives, we obtain new concepts, from the division by zero.

From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{9}$$

or

$$f'(x) = -\infty, \tag{10}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{11}$$

This property is derived from the fact that the gradient of the y axis is zero; that is,

$$\tan \frac{\pi}{2} = 0, \tag{12}$$

that was derived from many geometric properties in [11], and also in the formal way from the result $1/0 = 0$. Of course, by the division by zero calculus, we can derive the result.

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1 - x^2},$$

$$y' = \frac{-x}{\sqrt{1-x^2}},$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0.$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta, y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right) \tag{13}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

The differential equation

$$y' = -\frac{x}{y}$$

with a general solution

$$x^2 + y^2 = a^2$$

is satisfied for all the points of the solutions by the division by zero, however, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1$$

are not satisfied for all the points of the solutions, because they may not be considered at the points $(0, -a)$ and $(0, a)$ in the usual sense.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{14}$$

and so,

$$[y']_{x=0} = 0. \tag{15}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{16}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant C (see Figure 3)

$$y = \log x + C, \tag{17}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{18}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all the values are zero at the origin, in the sense of the division by zero.

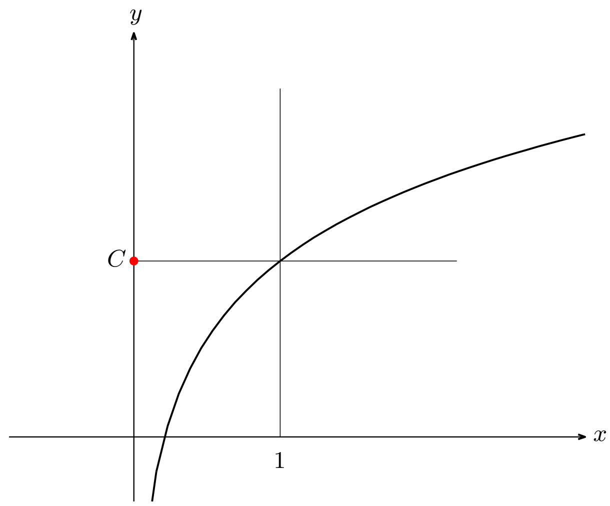

Figure 3.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable, but it has a singularity at the origin. For $x > 0$, there is no problem for (16) and (17). At $x = 0$, we see that we can not consider the limit in the usual sense. However, $x > 0$ we have (17) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{19}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have:

$$\left[(\log x)'\right]_{x=0} = 0 \tag{20}$$

and we will be able to understand its sense graphically.

4. Triangles and division by zero

In order to see how elementary of the division by zero, we will see the division by zero in triangles as the fundamental objects. Even the case of triangles, we can derive new concepts and results.

We will consider a triangle ABC with length a, b, c. Let θ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and not by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2 + b^2 - c^2 = 0$, then $C = \pi/2$. Then,

$$0 = \frac{\tan B}{\tan\frac{\pi}{2}} = \frac{\tan B}{0}.$$

Meanwhile, for the case $a^2 - b^2 + c^2 = 0$, then $B = \pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0} = \frac{\tan\frac{\pi}{2}}{\tan C} = 0.$$

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively (see Figure 4). Let θ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan\theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

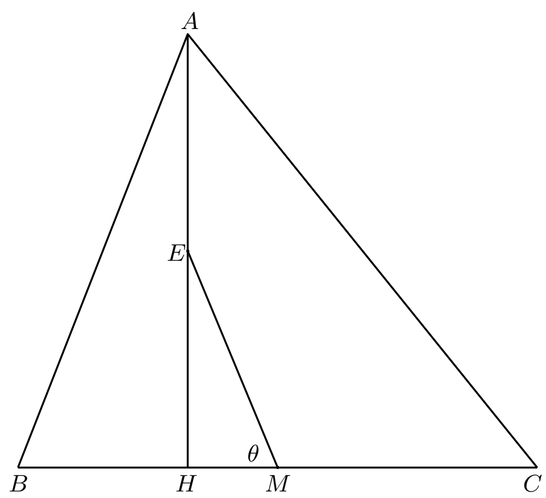

Figure 4.

Let r be the inradius of the triangle ABC, and r_A, r_B, r_C be the distances from A, B, C to the lines BC, CA, AB, respectively (see Figure 5). Then we have

$$\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

When the point A is the point at infinity, then, $r_A = 0$ and $r_B = r_C = 2r$ and the identity still holds (see Figure 6).

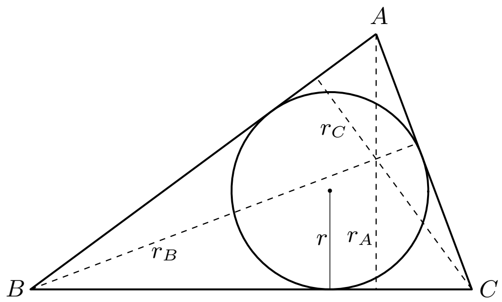

Figure 5.

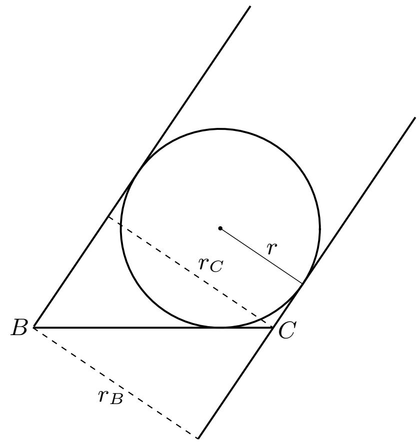

Figure 6.

We have identities for the circumradius R and the semiperimeter s of the triangle ABC,

$$S = \frac{ar_A}{2} = \frac{1}{2}bc\sin A = \frac{1}{2}a^2\frac{\sin B \sin C}{\sin A}$$

$$= \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = rs.$$

If A is the point at infinity, then, $S = s = r_A = b = c = 0$ and the above identities all valid.

For the identity

$$\tan\frac{A}{2} = \frac{r}{s-a},$$

if the point A is the point at infinity, $A = 0$, $s - a = 0$ and the identity holds as $0 = r/0$. Meanwhile, if $A = \pi$, then the identity holds as $\tan(\pi/2) = 0 = 0/s$.

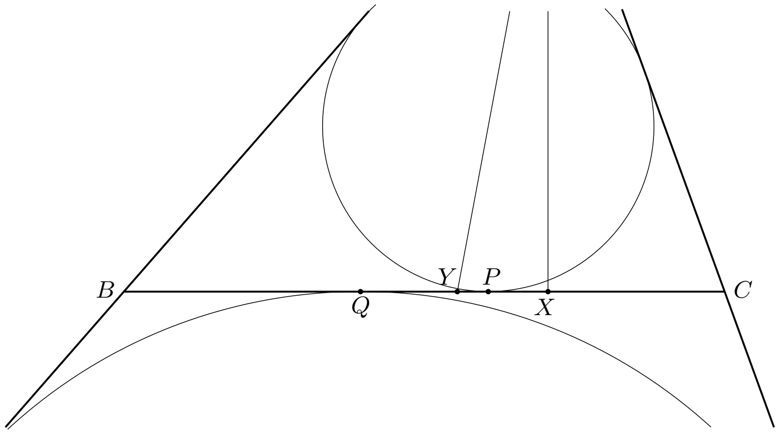

Figure 7.

Let X be the leg of the perpendicular line from A to the line BC and let Y be the common point of the angle bisector of A and the line BC (see Figure 7). Let P and Q be the tangential points on the line BC with the incircle of the triangle and the escribed circle touching BC from the side opposite to A, respectively. Then, we know that

$$\frac{XP}{PY} = \frac{XQ}{QY}.$$

If $AB = AC$, then, of course, X=Y=P=Q. Then, we have:

$$\frac{0}{0} = \frac{0}{0} = 0.$$

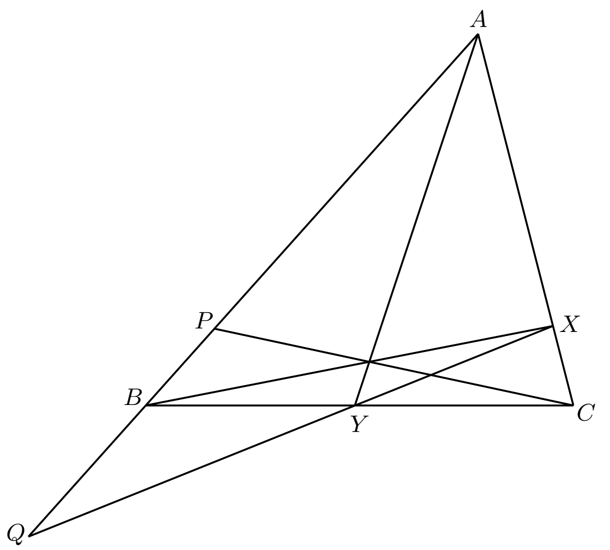

Figure 8.

Let X, Y, Q be the common points with a line and three lines AC, BC and AB, respectively. Let P be the common point with the line AB and the line through the point C and the common point of the lines AY and BX (see

Figure 8). Then, we know the identity

$$\frac{AP}{AQ} = \frac{BP}{BQ}.$$

If two lines XY and AB are parallel, then the point Q may be considered as the point at infinity. Then, by the interpretation $AQ = BQ = 0$, the identity is valid as

$$\frac{AP}{0} = \frac{BP}{0} = 0.$$

5. Euclidean spaces and division by zero

In this section, we will see the division by zero properties on the Euclidean spaces. Since the impact of the division by zero and division by zero calculus is widely expanded in elementary mathematics, here, elementary and typical topics will be introduced.

5.1. **Broken phenomena of figures by area and volume.** The strong discontinuity of the division by zero around the point at infinity will be appeared as the broken of various figures. These phenomena may be looked in many situations as the universe one. However, the simplest cases are disc and sphere (ball) with radius $1/R$. When $R \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $R = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry, as we see later. However, the results are clear already from the definition of the division by zero calculus.

For this fact, note the following:

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$ (see [2]). We thus see that

$$\lim_{z \to \infty} z = \infty, \tag{21}$$

however,

$$[z]_{z=\infty} = 0, \tag{22}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (21) and (22) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x \to +\infty} x = +\infty, \quad \lim_{x \to -\infty} x = -\infty,$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0.$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [11], we gave many examples for this property.

In particular, in $z \to \infty$ in (21), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (21) represents the limit by means of the ϵ - δ logic.

5.2. **Parallel lines.** We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2.$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right).$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the common point is always given by

$$(0, 0),$$

even the two lines are the same. This fact shows that the image of the Euclidean space is right, because any line is extended to the point at infinity and the point is represented by zero; the origin.

In particular, note that the concept of parallel lines is very important in the Euclidean plane and non-Euclidean geometry. In our sense, there is no parallel line and all lines pass the origin. This will be our world in the Euclidean plane. However, this property is not geometrical and has a strong discontinuity. This surprising property may be looked clearly by the polar representation of a line.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)},$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle OP ($P = (r, \theta)$ on the line) and the positive x axis. Then, if $\theta - \alpha = \pi/2$: that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

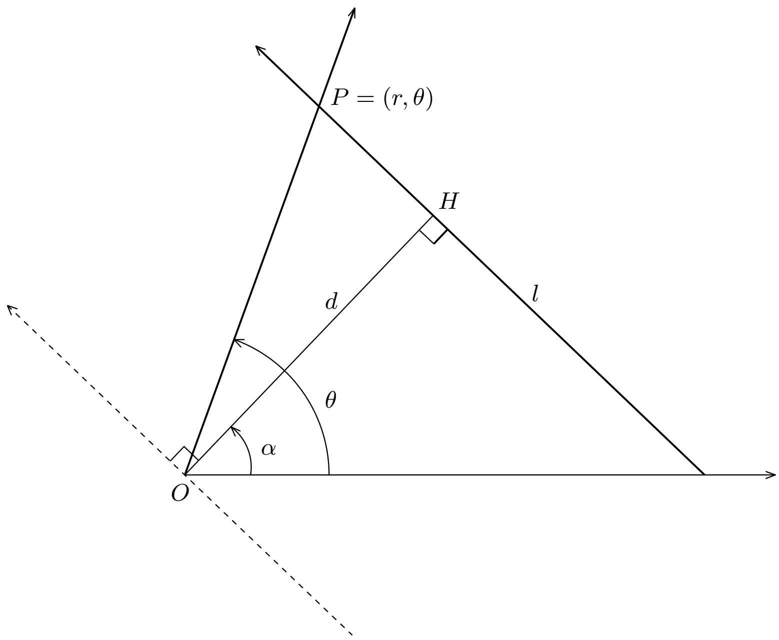

Figure 9.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics globally.

The distance d from the origin to the line determined by the two planes

$$\Pi_k : a_k x + b_k y + c_k z = 1, k = 1, 2,$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

If the two planes are coincident, then $d = 0$. Further, if the two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

5.3. **Tangential lines and** $\tan\frac{\pi}{2} = 0$**.** We looked the very fundamental and important formula $\tan\frac{\pi}{2} = 0$. For its importance we will furthermore see its various geometrical meanings.

We consider the high $\tan\theta$ $\left(0 \le \theta \le \frac{\pi}{2}\right)$ that is given by the common point of two lines $y = (\tan\theta)x$ and $x = 1$ on the (x, y) plane (see Figure 10). Then,

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$\tan\frac{\pi}{2} = 0,$$

by the division by zero. The result will show that, when $\theta = \pi/2$, two lines $y = (\tan\theta)x$ and $x = 1$ do not have a common point, because they are parallel in the usual sense. However, in the sense of the division by zero, parallel lines have the common point $(0,0)$. Therefore, we can see the result $\tan\frac{\pi}{2}=0$ following our new space idea.

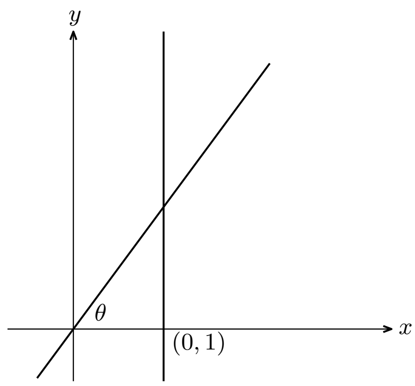

Figure 10.

We consider the unit circle with its center at the origin on the (x,y) plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y=(\tan\theta)x\ \left(0\leq\theta\leq\frac{\pi}{2}\right)$ (see Figure 11). Then, the distance R_θ between the common point and the common point of the tangential line and x-axis is given by

$$R_\theta=\tan\theta.$$

Then,

$$R_0=\tan 0=0,$$

and

$$\tan\theta\longrightarrow\infty;\quad \theta\longrightarrow\frac{\pi}{2}.$$

However,

$$R_{\pi/2}=\tan\frac{\pi}{2}=0.$$

This example shows also that by the stereoprojection mapping of the unit sphere with its center the origin $(0,0,0)$ onto the plane, the north pole corresponds to the origin $(0,0)$.

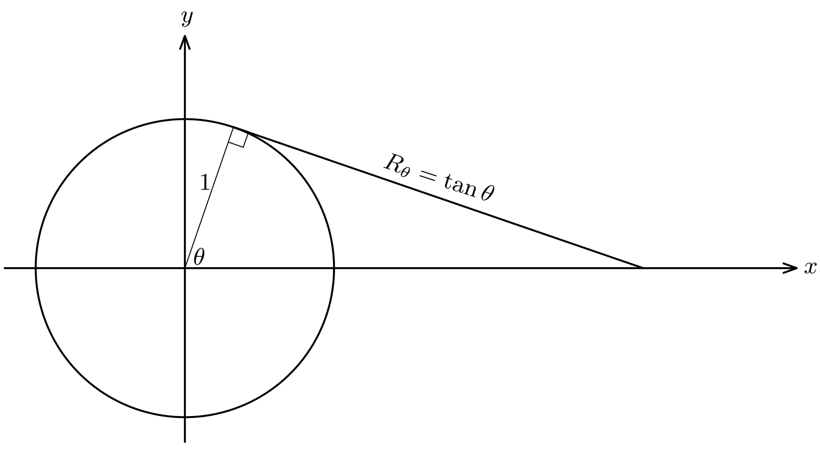

Figure 11.

In this case, we consider the orthogonal circle C_{R_θ} with the unit circle through at the common point and the symmetric point with respect to the x-axis with the center $((\cos\theta)^{-1}, 0)$ (see Figure 12). Then, the circle C_{R_θ} is as follows:

C_{R_0} is the point $(1, 0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the y-axis and its curvature is also zero. Meanwhile, by the division by zero, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

The point $(\cos\theta, 0)$ and $((\cos\theta)^{-1}, 0)$ are the symmetric points with respective to the unit circle, and the origin corresponds to the origin.

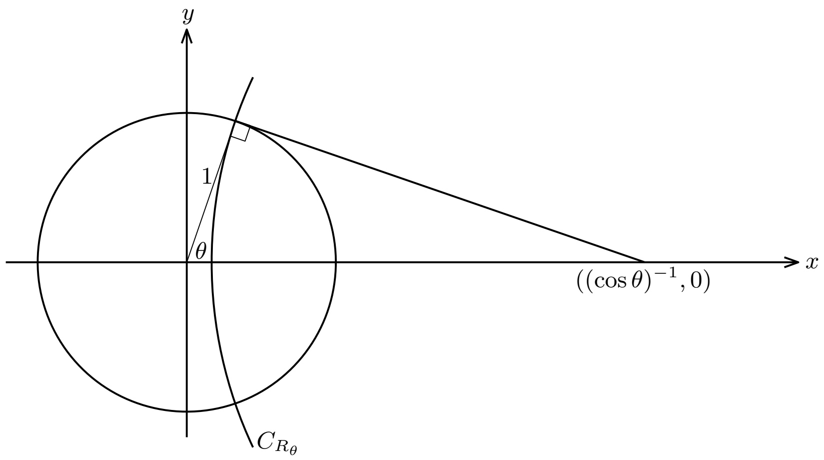

Figure 12.

In particular, the formal calculation

$$\sqrt{1 + R_{\pi/2}^2} = 1$$

is not good. The identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid for $\theta = \pi/2$.

However from

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

we have the right result for $\theta = \pi/2$.

On the point (p, q) $(0 \leq p, q \leq 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with x-axis and y-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area S_p of the triangle formed by the three points $(0, 0)$, $(1/p, 0)$ and $(0, 1/q)$ is given by

$$S_p = \frac{1}{2pq}.$$

Then,

$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$

however,

$$S_0 = 0$$

(H. Michiwaki: 2015.12.5.).

We denote the point on the unit circle on the (x, y) with $(\cos\theta, \sin\theta)$ for the angle θ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the x-axis for the case $\theta \neq \pi/2$. Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5.).

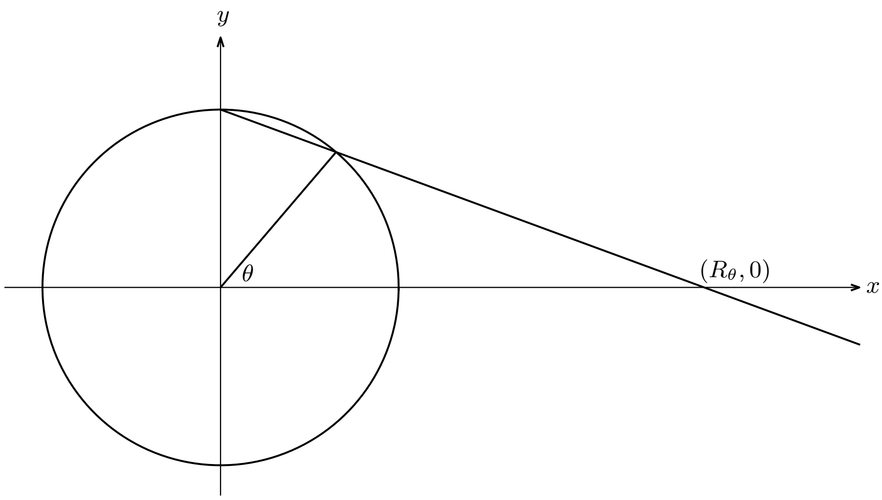

Figure 13.

The line through the points $(0, 1)$ and $(\cos\theta, \sin\theta)$ meets the x axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}.$$

Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = 0,$$

by the division by zero (see Figure 13). We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

5.4. **Two Circles.** We consider two circles with radii $a, b > 0$ with centers $(a, 0)$ and $(-b, 0)$, respectively. Then, the external common tangent $L_{a,b}$ (we assume that $a < b$) meets the x-axis in point $(R_a, 0)$ which is given by, by fixing b

$$R_a = \frac{2ab}{b-a}. \tag{23}$$

We consider the circle C_{R_a} with its center at $(R_a, 0)$ with radius R_a (see Figure 14). Then,

$$a \to b \Longrightarrow R_a \to \infty,$$

however, when $a = b$, then we have $R_b = -2b$ by the division by zero, from the identity

$$\frac{2ab}{b-a} = -2b - \frac{2b^2}{a-b}.$$

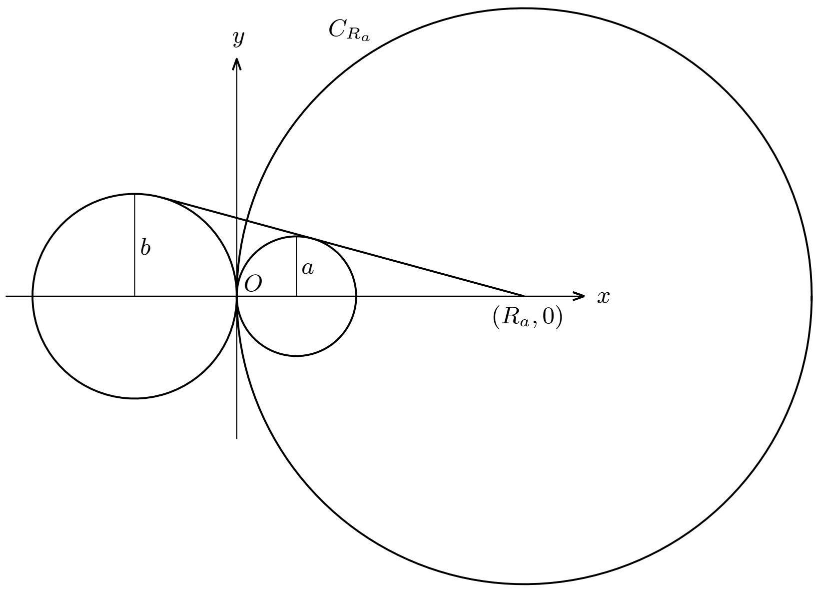

Figure 14.

Meanwhile, when we interpret (23) as

$$R_a = \frac{-1}{a-b} \cdot 2ab,$$

we have, for $a = b$, $R_b = 0$. It means that the circle C_{R_b} is the y axis with curvature zero through the origin $(0, 0)$.

The above formulas will show strong discontinuity for the change of the a and b from $a = b$ (H. Okumura: 2015.10.29.).

We denote the circles S_j:

$$(x - a_j)^2 + (y - b_j)^2 = r_j^2.$$

Then, the common point (X, Y) of the co- and exterior tangential lines of the circles S_j for $j = 1, 2$,

$$(X, Y) = \left(\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2}, \frac{r_1 b_2 - r_2 b_1}{r_1 - r_2} \right).$$

We will fix the circle S_2. Then, from the expansion

$$\frac{r_1a_2 - r_2a_1}{r_1 - r_2} = \frac{r_2(a_2 - a_1)}{r_1 - r_2} + a_2 \tag{24}$$

for $r_1 = r_2$, by the division by zero, we have

$$(X, Y) = (a_2, b_2).$$

Meanwhile, when we interpret (24) as

$$\frac{r_1a_2 - r_2a_1}{r_1 - r_2} = \frac{1}{r_1 - r_2} \cdot (r_1a_2 - r_2a_1),$$

we obtain that

$$(X, Y) = (0, 0),$$

that is reasonable. However, the both cases, the results show strong discontinuity.

5.5. **Newton's method.** The Newton's method is fundamental when we look for the solutions for some general equation $f(x) = 0$ numerically and practically. We will refer to its prototype case.

We will assume that a function $y = f(x)$ belongs to C^1 class. We consider the sequence $\{x_n\}$ for $n = 0, 1, 2, \dots, n, \dots$, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots. \tag{25}$$

When $f(x_n) = 0$, we have

$$x_{n+1} = x_n, \tag{26}$$

in the reasonable way (see Figure 15). Even the case $f'(x_n) = 0$, we have also the reasonable result (26), by the division by zero.

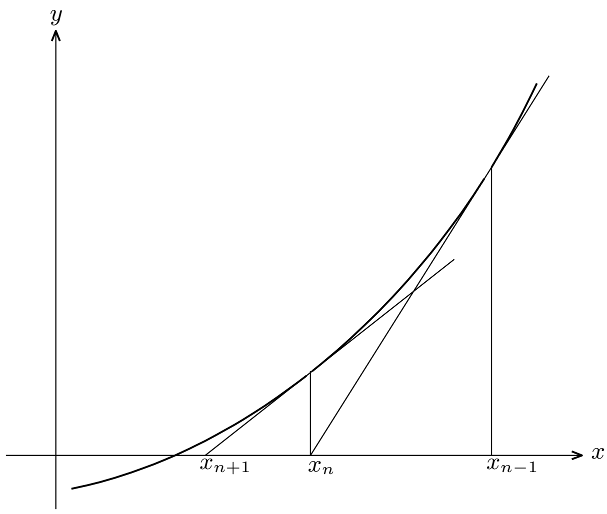

Figure 15.

5.6. **Cauchy's mean value theorem.** For the Cauchy mean value theorem: for $f, g \in Differ(a,b)$, differentiable, and $\in C^0[a,b]$, continuous and if $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, then there exists $\xi \in (a,b)$ satisfying that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)},$$

we do not need the assumptions $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, by the division by zero. Indeed, if $g(a) = g(b)$, then, by the Rolle theorem, there exists $\xi \in (a,b)$ such that $g'(\xi) = 0$. Then, the both terms are zero and the equality is valid.

For $f, g \in C^2[a,b]$, there exists a $\xi \in (a,b)$ satisfying

$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f''(a)}{g''(a)}.$$

Here, we do not need the assumption

$$g(b) - g(a) - (b-a)g'(a) \neq 0,$$

by the division by zero.

5.7. **Length of tangential lines.** We will consider a function $y = f(x)$ of C^1 class on the real line. We consider the tangential line through $(x, f(x))$

$$Y = f'(x)(X - x) + f(x).$$

Then, the length (or distance) $d(x)$ between the point $(x, f(x))$ and $\left(x - \frac{f(x)}{f'(x)}, 0\right)$ is given by, for $f'(x) \neq 0$

$$d(x) = |f(x)|\sqrt{1 + \frac{1}{f'(x)^2}}.$$

How will be the case $f'(x^*) = 0$? Then, the division by zero shows that

$$d(x^*) = |f(x^*)|.$$

Meanwhile, the x axis point $(X_t, 0)$ of the tangential line at (x, y) and y axis point $(0, Y_n)$ of the normal line at (x, y) are given by

$$X_t = x - \frac{f(x)}{f'(x)}$$

and

$$Y_n = y + \frac{x}{f'(x)},$$

respectively. Then, if $f'(x) = 0$, we obtain the reasonable results:

$$X_t = x, \quad Y_n = y.$$

5.8. **Curvature and center of curvature.** We will assume that a function $y = f(x)$ is of class C^2. Then, the curvature radius ρ and the center $O(x, y)$ of the curvature at point $(x, f(x))$ are given by

$$\rho(x,y) = \frac{(1 + (y')^2)^{3/2}}{y''}$$

and

$$O(x,y) = \left(x - \frac{1 + (y')^2}{y''} y', y + \frac{1 + (y')^2}{y''} \right),$$

respectively. Then, if $y'' = 0$, we have:

$$\rho(x,y) = 0$$

and

$$O(x,y) = (x,y),$$

by the division by zero. They are reasonable.

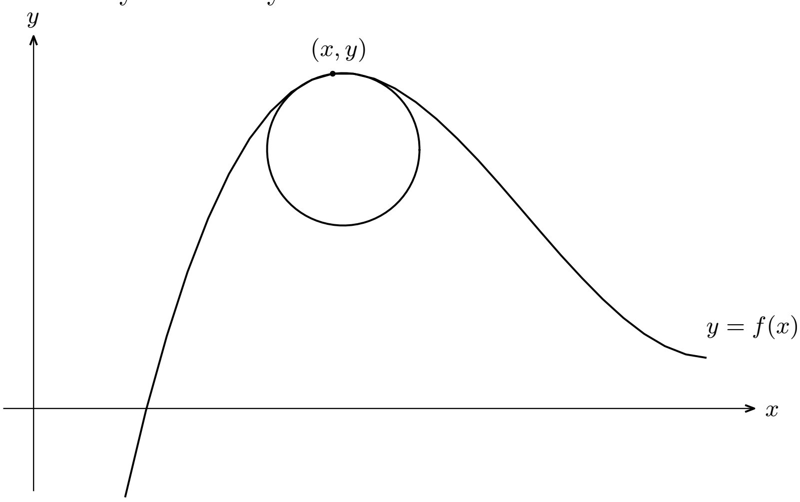

Figure 16.

We will consider a curve $\mathbf{r} = \mathbf{r}(s), s = s(t)$ of class C^2. Then,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \mathbf{t} = \frac{d\mathbf{r}(\mathbf{s})}{ds}, v = \frac{ds}{dt}, \frac{d\mathbf{t}(\mathbf{s})}{ds} = \frac{1}{\rho}\mathbf{n},$$

by the principal normal unit vector $\mathbf{n}$. Then, we see that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{t} + \frac{v^2}{\rho}\mathbf{n}.$$

If $\rho(s_0) = 0$, then

$$\mathbf{a}(s_0) = \left[\frac{dv}{dt}\mathbf{t}\right]_{s=s_0}$$

and

$$\left[\frac{v^2}{\rho}\right]_{s=s_0} = \infty$$

will be funny. It will be the zero.

5.9. $n = 2, 1, 0$ **regular polygons inscribed in a disc.** We consider n regular polygons inscribed in a fixed disc with radius a. Then we note that their area S_n and the lengths L_n of the sum of the sides are given by

$$S_n = \frac{na^2}{2} \sin \frac{2\pi}{n}$$

and

$$L_n = 2na \sin \frac{\pi}{n},$$

respectively (see Figure 17). For $n \geq 3$, the results are clear.

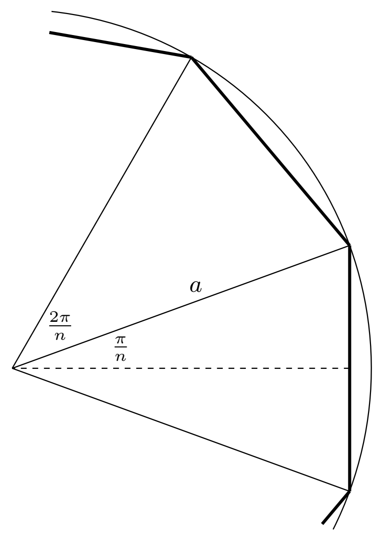

Figure 17.

For $n = 2$, we will consider two diameters that are the same. We can consider it as a generalized regular polygon inscribed in the disc as a degenerate case. Then, $S_2 = 0$ and $L_2 = 4a$, and the general formulas are valid.

Next, we will consider the case $n = 1$. Then the corresponding regular polygon is a just diameter of the disc. Then, $S_1 = 0$ and $L_1 = 0$ that will mean that any regular polygon inscribed in the disc may not be formed and so its area and length of the side are zero.

For an $n = 1$ triangle, if 1 means one side, then we can interpretate as in the above, however, if we consider 1 as one vertex, the above situation may be consider as one point on the circle which coincides with $S_l = L_l = 0$.

Now we will consider the case $n = 0$. Then, by the division by zero calculus, we obtain that $S_0 = \pi a^2$ and $L_0 = 2\pi a$. Note that they are the area and the length of the disc. How to understand the results? Imagine contrary n tending to infinity, then the corresponding regular polygons inscribed in the disc tend to the disc. Recall our new idea that the point at infinity is represented by 0. Therefore, the results say that $n = 0$ regular polygons are $n = \infty$ regular polygons inscribed in the disc in a sense and they are the disc. This is our interpretation of the theorem:

Theorem. $n = 0$ *regular polygons inscribed in a disc are the whole disc.*

In addition, note that each inner angle A_n of a general n regular polygon inscribed in a fixed disc with radius a is given by

$$A_n = \left(1 - \frac{2}{n}\right)\pi. \tag{27}$$

The circumstances are similar for n regular polygons circumscribed in the disc, because the corresponding data are given by

$$S_n = na^2 \tan\frac{\pi}{n}$$

and

$$L_n = 2na \tan\frac{\pi}{n},$$

and (27), respectively.

We consider a disc with radius $R > 0$ and an n regular inscribed polygon with vertexes $A_1, A_2, ..., A_n$. We consider the inner circle radius r_k of the triangle $A_1A_{k+1}A_{k+2}$, that is, for $\alpha = \pi/n$,

$$r_k \cos\frac{\alpha}{2} = \frac{\alpha}{2}\left\{\sin\frac{(2k+1)\alpha}{2} - \sin\frac{\alpha}{2}\right\}.$$

The total sum L_n is given, for $k = 1, 2, ..., n-2$ by

$$L_n = 2R\left(1 - n\sin^2\frac{\pi}{2n}\right).$$

Note that $L_2 = L_1 = 0$, however, by the division by zero calculus,

$$L_0 = 2R.$$

This is the same that

$$\lim_{n\to\infty} L_n = 2R.$$

5.10. **Our life figure.** As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{|\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2}\right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k, 0)$ with radius r. Then, we have

$$r = k\sin\alpha.$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately that

$$[r]_{r=\infty} = 0.$$

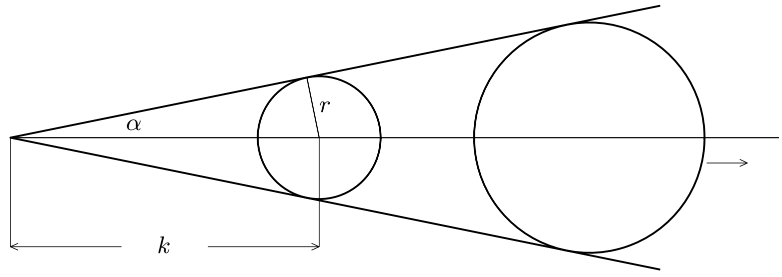

Figure 18: θ: const, $r \to \infty$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

5.11. **H. Okumura's example.** The surprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a fixed circle $C_a, a > 0$ with its radius a inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive imaginary line is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

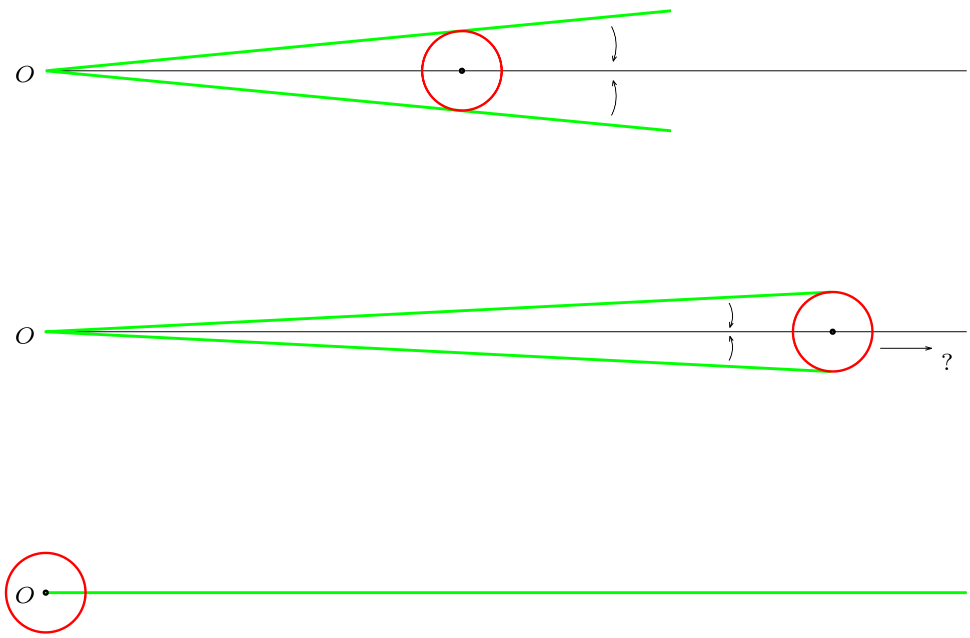

Figure 19.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

5.12. **Interpretation by analytic geometry.** For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{28}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero; that is, the circle is a line (degenerate).

The center of the circle (28) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a}\right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

We consider the functions

$$S_j(x, y) = a_j(x^2 + y^2) + 2g_jx + 2f_jy + c_j.$$

The distance d of the centers of the circles $S_1(x, y) = 0$ and $S_2(x, y) = 0$ is given by

$$d^2 = \frac{g_1^2 + f_1^2}{a_1^2} - 2\frac{g_1g_2 + f_1f_2}{a_1a_2} + \frac{g_2^2 + f_2^2}{a_2^2}.$$

If $a_1 = 0$, then by the division by zero

$$d^2 = \frac{g_2^2 + f_2^2}{a_2^2}.$$

Then, $S_1(x, y) = 0$ is a line and its center is the origin $(0, 0)$. Therefore, the result is very reasonable.

6. Mirror image with respect to a circle

For simplicity, we will consider the unit circle $|z| = 1$ on the complex $z = x + iy$ plane. Then, we have the reflection formula

$$z^* = \frac{1}{\overline{z}} \tag{29}$$

for any point z, as well-known ([2]). For the reflection point z^*, there is no problem for the points $z \neq 0, \infty$. As the classical result, the reflection of zero is the point at infinity and conversely, for the point at infinity we have the zero point. The reflection is a one to one and onto mapping between the inside and the outside of the unit circle, by considering the point at infinity.

Are these correspondences, however, suitable? Does there exist the point at ∞, really? Is the point at infinity corresponding to the zero point, by the reflection? Is the point at ∞ reasonable from the practical point of view? Indeed, where can we find the point at infinity? Of course, we know pleasantly the point at infinity on the Riemann sphere, however, on the complex z-plane it seems that we can not find the corresponding point. When we approach to the origin on a radial line, it seems that the correspondence reflection points approach to *the point at infinity* with the direction (on the radial line).

On the concept of the division by zero, there is no the point at infinity ∞ as the numbers. For any point z such that $|z| > 1$, there exists the unique point z^* by (29). Meanwhile, for any point z such that $|z| < 1$ except $z = 0$, there exits the unique point z^* by (29). Here, note that for $z = 0$, by the division by zero, $z^* = 0$. Furthermore, we can see that

$$\lim_{z\to 0} z^* = \infty, \tag{30}$$

however, for $z = 0$ itself, by the division by zero, we have $z^* = 0$. This will mean a strong discontinuity of the functions $W = \frac{1}{z}$ and (29) at the origin $z = 0$; that is a typical property of the division by zero. This strong discontinuity may be looked in the above reflection property, physically.

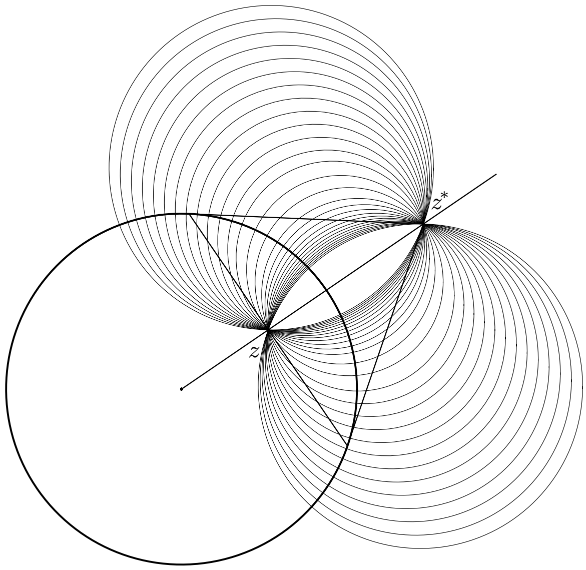

Figure 20.

The result is a surprising one in a sense; indeed, by considering the geometrical corresponding of the mirror image, we will consider the center corresponds to the point at infinity that is represented by the origin $z = 0$. This will show that the mirror image is not followed by this concept; the corresponding seems to come from the concept of one-to-one and onto mapping.

Should we exclude the point at infinity, from the numbers? We were able to look the strong discontinuity of the division by zero in the reflection with respect to circles, physically (geometrical optics). The division by zero gives a one to one and onto mapping of the reflection from the whole complex plane onto the whole complex plane.

The infinity ∞ may be considered as in as the usual sense of limits, however, the infinity ∞ is not a definite number.

On the x, y plane, we shall consider the inversion relation with respect to the circle with its radius R and with its center at the origin:

$$x' = \frac{xR^2}{x^2 + y^2}, \quad y' = \frac{yR^2}{x^2 + y^2}.$$

Then, the line

$$ax + by + c = 0$$

is transformed to the line

$$R^2(ax' + by') + c((x')^2 + (y')^2) = 0.$$

In particular, for $c = 0$, the line $ax + by = 0$ is transformed to the line $ax' + by' = 0$. This corresponding is one-to-one and onto, and so the origin $(0, 0)$ have to correspond to the origin $(0, 0)$.

7. Stereographic projection

For a great meaning and importance, we will see that the point at infinity is represented by zero.

7.1. **The point at infinity is represented by zero.** By considering the stereographic projection, we will be able to see that the point at infinity is represented by zero.

Consider the sphere (ξ, η, ζ) with radius $1/2$ put on the complex $z = x+iy$ plane with its center $(0, 0, 1/2)$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the sphere onto the complex $z(= x + iy)$ plane; that is,

$$x = \frac{\xi}{1 - \zeta}, \quad y = \frac{\eta}{1 - \zeta}.$$

If $\zeta = 1$, then, by the division by zero, the north pole corresponds to the origin $(0, 0) = 0$.

Here, note that

$$x^2 + y^2 = \frac{\zeta}{1 - \zeta}.$$

For $\zeta = 1$, we should consider as $1/0 = 0$, not by the division by zero calculus,

$$\frac{\zeta}{1 - \zeta} = -1 - \frac{1}{\zeta - 1}.$$

We will consider the unit sphere $\{(x_1, x_2, x_3); x_1^2+x_2^2+x_3^2 = 1\}$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(x_1, x_2, x_3)$ on the sphere onto the (x, y) plane; that is,

$$(x_1, x_2, x_3) =$$
$$\left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{1 - 1/(x^2 + y^2)}{1 + 1/(x^2 + y^2)}\right).$$

Then, we see that the north pole corresponds to the origin.

Next, we will consider the semi-sphere (ξ, η, ζ) with its center $C(0, 0, 1)$ on the origin on the (x, y) plane. From the center $C(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the semi- sphere onto the complex (x, y) plane; that is,

$$x = \frac{\xi}{1 - \zeta}, y = \frac{\eta}{1 - \zeta}.$$

If $\zeta = 1$, then, by the division by zero, the center C corresponds to the origin $(0, 0)$.

Meanwhile, we will consider the mapping from the open unit disc onto $\mathbf{R}^2$ in one to one and onto

$$\xi = \frac{x\sqrt{x^2+y^2}}{1+x^2+y^2}, \quad \eta = \frac{y\sqrt{x^2+y^2}}{1+x^2+y^2}$$

or

$$x = \frac{\xi}{\sqrt{\rho(1-\rho)}}, y = \frac{\eta}{\sqrt{\rho(1-\rho)}}; \quad \rho^2 = \xi^2 + \eta^2.$$

Note that the point $(x, y) = (0, 0)$ corresponds to $\rho = 0; (\xi, \eta) = (0, 0)$ and $\rho = 1$.

7.2. **A contradiction of classical idea for $1/0 = \infty$.** The infinity ∞ may be considered by the idea of the limiting, however, we had considered it as a number, for sometimes, typically, the point at infinity was represented by ∞ for some long years. For this fact, we will show a formal contradiction.

We will consider the stereographic projection by means of the unit sphere

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = 1$$

from the complex $z = x + iy$ plane onto the sphere. Then, we obtain the correspondences

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}$$

and

$$\xi = \frac{1}{2}\frac{z+\overline{z}}{z\overline{z}+1}, \eta = \frac{1}{2i}\frac{z-\overline{z}}{z\overline{z}+1}, \zeta = \frac{z\overline{z}}{z\overline{z}+1}.$$

In general, two points P and Q_1 on the diameter of the unit sphere correspond to z and z_1, respectively if and only if

$$z\overline{z_1} + 1 = 0. \tag{31}$$

Meanwhile, two points P and Q_2 on the symmetric points on the unit sphere with respect to the plane $\zeta = \frac{1}{2}$ correspond to z and z_2, respectively if and only if

$$z\overline{z_2} - 1 = 0. \tag{32}$$

If the point P is the origin or the north pole, then the points Q_1 and Q_2 are the same point. Then, the identities (31) and (32) are not valid that show a contradiction.

Meanwhile, if we write (31) and (32)

$$z = -\frac{1}{\overline{z_1}}$$

and

$$z = \frac{1}{\overline{z_2}},$$

respectively, we see that the division by zero is valid.

7.3. **Natural meanings of $1/0 = 0$.** For constants a and b satisfying

$$\frac{1}{a} + \frac{1}{b} = k, \quad (\neq 0, \text{const.})$$

the function

$$\frac{x}{a} + \frac{y}{b} = 1$$

passes the point $(1/k, 1/k)$. If $a = 0$, then, by the division by zero, $b = 1/k$ and $y = 1/k$; this result is natural.

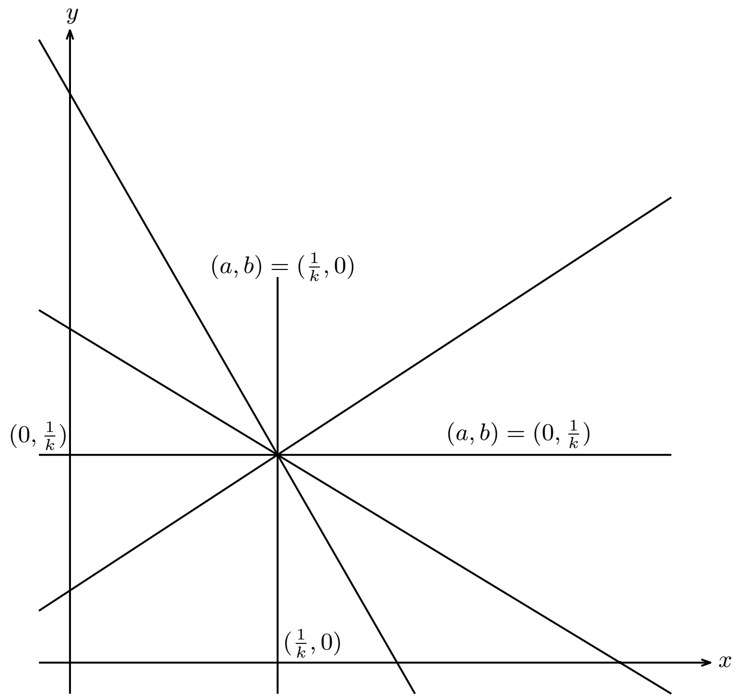

Figure 21.

We will consider the line $y = m(x-a)+b$ through a fixed point (a, b); $a, b > 0$ with gradient m (see Figure 22). We set $A(0, -am+b)$ and $B(a-(b/m), 0)$ that are common points with the line and both lines $x = 0$ and $y = 0$, respectively. Then,

$$\overline{AB}^2 = (-am + b)^2 + \left(a - \frac{b}{m}\right)^2.$$

If $m = 0$, then $A(0, b)$ and $B(a, 0)$, by the division by zero, and furthermore

$$\overline{AB}^2 = a^2 + b^2.$$

Then, the line AB is a corresponding to the line between the origin and the point (a, b). Note that this line has only one common point with the both lines $x = 0$ and $y = 0$. Therefore, this result will be very natural in a sense. – Indeed, we can understand that the line $\overline{AB}$ is broken as the two lines $(0, b) - (a, b)$ and $(a, b) - (a, 0)$, suddenly.

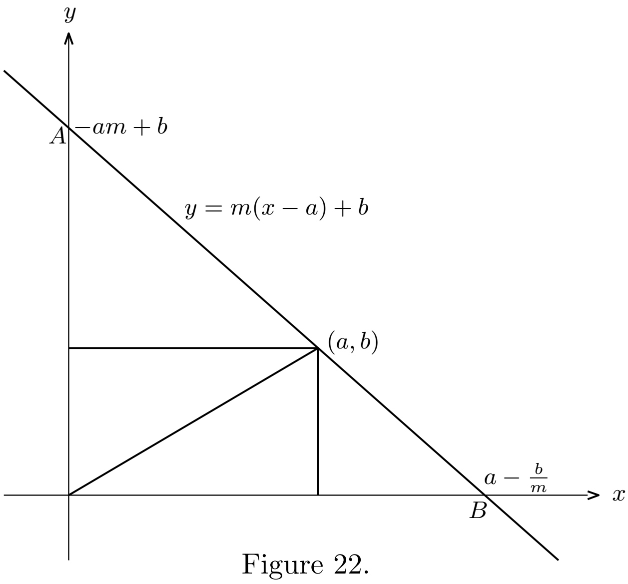

Figure 22.

The general line equation with gradient m is given by, with a constant b

$$y = m(x-a) + b \tag{33}$$

or

$$\frac{y}{m} = x - a + \frac{b}{m}.$$

By $m = 0$, we obtain the equation $x = a$, by the division by zero. This equation may be considered the cases $m = \infty$ and $m = -\infty$, and these cases may be considered by the strictly right logic with the division by zero.

By the division by zero, we can consider the equation (33) as a general line equation.

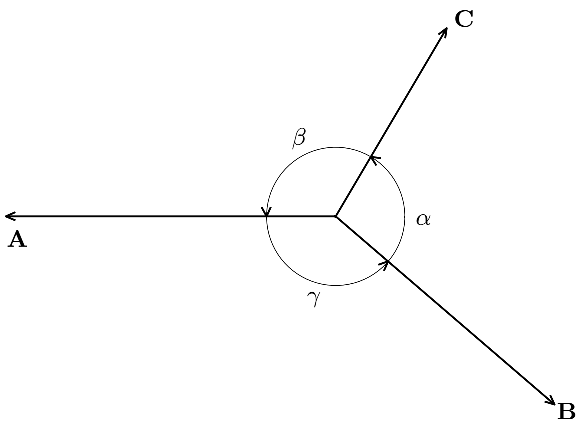

Figure 23.

In the Lami's formula for three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ satisfying

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0},$$

$$\frac{\|\mathbf{A}\|}{\sin\alpha} = \frac{\|\mathbf{B}\|}{\sin\beta} = \frac{\|\mathbf{C}\|}{\sin\gamma},$$

if $\alpha = 0$, then we obtain:

$$\frac{\|\mathbf{A}\|}{0} = \frac{\|\mathbf{B}\|}{0} = \frac{\|\mathbf{C}\|}{0} = 0.$$

Here, of course, α is the angle of $\mathbf{B}$ and $\mathbf{C}$, β is the angle of $\mathbf{C}$ and $\mathbf{A}$, and γ is the angle of $\mathbf{A}$ and $\mathbf{B}$ (see Figure 23).

For the Newton's formula; that is, for a C^2 class function $y = f(x)$, the curvature K at the origin is given by

$$K = \lim_{x \to 0} \left| \frac{x^2}{2y} \right| = \left| \frac{1}{f''(0)} \right|,$$

we have: for $f''(0) = 0$,

$$K = \frac{1}{0} = 0.$$

7.4. **Double natures of the zero point $z = 0$.** Any line on the complex plane arrives at the point at infinity and the point at infinity is represented by zero. That is, a line is, indeed, contains the origin; the true line should be considered as the union of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it it is represented by $z = 0$. We looked this property by analytic geometry and the division by zero calculus in many situations.

In addition, for the general line equation

$$ax + by + c = 0,$$

by using the polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$r = \frac{-c}{a\cos\theta + b\sin\theta}.$$

When $a\cos\theta + b\sin\theta = 0$, by the division by zero, we have $r = 0$; that is, we can consider that the line contains the origin in our sense.

The envelop of the linear lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{34}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{35}$$

The origin of this parabolic function is missing from the envelop of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis. Note that both the x axis $y = 0$ and the parabolic function have the zero gradient at the origin; that will mean that in the reasonable sense the x axis is a tangential line of the parabolic function. Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is an ordinally tangential line of the parabolic function. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (34) for the line (35) is given by

$$\left(\frac{p}{m}, \frac{2p}{m}\right)$$

and it is $(0, 0)$ for $m = 0$.

We can see that the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is reflected to the origin of the point at infinity.

8. Interesting examples in the division by zero

We will give interesting examples in the division by zero. Indeed, the division by zero may be looked in the elementary mathematics and also in the universe.

- For the line
$$\frac{x}{a} + \frac{y}{b} = 1,$$
if $a = 0$, then by the division by zero, we have the line $y = b$. This is a very interesting property creating new phenomena at the term x/a for $a = 0$.

  Meanwhile, from
$$x + a\frac{y}{b} = a,$$
by setting $a = 0$, we have the reasonable result $x = 0$.

  Note that here we can not consider the case $a = b = 0$.
- For the area $S(a, b) = ab$ of the rectangle with sides of lengths a, b, we have
$$a = \frac{S(a, b)}{b}$$
and for $b = 0$, formally
$$a = \frac{0}{0}.$$
However, there exists a contradiction. $S(a, b)$ depends on b and by the division by zero calculus, we have, for the case $b = 0$, the right result
$$\frac{S(a, b)}{b} = a.$$
- We consider 4 lines
$$a_1x + b_1y + c_1 = 0,$$
$$a_1x + b_1y + c_1' = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_2x + b_2y + c_2' = 0,$$

Then, the area S surrounded by these lines is given by the formula

$$S = \frac{|c_1 - c_1'| \cdot |c_1 - c_1'|}{|a_1 b_2 - a_2 b_1|}.$$

Of course, if $|a_1 b_2 - a_2 b_1| = 0$, then $S = 0$.

- $\frac{1}{\sin 0} = \frac{1}{\cos \pi/2} = 0$**.** Consider the linear equation with a fixed positive constant a

$$\frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1.$$

Then, the results are clear from the graphic meanings.

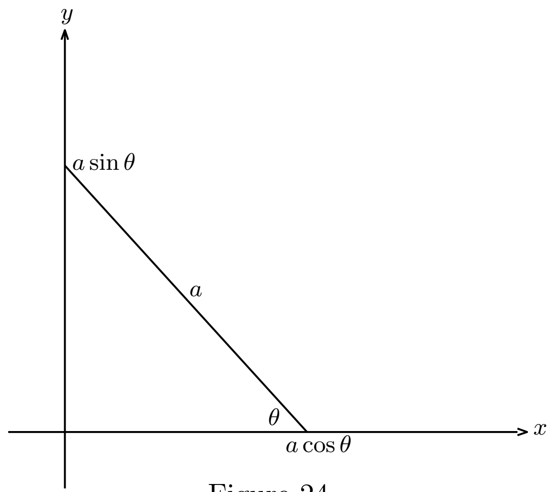

Figure 24.

- For the tangential line at a point $(a\cos\theta, b\sin\theta)$ on the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{36}$$

we have $Q(a/(\cos\theta), 0)$ and $R(0, b/(\sin\theta))$ as the common points with x and y axises, respectively (see Figure 25). if $\theta = 0$, then $Q(a, 0)$ and $R(0, 0)$. If $\theta = \pi/2$, then $Q(0, 0)$ and $R(0, b)$.

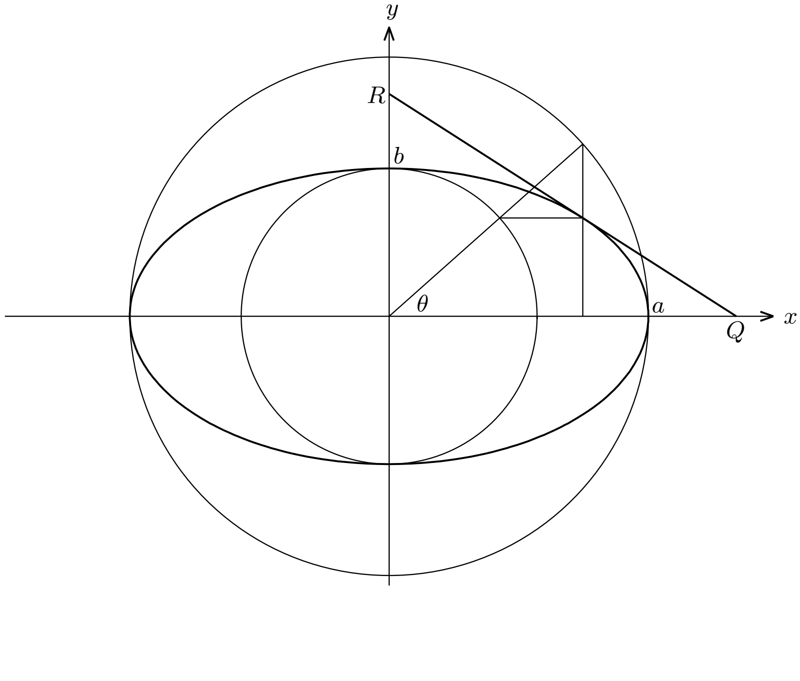

Figure 25.

For the representation

$$\frac{y/b}{1-(x/a)} = \frac{1+(x/a)}{y/b)},$$

we see that the points $(-a,0)$ and $(a,0)$ are also represented by this equation, by the division by zero. If we do not consider the division by zero, these two points are not represented by this equation.

- For the tangential line at the point $(a\cos\theta, b\sin\theta)$ on the elliptic curve, we shall consider the area $S(\theta)$ of the triangle formed by this line and x, y axises

$$S(\theta) = \frac{ab}{|\sin 2\theta|}.$$

Then, by the division by zero calculus, we have $S(0) = 0$.

- The common point of B (resp. B') of a tangential line (36) and the line $x = a$ (resp. $x = -a$) is given by

$$B\left(a, b\tan\frac{\theta}{2}\right) \quad \left(resp.\ \ B'\left(-a, b\cot\frac{\theta}{2}\right)\right)$$

(see Figure 26). The circle with diameter BB' is given by

$$x^2 + y^2 - \frac{2b}{\sin\theta} y - (a^2 - b^2) = 0.$$

Note that this circle passes two forcus points of the elliptic curve. Note that for $\theta = 0$, we have the reasonable result, by the division by zero calculus

$$x^2 + y^2 - (a^2 - b^2) = 0.$$

In the classical theory for quadratic curves, we have to arrange globally it by the division by zero calculus.

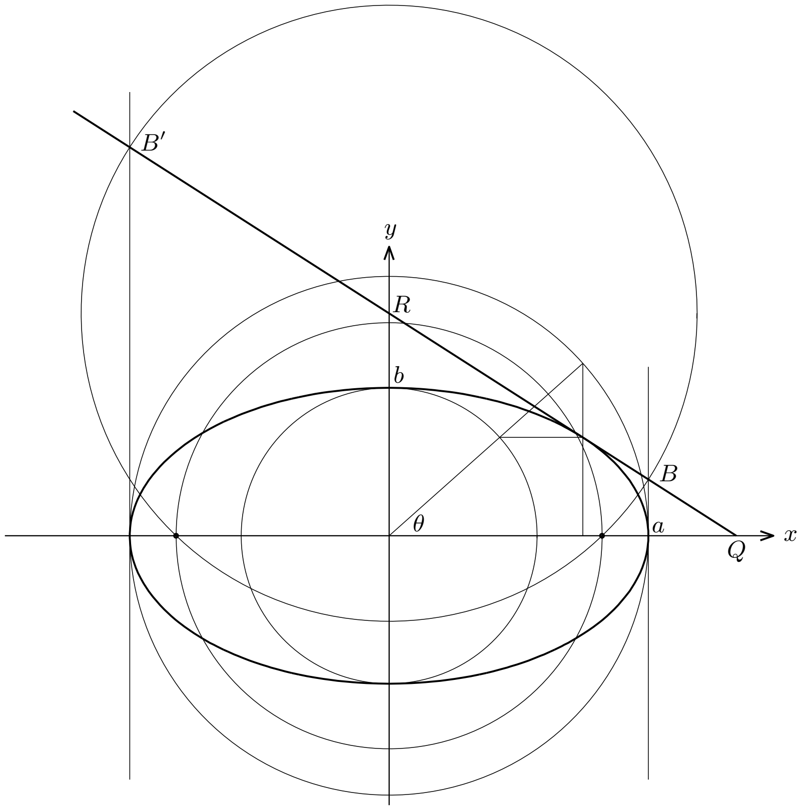

Figure 26.

- The area $S(x)$ surrounded by two x, y axises and the line passing through a fixed point $(a, b), a, b > 0$ and a point $(x, 0)$ is given by

$$S(x) = \frac{bx^2}{2(x-a)}$$

(see Figure 27). For $x = a$, we obtain, by the division by zero calculus, the very interesting value

$$S(a) = ab.$$

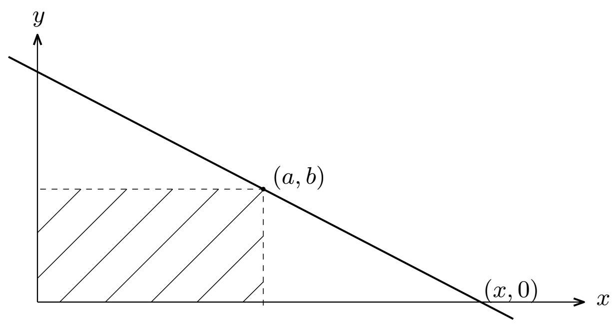

Figure 27.

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For example, for a fixed point $(a,b); a,b > 0$ and a fixed line $y = (\tan\theta)x$, $0 < \theta < \pi$, we will consider the line $L(x)$ passing through the two points (a,b) and $(x,0)$ (see Figure 28). Then, the area $S(x)$ of the triangle surround by the three lines $y = (\tan\theta)x$, $L(x)$ and the x axis is given by

$$S(x) = \frac{b}{2}\frac{x^2}{x-(a-b\cot\theta)}.$$

For the case $x = a - b\cot\theta$, by the division by zero calculus, we have

$$S(a-b\cot\theta) = b(a-b\cot\theta).$$

Note that this is the area of the parallelogram through the origin and the point (a,b) formed by the lines $y = (\tan\theta)x$ and the x axis.

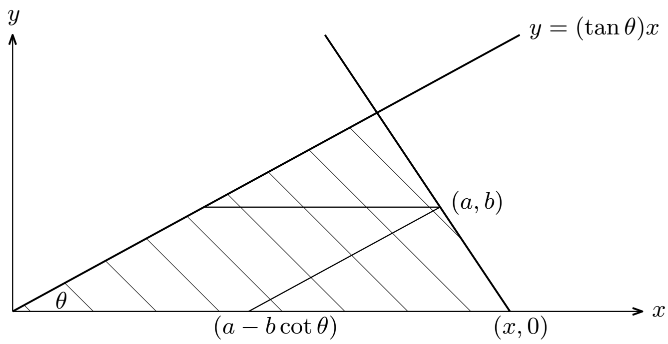

Figure 28.

- We consider an equilateral triangle with vertices $(\pm a/2, \sqrt{3}a/2)$ and the origin. The area $S(h)$ of the triangle surrounded by the three lines that the line through $(0, h+\sqrt{3}a/2)$ and $(-a/2, \sqrt{3}a/2)$, the line through $(0, h+\sqrt{3}a/2)$ and $(a/2, \sqrt{3}a/2)$ and the x- axis is given by

$$S(h) = \frac{\left(h + (\sqrt{3}/2)a\right)^2}{2h}.$$

Then, by the division by zero calculus, we have, for $h = 0$,

$$S(0) = \frac{\sqrt{3}}{2}a^2.$$

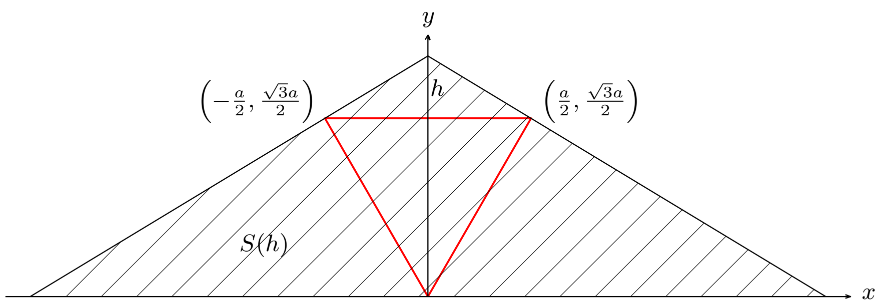

Figure 29.

- Similarly, we will consider the cone formed by the rotation of the line
$$\frac{kx}{a(k+h)}+\frac{y}{k+h}=1$$
and the x, y plane with center the z- axis ($a, h>0$, and a, h are fixed). Then, the volume $V(x)$ is given by
$$V(k)=\frac{\pi}{3}\frac{a^2(k+h)^3}{k^2}.$$
Then, by the division by calculus, we have the reasonable value
$$V(0)=\pi a^2 h.$$
- As in the line case, in the hyperbolic curve
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad a, b>0,$$
by the representations by parameters
$$x=\frac{a}{\cos\theta}=\frac{a}{2}\left(\frac{1}{t}+t\right)$$
and
$$y=\frac{b}{\tan\theta}=\frac{b}{2}\left(\frac{1}{t}-t\right),$$
the origin $(0,0)$ may be included as the point of the hyperbolic curve, as we see from the cases $\theta=\pi/2$ and $t=0$.

In addition, from the fact, we will be able to understand that the asymptotic lines are the tangential lines of the hyperbolic curve.

The two tangential lines of (36) with gradient m is given by
$$y=mx\pm\sqrt{a^2m^2-b^2} \tag{37}$$
and the gradients of the asymptotic lines are
$$m=\pm\frac{b}{a}.$$
Then, we have asymptotic lines $y=\pm\frac{b}{a}x$ as tangential lines in (36).

The common points of (36) and (37) are given by
$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2m}{\sqrt{a^2m^2-b^2}}\right).$$
For the case $a^2m^2-b^2=0$, we have they are $(0,0)$.

- We fix a circle
$$x^2 + (y-a)^2 = a^2, \quad a > 0.$$
From the point $(0, 2a+h)$ $(h > 0)$, we consider two tangential lines to the circle, which meet the x-axis in points $(\pm a\sqrt{(2a+h)/h}, 0)$. Let 2θ be the angle between the two tangential lines at the point $(0, 2a+h)$. From $L(h) = L(\theta) = a\sqrt{(2a+h)/h} = (2a+h)\tan\theta$, we have $h = a(1/\sin\theta - 1)$. Hence we have
$$L(h) = L(\theta) = a\left(\frac{1}{\cos\theta} + \tan\theta\right).$$
Let $S(h)$ be the area of the triangle formed by the tangential lines and the x-axis. Then we have
$$S(h) = S(\theta) = \frac{a}{\sqrt{h}}(h+2a)^{\frac{3}{2}}$$
$$= \frac{a^2}{\cos\theta}\left(\sin\theta + 2 + \frac{1}{\sin\theta}\right).$$
For $h = 0$, by division by zero calculus, we see $L = S = 0$. However for $\theta = 0$, we have $L = a$ and $S = 2a^2$.

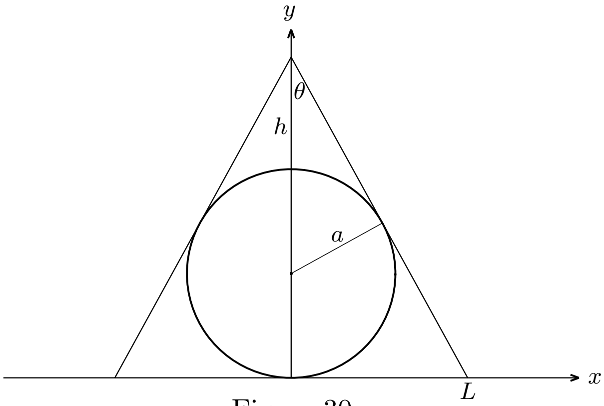

Figure 30.

- We consider two spheres defined by
$$x^2 + y^2 + z^2 + 2a_jx + 2b_jy + 2c_jz + 2d_j = 0, \quad j = 1, 2.$$
Then, the angle θ by two spheres is given by
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2 - (d_1 + d_2)}{\sqrt{a_1^2 + b_1^2 + c_1^2 - 2d_1}\sqrt{a_2^2 + b_2^2 + c_2^2 - 2d_2}}.$$
If two spheres are orthogonal or one sphere is a point sphere, then, $\cos\theta = 0$.
- For the parabolic equation
$$y^2 = 4px,$$
assume that the normal at a point $(ps^2, 2ps)$ meets the parabola again in a point $(pt^2, 2pt)$ (see Figure 31). Then we have
$$(s-t)\{t(s+t)+2\} = 0; \quad s = -t - \frac{2}{t}.$$

The distance r between the two points, which is called a diameter of the parabola, satisfies

$$r^2 = p^2(t-s)^2\{(t+s)^2+4\}.$$

Here, we should consider the case $t = s = 0$ as $r = 0$ and

$$0 = -0 - \frac{2}{0},$$

and the x and y axises are the orthogonal two tangential lines of the parabolic equation.

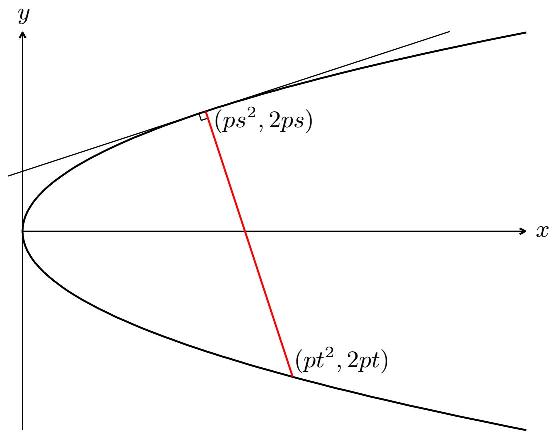

Figure 31.

9. Applications to Wasan geometry

We will introduce typical applications of the division by zero calculus to Wasan geometry (traditional Japanese geometry), however, the results and their impacts will create some new fields in mathematics.

9.1. **Circle and line.** Generalizing a problem in Wasan geometry in [3], we have the following proposition (see Figure 32).

Proposition[16]. *Let α, β, γ be circles of radii a, b, c, respectively. If s and t are tangents of β parallel to each other, α touches s from the same side as β and β externally, and γ touches t from the same side as β and α and β externally, then the following relation holds:*

$$c = \frac{b^2}{4a}. \tag{38}$$

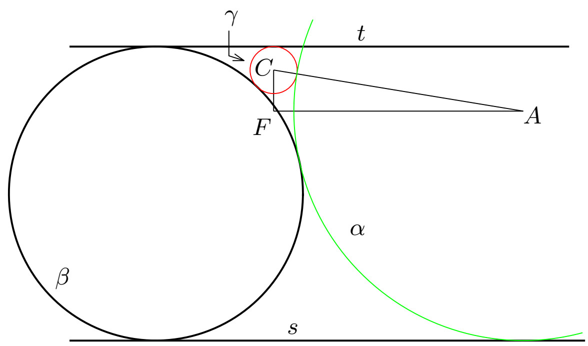

Figure 32.

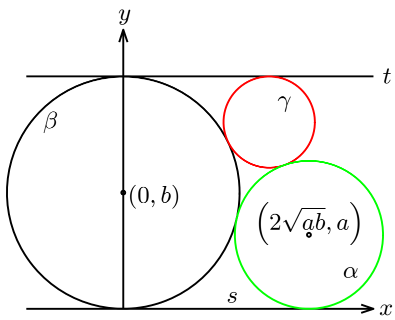

Figure 33.

We now consider the case in which the circle α is a point or a line. It is equivalent to $a = 0$. We setup a rectangular coordinate system with origin at the point of tangency of the circle β and the line s so that the centers of the circles β and α have coordinates $(0, b)$ and $\left(2\sqrt{ab}, a\right)$, respectively (see Figure 33). Then α has an equation

$$\left(x - 2\sqrt{ab}\right)^2 + (y - a)^2 - a^2 = 0. \tag{39}$$

The equation is arranged as

$$\frac{x^2 + y^2}{\sqrt{a}} - 4x\sqrt{b} - 2\sqrt{a}(y - 2b) = 0, \tag{40}$$

and

$$\frac{x^2 + y^2}{a} - 4x\sqrt{\frac{b}{a}} - 2(y - 2b) = 0. \tag{41}$$

If $a = 0$, then the equations (39), (40), (41) imply

$$x^2 + y^2 = 0, \tag{42}$$

$$x = 0, \tag{43}$$

$$y = 2b, \tag{44}$$

respectively. The last three equations show that α is the origin, the y-axis, the line t, respectively. Notice that we can consider that the y-axis touches the circle β. Therefore the three conclusions are reasonable.

We now consider the circle γ in the same case. It has an equation $\left(x - 2\sqrt{bc}\right)^2 + (y - 2b + c)^2 = c^2$. Since $c = b^2/(4a)$, the equation is arranged as

$$a(x^2 + (y - 2b)^2) - 2bx\sqrt{ab} + \frac{b^2 y}{2} = 0, \tag{45}$$

$$\sqrt{a}(x^2 + (y - 2b)^2) - 2bx\sqrt{b} + \frac{b^2 y}{2\sqrt{a}} = 0, \tag{46}$$

$$x^2 + (y - 2b)^2 - 2bx\sqrt{\frac{b}{a}} + \frac{b^2 y}{2a} = 0. \tag{47}$$

If $a = 0$, the equations (45), (46), (47) give

$$y = 0, \tag{48}$$

$$x = 0 \tag{49}$$

$$x^2 + (y - 2b)^2 = 0, \tag{50}$$

respectively by (d1). Hence γ is the x-axis, the y-axis, the point $(0, 2b)$, respectively.

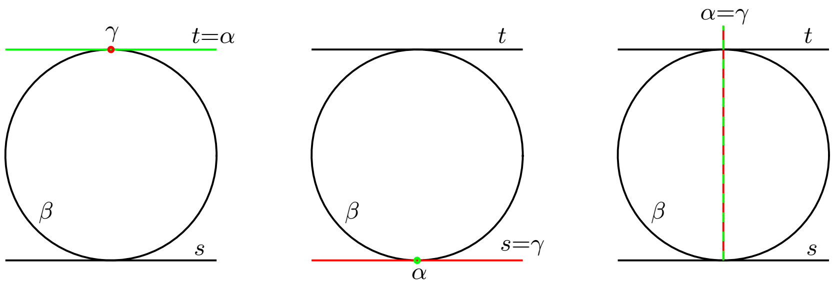

Figure 34. Figure 35. Figure 36.

If α approaches to t, then γ approaches to the point $(0, 2b)$. Therefore we can easily consider that γ is $(0, 2b)$ if α coincides with t (see Figure 34). Symmetrically γ is the line s, if α is the origin (see Figure 35). In the rest of the case, both α and γ coincide with the y-axis (see Figure 36). In all the cases the circle γ is a point or a line, i.e., $c = 0$ by (d2). Therefore (38) still holds in all the three cases.

9.2. **Three externally touching circles.** For real numbers z, and $a, b > 0$, the point $(0, 2\sqrt{ab}/z)$ is denoted by V_z. H. Okumura and M. Watanabe gave the theorem in [13]:

Theorem 7. *The circle touching the circle α: $(x - a)^2 + y^2 = a^2$ and the circle β: $(x + b)^2 + y^2 = b^2$ at points different from the origin O and passing through $V_{z\pm1}$ is represented by*

$$\left(x - \frac{b - a}{z^2 - 1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2 - 1}\right)^2 = \left(\frac{a + b}{z^2 - 1}\right)^2 \tag{51}$$

for a real number $z \neq \pm 1$.

The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{52}$$

Following our concept of the division by zero calculus, we will consider the case $z^2 = 1$ for the singular points in the general parametric representation of the touching circles.

9.2.1. *Results.* First, for $z = 1$ and $z = -1$, respectively by the division by zero calculus, we have from (51), surprisingly

$$x^2 + \frac{b-a}{2}x + y^2 \mp \sqrt{ab}y - ab = 0, \tag{53}$$

respectively [12].

Secondly, multiplying (51) by (z^2-1), we immediately obtain surprisingly (52) for $z = 1$ and $z = -1$, respectively by the division by zero calculus.

In the usual way, when we consider the limiting $z \to \infty$ for (51), we obtain the trivial result of the point circle of the origin. However, the result may be obtained by the division by zero calculus at $w = 0$ by setting $w = 1/z$.

9.2.2. *On the circle appeared.* Let ζ be the circle expressed by (53) with minus sign. Then ζ meets the circles α in two points

$$P_a\left(2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{a}{b}}\right), \quad Q_a\left(\frac{2ab}{9a+b}, -\frac{6a\sqrt{ab}}{9a+b}\right),$$

where $r_{\mathrm{A}} = ab/(a+b)$ (see Figure 37). Also it meets β in points

$$P_b\left(-2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{b}{a}}\right), \quad Q_b\left(\frac{-2ab}{a+9b}, -\frac{6b\sqrt{ab}}{a+9b}\right).$$

The line P_aP_b is the external common tangent of the two circles α and β on the upper half plane. The lines P_aQ_a and P_bQ_b intersect at the point $R : \left(0, -\sqrt{ab}\right)$, which lies on the remaining external common tangent of α and β. Furthermore, ζ is orthogonal to the circle with center R passing through the origin.

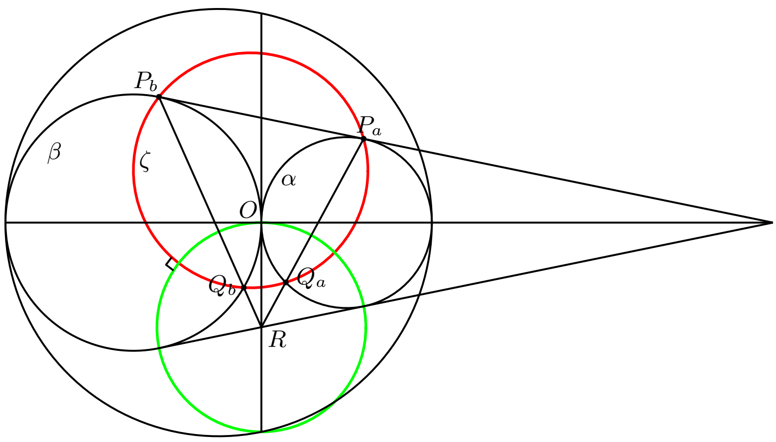

Figure 37.

9.3. **The Descartes circle theorem.** We recall the famous and beautiful theorem ([6, 26]):

Theorem (Descartes) *Let C_i $(i = 1, 2, 3)$ be circles touching to each other of radii r_i. If a circle C_4 touches the three circles, then its radius r_4 is given by*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{54}$$

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

We would like to consider all the cases for the Descartes theorem for lines and point circles, step by step.

9.3.1. *One line and two circles case.* We consider the case in which the circle C_3 is one of the external common tangents of the circles C_1 and C_2. This is a typical case in this paper. We assume $r_1 \geq r_2$. We now have $r_3 = 0$ in (54). Hence

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{0} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 \cdot 0} + \frac{1}{0 \cdot r_1}} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1 r_2}}.$$

This implies

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

in the plus sign case. The circle C_4 is the incircle of the curvilinear triangle made by C_1, C_2 and C_3 (see Figure 38). In the minus sign case we have

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_2}} - \frac{1}{\sqrt{r_1}}.$$

In this case C_2 is the incircle of the curvilinear triangle made by the other three (see Figure 39).

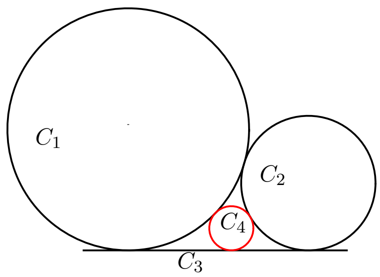

Figure 38.

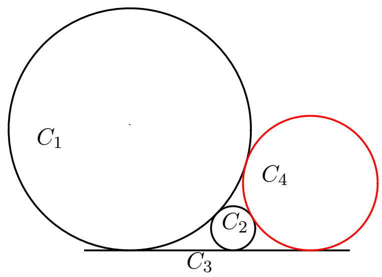

Figure 39.

Of course, the result is known. The result was also well-known in Wasan geometry [29] with the Descartes circle theorem itself.

9.3.2. *Two lines and one circle case.* In this case, the two lines have to be parallel, and so, this case is trivial, because then other two circles are the same size circles, by the division by zero $1/0 = 0$.

9.3.3. *One point circle and two circles case.* This case is another typical case for the theorem. Intuitively, for $r_3 = 0$, the circle C_3 is the common point of the circles C_1 and C_2. Then, there does not exist any touching circle of the three circles $C_j; j = 1, 2, 3$.

For the point circle C_3, we will consider it by limiting of circles attaching to the circles C_1 and C_2 to the common point. Then, we will examine the circles C_4 and the Descartes theorem.

In Theorem 7, by setting $z = 1/w$, we will consider the case $w = 0$; that is, the case $z = \infty$ in the classical sense; that is, the circle C_3 is reduced to the origin.

We look for the circles C_4 attaching with three circles $C_j; j = 1, 2, 3$. We set

$$C_4 : (x - x_4)^2 + (y - y_4)^2 = r_4^2. \tag{55}$$

Then, from the touching property we obtain:

$$x_4 = \frac{r_1 r_2 (r_2 - r_1) w^2}{D},$$

$$y_4 = \frac{2 r_1 r_2 \left(\sqrt{r_1 r_2} + (r_1 + r_2) w\right) w}{D}$$

and

$$r_4 = \frac{r_1 r_2 (r_1 + r_2) w^2}{D},$$

where

$$D = r_1 r_2 + 2\sqrt{r_1 r_2}(r_1 + r_2) w + (r_1^2 + r_1 r_2 + r_2^2) w^2.$$

By inserting these values to (55), we obtain

$$f_0 + f_1 w + f_2 w^2 = 0,$$

where

$$f_0 = r_1 r_2 (x^2 + y^2),$$

$$f_1 = 2\sqrt{r_1 r_2}\big((r_1 + r_2)(x^2 + y^2) - 2 r_1 r_2 y\big)$$

and

$$f_2 = (r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2 r_1 r_2 (r_2 - r_1) x - 4(r_1 + r_2) y + 4 r_1^2 r_2^2.$$

By using the division by zero calculus for $w = 0$, we obtain, for the first, for $w = 0$, the second by setting $w = 0$ after dividing by w and for the third case, by setting $w = 0$ after dividing by w^2,

$$x^2 + y^2 = 0, \tag{56}$$

$$(r_1 + r_2)(x^2 + y^2) - 2 r_1 r_2 y = 0 \tag{57}$$

and

$$(r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2 r_1 r_2 (r_2 - r_1) x - 4 r_1 r_2 (r_1 + r_2) y + 4 r_1^2 r_2^2 = 0. \tag{58}$$

Note that (57) is the red circle in Figure 40 and its radius is

$$\frac{r_1 r_2}{r_1 + r_2} \tag{59}$$

and (58) is the green circle in Figure 40 whose radius is

$$\frac{r_1 r_2 (r_1 + r_2)}{r_1^2 + r_1 r_2 + r_2^2}.$$

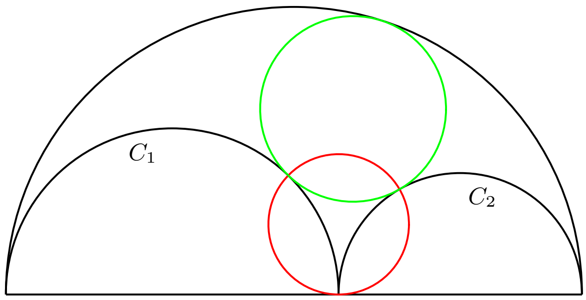

Figure 40.

When the circle C_3 is reduced to the origin, of course, the inscribed circle C_4 is reduced to the origin, then the Descartes theorem is not valid. However, by the division by zero calculus, then the origin of C_4 is changed suddenly for the cases (56), (57) and (58), and for the circle (57), the Descartes theorem is valid for $r_3 = 0$, surprisingly.

Indeed, in (9.4) we set $\xi = \sqrt{r_3}$, then (54) is as follows:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\xi^2} \pm 2\frac{1}{\xi}\sqrt{\frac{\xi^2}{r_1 r_2} + \left(\frac{1}{r_1} + \frac{1}{r_2}\right)}.$$

and so, by the division by zero calculus at $\xi = 0$, we have

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2}$$

which is (59). Note, in particular, that the division by zero calculus may be applied in many ways and so, for the results obtained should be examined some meanings. This circle (57) may be looked a circle touching the origin and two circles C_1 and C_2, because by the division by zero calculus

$$\tan\frac{\pi}{2} = 0,$$

that is a popular property.

Meanwhile, the circle (58) is the attaching circle with the circles C_1, C_2 and the beautiful circle with center $((r_2 - r_1), 0)$ with radius $r_1 + r_2$. The each of the areas surrounded by the three cicles C_1, C_2 and the circle of radius $r_1 + r_2$ is called an arbelos, and the circle (57) is the famous Bankoff circle of the arbelos.

For $r_3 = -(r_1 + r_2)$, from the Descartes identity (10.4), we have (10.4). That is, when we consider that the circle C_3 is changed to the circle with center $((r_2 - r_1), 0)$ with radius $r_1 + r_2$, the Descartes identity holds. Here, the minus sign shows that the circles C_1 and C_2 touch C_3 internally from the inside of C_3.

9.3.4. *Two point circles and one circle case.* This case is trivial, because, the exterior touching circle is coincident with one circle.

9.3.5. *Three points case and three lines case.* In these cases we have $r_j = 0, j = 1, 2, 3$ and the formula (54) shows that $r_4 = 0$. This statement is trivial in the general sense.

As the solution of the simplest equation

$$ax = b, \tag{60}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (60) is impossible. The zero will represent some **impossibility**.

In the Descartes theorem, three lines and three points cases, we can understand that the attaching circle does not exist, or it is the point and so the Descartes theorem is valid.

9.4. **Circles and a chord.** We recall the following result of the old Japanese geometry [28, 26, 13] (see Figure 41):

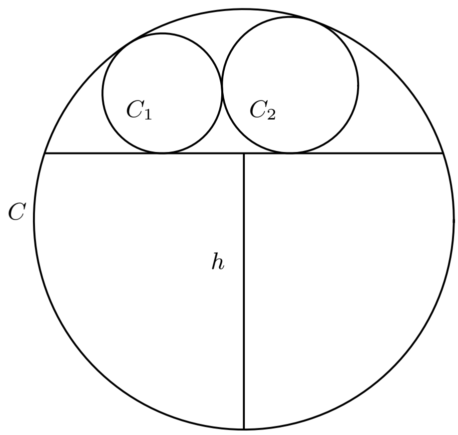

Figure 41.

Lemma 10. *Assume that the circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

We are interesting in the limit case $r_1 = 0$ or $r_2 = 0$.

9.4.1. *Results.* We introduce the coordinates in the following way: the bottom of the circle C is the origin and tangential line at the origin of the circle C is the x axis and the y axis is given as in the center of the circle C is $(0, r)$. We denote the centers of the circles $C_j; j = 1, 2$ by (x_j, y_j), then we have

$$y_1 = h + r_1, \quad y_2 = h + r_2.$$

Then, from the attaching conditions, we obtain the three equations:

$$(x_2 - x_1)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2,$$
$$x_1^2 + (h - r + r_1)^2 = (r - r_1)^2$$

and

$$x_2^2 + (h - r + r_2)^2 = (r - r_2)^2.$$

Solving the equations for x_1, x_2 and r_2, we get four sets of the solutions. Let $h = 2r_3$, $v = r - r_1 - r_3$. Then two sets are:

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \pm 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

The other two sets are

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \mp 2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

We now consider the solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{g_0 + g_1 r_1 + g_2 r_1^2 + g_3}{(r_1 + r_3)^2},$$

where

$$g_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$
$$g_1 = 2r_3((x - \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$
$$g_2 = (x - 2\sqrt{rr_3})^2 + y^2 - 2r_3 y,$$

and

$$g_3 = 4r_3\sqrt{v}(r_1(\sqrt{r}y - \sqrt{r_3}x) - r_3\sqrt{r_3}x).$$

We now consider another solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= -2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x-x_2)^2+(y-y_2)^2-r_2^2=\frac{k_0+k_1r_1+k_2r_1^2+k_3}{(r_1+r_3)^2},$$

where

$$k_0=r_3^2(x^2+y(y-4r_3)+4rr_3),$$

$$k_1=2r_3((x+\sqrt{rr_3})^2+y^2-(2r+3r_3)y+3rr_3),$$

$$k_2=(x+2\sqrt{rr_3})^2+y^2-2r_3y,$$

and

$$k_3=-4r_3\sqrt{v}(r_1(\sqrt{r}y+\sqrt{r_3}x)+r_3\sqrt{r_3}x).$$

We thus see that the circle C_2 is represented by

$$(g_0+g_3)+g_1r_1+g_2r_1^2=0$$

and

$$(k_0+k_3)+k_1r_1+k_2r_1^2=0.$$

For the symmetry, we consider only the above case. We obtain the division by zero calculus, first by setting $r_1=0$, the next by setting $r_1=0$ after dividing by r_1 and the last by setting $r_1=0$ after dividing by r_1^2,

$$g_0+g_3=0,$$

$$g_1=0,$$

and

$$g_2=0.$$

That is,

$$\left(x-\sqrt{2rh-h^2}\right)^2+(y-h)^2=0,$$

$$\left(x-\sqrt{\frac{rh}{2}}\right)^2+\left(y-\left(r+\frac{3h}{4}\right)\right)^2=r^2+\frac{9}{16}h^2,$$

and

$$\left(x-\sqrt{2rh}\right)^2+\left(y-\frac{h}{2}\right)^2=\left(\frac{h}{2}\right)^2.$$

The first equation represents one $(\sqrt{2rh-h^2},h)$ of the points of intersection of the circle C and the chord t (see Figure 42). The second equation expresses the red circle in the figure. The third equation expresses the circle touching C externally, the x-axis and the extended chord t denoted by the green circle in the figure. The last two circles are orthogonal to the circle with center origin passing through the points of intersection of C and t.

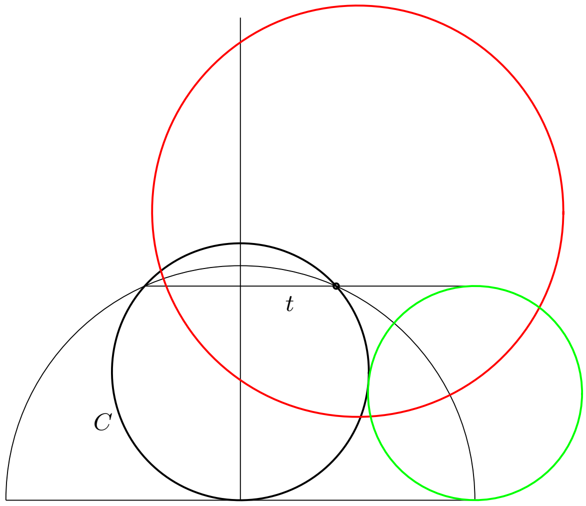

Figure 42

Now for the beautiful identity in the lemma, for $r_1 = 0$, we have, by the division by zero,

$$\frac{1}{0} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{0 \cdot r_2 h}}$$

and

$$r_2 = -\frac{h}{2}.$$

Here, the minus sigh will mean that the blue circle is attaching with the circle C in the outside of the circle C; that is, we can consider that when the circle C_1 is reduced to the point $(\sqrt{2rh - h^2}, h)$, then the circle C_2 is suddenly changed to the blue circle and the beautiful identity is still valid. Note, in particular, the blue circle is attaching with the circle C and the cord t.

Meanwhile, for the curious red circle, we do not know its property, however, we know curiously that it is orthogonal with the circle with the center at the origin and with radius $\sqrt{2rh}$ passing through the points $(\pm\sqrt{2rh - h^2}, h)$.

This subsection is based on the paper [18].

10. Conclusion

Apparently, the common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero.

The division by zero theory may be developed and expanded greatly.

We have to arrange globally our modern mathematics with our division by zero in our undergraduate level.

We have to change our basic ideas for our space and world.

We have to change globally our textbooks and scientific books on the division by zero.

For a systematic development, we are founding the new international journal on the division by zero calculus as in [19, 25].

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