ANALYTIC EXPANSIONS AND AN APPLICATION TO FUNCTION THEORY

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Abstract. In this paper we introduce and study the notion of singularity, the kernel and analytic expansions. We provide an application to the existence of singularities of solutions to certain polynomial equations.

1. **Introduction**

In this section we introduce the notion of an expansion in a **mixed** and **specific** directions. We launch the following extension program

Definition 1.1. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $\mathcal{S} \in \mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ in the direction x_i for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \longrightarrow \mathcal{F}$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

with

$$\nabla_{[x_i]}(\mathcal{S}) = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i} \right).$$

The value of the l th expansion at a given value a of x_i is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i](a)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i](a)}(\mathcal{S})$ is a tuple of polynomials in $\mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Similarly by an expansion in the mixed direction $\otimes_{i=1}^{l}[x_{\sigma(i)}]$ we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^{l}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

for any permutation $\sigma : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value a_i of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$ is tuple of real numbers $\mathbb{R}$.

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2. Some notions from single variable expansity theory

In this section we recall some notions under single variable expansivity theory developed earlier [1]. These notions will serve as a model to proving the main result in this section.

Definition 2.1. Let $\mathcal{F} = \{\mathcal{S}_m\}_{m=1}^{\infty}$ be a family of tuples of polynomials in the ring $\mathbb{R}[x]$, each having at least two entries with distinct degrees. Then the value of n such that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) \neq \mathcal{S}_0$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}) = \mathcal{S}_0$ where $\mathcal{S}_0 = (0, 0, \ldots, 0)$ is called the degree of expansion and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S})$ is the rank of expansion, denoted by $\mathcal{R}(\mathcal{S})$.

Theorem 2.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be tuples of polynomials in the ring $\mathbb{R}[x]$, with $deg(\mathcal{S}_1) > deg(\mathcal{S}_2)$, satisfying certain initial conditions at each phase of expansion. If $\mathcal{R}(\mathcal{S}_1) = \mathcal{R}(\mathcal{S}_2)$, then there exist some j satisfying $1 \leq j < deg(\mathcal{S}_1)$ such that $\mathcal{S}_1^j = \mathcal{S}_2$.*

Proof. Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are tuples of polynomials in the ring $\mathbb{R}[x]$. Let $deg(\mathcal{S}_1) = k_1$ and $deg(\mathcal{S}_2) = k_2$. By definition 2.1, we can write $\mathcal{R}(\mathcal{S}_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1}(\mathcal{S}_1)$ and $\mathcal{R}(\mathcal{S}_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2}(\mathcal{S}_2)$. Under the assumption that $\mathcal{R}(\mathcal{S}_1) = \mathcal{R}(\mathcal{S}_2)$, we must have that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2}(\mathcal{S}_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1}(\mathcal{S}_1)$ if and only if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1 - k_2}(\mathcal{S}_1) = \mathcal{S}_2$. Since $1 \leq k_1 - k_2 < k_1 = deg(\mathcal{S}_1)$, the result follows immediately. □

Definition 2.3. Let $\{\mathcal{S}^m\}_{m=1}^{\infty}$ be a family of expanded tuples of $\mathcal{S}$, having at least two entries with distinct degrees. Then the limit of expansion of $\mathcal{S}$ is the first expanded tuple $\mathcal{S}^j = (g_1, g_2, \ldots, g_n)$ such that $\deg(g_1) = \deg(g_2) = \cdots = \deg(g_n)$ for $n \geq 3$ and $1 \leq j \leq m$. Notation-wise, we denote simply by

$$\lim(\mathcal{S}^n) = \mathcal{S}^j,$$

the limit of the expansion.

Theorem 2.4. *Let $\{\mathcal{S}^m\}_{m=1}^{\infty}$ be a family of expansions of the tuple $\mathcal{S}$ of polynomials in the ring $\mathbb{R}[x]$, such that at least two entries have distinct degree. Then the limit of expansions $\lim(\mathcal{S}^n)$ of $\mathcal{S}$ exists.*

Proof. Let $\{\mathcal{S}^m\}_{m=1}^{\infty}$ be a family of expansions of the tuple $\mathcal{S}$ of polynomials in the ring $\mathbb{R}[x]$, having at least two entries with distinct degree. Suppose the limit of expansion does not exist, and let $\mathcal{S}^1 = (f_1, f_2, \ldots, f_n)$ be the first phase expansion of $\mathcal{S}$, then it follows that $\deg(f_i) \neq \deg(f_j)$ for some $1 \leq i, j \leq n$ with $i \neq j$. It follows in particular that $\mathcal{S}^1 \neq \mathcal{R}(\mathcal{S})$ and $\mathcal{S}^1 \neq \mathcal{S}_0$. Thus the second phase expansion exists and let $\mathcal{S}^2 = (g_1, g_2, \ldots, g_n)$ be the second phase expanded tuple. Again, it follows from the hypothesis that $\deg(g_i) \neq \deg(g_j)$ for some $1 \leq i, j \leq n$, with $i \neq j$, and it follows in particular that $\mathcal{S}^2 \neq \mathcal{R}(\mathcal{S})$ and $\mathcal{S}^2 \neq \mathcal{S}_0$. Thus the third phase expansion exist. By induction it follows that the tuple $\mathcal{S}$ of $\mathbb{R}[x]$ admits infinite number of expansions, thereby contradicting Proposition **??**. □

Theorem 2.5. *Let $\{\mathcal{S}^n\}_{n=1}^{\infty}$ be a family of expanded tuples of the tuple $\mathcal{S}$ of polynomials in the ring $\mathbb{R}[x]$, such that at least two entries have distinct degrees and satisfying certain initial conditions at each phase of expansion. Then there exist some number k called the dimension of expansion ($dim(\mathcal{S})$), such that $\lim(\mathcal{S}^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$ for some $k < deg(\mathcal{S})$.*

Proof. Let $\mathcal{S}$ be any tuple of polynomials in the ring $\mathbb{R}[x]$ that can be expanded, with at least two entries having distinct degree. Then, the limit exists by Theorem 2.4 and since an expansion can only be applied at a finite number of time and the map $\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is a recovery which exist, it is clear there will exist such number k, so that $\lim(\mathcal{S}^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$. We only need to show that k lies in the stated range. In anticipation of a contradiction, let us suppose $\lim(\mathcal{S}^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$ for any $k \geq deg(\mathcal{S})$. Since the map is a bijection, it follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k(\lim(\mathcal{S}^n)) = \mathcal{R}(\mathcal{S})$. It is easy to see that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k(\lim(\mathcal{S}^n) = \mathcal{S}_0$, in which case we have that $\mathcal{R}(\mathcal{S}) = \mathcal{S}_0$, and so the rank of an expansion is null, which is a contradiction by definition 2.1. □

Definition 2.6. Let $\mathcal{S}$ be a tuple of polynomial in the ring $\mathbb{R}[x]$ and $\{\mathcal{S}^m\}_{m=1}^{\infty}$ the family of expanded tuple of $\mathcal{S}$. Then by the local number of expansion, denoted $\mathcal{L}(\mathcal{S})$, we mean the value of n such that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) = \lim(\mathcal{S}^m)$.

Invoking Theorem 2.5, It follows from the above definition that for any tuple of polynomial in the ring $\mathbb{R}[x]$ satisfying certain initial conditions at each phase of expansion,

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$$

if and only if

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+k}(\mathcal{S}) = \mathcal{R}(\mathcal{S}).$$

By the definition of the rank of an expansion, it follows that

$$n + k = deg(\mathcal{S})$$

which we call the principal equation and where $\mathcal{L}(\mathcal{S}) = n$, $dim(\mathcal{S}) = k$ and $deg(\mathcal{S})$ are the local number, the dimension and the degree of expansion, respectively, on $\mathcal{S}$. It is interesting to recognize that the value of the local number $\mathcal{L}(\mathcal{S})$ in any case is bounded cannot be more than the dimension of expansion. This assertion is confirmed in the following sequel.

Lemma 2.7. *Let $\mathcal{S}$ be a tuple of polynomials in the ring $\mathbb{R}[x]$, satisfying certain initial conditions at each phase with $deg(\mathcal{S}) \geq 4$. If $dim(\mathcal{S}) > 2$, then the local number $\mathcal{L}(\mathcal{S})$ must satisfy the inequality*

$$0 \leq \mathcal{L}(\mathcal{S}) \leq 2.$$

Proof. Let us suppose on the contrary $\mathcal{L}(\mathcal{S}) > 2$. Then it follows from the principal equation that $dim(\mathcal{S}) < deg(\mathcal{S}) - 2$, so that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{dim(\mathcal{S})+2}(\mathcal{S}) \neq \mathcal{R}(\mathcal{S})$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{dim(\mathcal{S})+2}(\mathcal{S}) \neq \mathcal{S}_0$. It follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{dim(\mathcal{S})+2}(\mathcal{S}) = \mathcal{S}_1$. Theorem 2.2 gives $\mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{S}_1)$, and we have that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{deg(\mathcal{S})}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{deg(\mathcal{S}_1)}(\mathcal{S}_1),$$

if and only if

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{deg(\mathcal{S})-deg(\mathcal{S}_1)}(\mathcal{S}) = \mathcal{S}_1.$$

It follows therefore that $deg(\mathcal{S}) - deg(\mathcal{S}_1) = dim(\mathcal{S}) + 2$. Again, using the principal equation, we find that

$$\mathcal{L}(\mathcal{S}) = deg(\mathcal{S}_1) + 2.$$

It follows from the above equation that $deg(\mathcal{S}_1)+2=\mathcal{L}(\mathcal{S})=deg(\mathcal{S})-dim(\mathcal{S})<deg(\mathcal{S})-2$, so that $deg(\mathcal{S}_1)+4<deg(\mathcal{S})$. Since $deg(\mathcal{S})\geq 4$, it must be that $deg(\mathcal{S}_1)+4\leq 4$, and we have that $deg(\mathcal{S}_1)\leq 0$. This leaves us with the only choice that $deg(\mathcal{S}_1)=0$, contradicting the fact that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})\neq\mathcal{R}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})\neq\mathcal{S}_0$, and the proof is complete. □

3. The kernel of an expansion

In this section we introduce the notion of the kernel of an expansion. One could draw some parallels with this notion and the notion of the boundary points of an expansion under the single variable theory. This choice of terminology is appropriate for this context, since we are no longer considering points as being solutions to our tuple equation but instead tuples consisting of solutions to certain partial differential equation. We launch formally the following languages.

Definition 3.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of l-tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. By the **kernel** of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S})$, denoted $\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S})]$ we mean

$$\begin{aligned}\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S})]=\Big\{&(f_1,f_2,\ldots,f_l)\mid\\ &f_r\in\mathbb{F}_{\mathbb{C}}[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n],\\ &(1\leq r\leq l),\mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i](f_r)}(\mathcal{S})]=0\Big\}\end{aligned}$$

where $\mathbb{F}_{\mathbb{C}}$ is a function field with complex number $\mathbb{C}$ base space. We call each tuple in the kernel an **annihilator** of the given expansion.

Definition 3.2. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of l-tuples of polynomials in the ring $\mathbb{C}[x_1,x_2,\ldots,x_n]$. We denote by

$$\mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S})](f_r)_{x_i}$$

the value of

$$\mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S})]$$

at $x_i=f_r$.

Proposition 3.1. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of l-tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. If for $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{F}$*

$$\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_1)]=\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_2)]$$

then $\mathcal{S}_1=\mathcal{S}_2+\mathcal{S}_{\mathbb{C}[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]}$ and where $\mathcal{S}_{\mathbb{C}[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]}$ is an l tuple of polynomials in the ring $\mathbb{C}[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]$.

Proof. Let us suppose $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{F}$ and

$$\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_1)]=\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_2)].$$

For any $(f_1,f_2,\ldots,f_l)\in\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_1)]$ then $(f_1,f_2,\ldots,f_l)\in\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_2)]$ so that we can write

$$\mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i](f_r)}(\mathcal{S}_1)]=\mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i](f_r)}(\mathcal{S}_2)]=0$$

for $1 \leq r \leq l$. Appealing to Definition 3.2 we can write

$$\begin{aligned}\mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_1)](f_r)_{x_i} &= \mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i](f_r)}(\mathcal{S}_1)]\\ &= 0\\ &= \mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i](f_r)}(\mathcal{S}_2)]\\ &= \mathrm{Id}_r[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_2)](f_r)_{x_i}\end{aligned}$$

for $1 \leq r \leq l$. It follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_1) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_2)$$

in

$$\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_1)] = \mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_i]}(\mathcal{S}_2)]$$

so that by the linearity of an expansion in a specific direction, it follows that

$$\mathcal{S}_1 = \mathcal{S}_2 + \mathcal{S}_{\mathbb{R}[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]}$$

since an expansion in a specific direction is uniquely determined by their kernel. □

Remark 3.3. Next we highlight the possibility two separate disparate expansions in separate directions at spots not quite equivalent can have the same kernel. The proposition below underscores this possibility. Put it differently, any two expansions need not happen in the same direction to have the chance of having the same kernel. That is to say, all hybrids expansions should in principle have the same kernel of their expansions.

Definition 3.4. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. We say the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is **diagonalizable** in the direction $[x_j]$ $(1 \leq j \leq n)$ at the spot $\mathcal{S}_r \in \mathcal{F}$ with order k with $\mathcal{S} - \mathcal{S}_r$ not a tuple of $\mathbb{R}$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^k_{[x_j]}(\mathcal{S}_r).$$

We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)$ the **diagonal** of the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ of **order** $k \geq 1$. We denote with $\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)]$ the order of the diagonal.

Lemma 3.5. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials belonging to the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then the mixed expansion*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

is diagonalizable in each direction $[x_{\sigma(i)}]$ for $1 \leq i \leq l$.

Proof. Let us consider the mixed expansion

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

and let $[x_{\sigma(j)}]$ for $1 \leq j \leq l$ be our targeted direction, then by appealing to the commutative property of an expansion we have

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{\substack{i=1\\ i\neq j}}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$

Next let us consider the residual mixed expansion

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{\substack{i=1\\i\neq j}}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{\substack{i=1\\i\neq j}}^{l}[x_{\sigma(i)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}).$$

If there exist some tuple $\mathcal{S}_a\in\mathcal{F}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}(\mathcal{S}_a)$$

then we make a substitution and obtain two copies of the expansion operator $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}$ by virtue of the commutative property of an expansion. Otherwise we choose

$$\mathcal{S}_b=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

and apply the remaining operators on it. By repeating the iteration in this manner, we will obtain

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_{\sigma(j)}]}(\mathcal{S}_t)$$

for $k\geq 1$ and for some $\mathcal{S}_t\in\mathcal{F}$. This completes the proof of the proposition. ☐

Proposition 3.2. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. If for $k\neq j$ with $1\leq j,k\leq l$, then there exists some $\mathcal{S}_t,\mathcal{S}_r\in\mathcal{F}$ with $\mathcal{S}_t-\mathcal{S}_r\neq\mathcal{S}_{\mathbb{R}}$ and some $u,v\geq 1$ such that*

$$\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_{\sigma(i)}]}(\mathcal{S}_t)]=\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_{\sigma(j)}]}(\mathcal{S}_r)].$$

Proof. Appealing to Lemma 3.5 then under the assumption $k\neq j$ with $1\leq j,k\leq l$ there exists some $u,v\geq 1$ and $\mathcal{S}_t,\mathcal{S}_r\in\mathcal{F}$ with $\mathcal{S}_t-\mathcal{S}_r\neq\mathcal{S}_{\mathbb{R}}$ such that we can write for $\mathcal{S}\in\mathcal{F}$

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_{\sigma(k)}]}(\mathcal{S}_t)$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_{\sigma(j)}]}(\mathcal{S}_r)$$

so that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_{\sigma(k)}]}(\mathcal{S}_t)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_{\sigma(j)}]}(\mathcal{S}_r)$$

and the claim follows immediately. ☐

4. Singularity and singular points of an expansion

In this section we introduce the notion of **singularity** and associated **singular** points of an expansion in a specific direction. We launch the following terminology.

Definition 4.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of l-tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$ and $\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S})]$ be the **kernel** of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S})$. Then by a **singular** point of the expansion we mean a tuple $\mathcal{S}=(a_1,\ldots a_{i-1},a_{i+1},\ldots,a_n)$ with $a_j\in\mathbb{C}$ such that for some annihilator

$$(f_1,f_2,\ldots,f_l)\in\mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S})]$$

then

$$f_i[(a_1,\ldots a_{i-1},a_{i+1},\ldots,a_n)]=\infty$$

for some $1 \leq i \leq l$. We call the collection of all such points the singularity of the expansion and denote with

$$\mathrm{Sing}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S})] = \bigg\{(a_1, \ldots a_{i-1}, a_{i+1}, \ldots, a_n) \in \mathbb{C}^{n-1} | \quad f_i[(a_1, \ldots a_{i-1}, a_{i+1}, \ldots, a_n)] = \infty\bigg\}.$$

5. **Analytic Expansions**

In this section we introduce and study the notion analytic expansions in specified directions in a particular domain in space.

Definition 5.1. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $\mathcal{D} \subset \mathbb{C}^{n-1}$. Then we say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S})$ is analytic in $\mathcal{D}$ if

$$\mathcal{D} \cap \mathrm{Sing}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S})] = \emptyset.$$

If the expansion is analytic in the entire $\mathcal{C}^{n-1}$ then we say for simplicity it is analytic.

Definition 5.2. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ be an expansion. Then by the **unionization** stage of the expansion, we mean the least of value of j such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{j}_{[x_i](0)}(\mathcal{S}) = \mathcal{S}_o.$$

Definition 5.3. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of l-tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ be an expansion. By the **normalization** stage of the expansion, denoted $\varrho[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]$, we mean the smallest value of k such that

$$\mathrm{Ind}_{\mathrm{Id}_r[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S})]}(x_i) = \mathrm{Ind}_{\mathrm{Id}_s[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S})]}(x_i)$$

for all $1 \leq r, s \leq l$ with $r \neq s$. We call the corresponding expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S})$$

the **fibre** of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$.

Remark 5.4. Throughout this paper we will assume the normalization stage of an expansion is satisfies $\varrho[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})] > 0$ by working with tuples of multivariate polynomials with at least entries of distinct degree of the underlying direction.

Proposition 5.1. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ be an expansion. Then the **unionization** stage of the expansion satisfies the inequality*

$$j \geq \left\lfloor \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]}{\varrho[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]} \right\rfloor.$$

Proof. The normalization stage of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ is given by $\varrho[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]$ so that the unionization stage is the index of the normalization stage of the expansion obtained as

$$j \geq \left\lfloor \frac{\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]}{\varrho[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})]} \right\rfloor.$$

□

It is important to notice that the notion of the **normalization** stage is analogous to and has parallels with the notion of the limit of an expansion under the single variable theory. As such the notion of the local number runs exactly parallel to the notion of the **normalization** stage in multivariate expansivity theory. Next we prove the following Proposition, which will eventually feature in closing our argument.

Proposition 5.2. *Let $\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_i)\}_{i=1}^{\infty}$ be a collection of expansions in the direction x_i of l-tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$ and $\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S}_i')\}_{i=1}^{\infty}$ be a collection of expansions of tuples of polynomials in the ring $\mathbb{R}[x]$. Then the map*

$$\chi_{(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)}:\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_i)\}_{i=1}^{\infty}\longrightarrow\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S}_i')\}_{i=1}^{\infty}$$

for a fixed $(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)\in\mathbb{R}^{n-1}$ such that for any $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})\in\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_i)\}_{i=1}^{\infty}$ then

$$\chi_{(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S})(a_1,a_2,\ldots,a_{i-1},\\ a_{i+1},\ldots,a_n)\in\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S}_i')\}_{i=1}^{\infty}$$

is an isomorphism. We denote the isomorphism by

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_i)\simeq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S}_i').$$

Proposition 5.3. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of l-tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ be an expansion. Then*

$$\varrho[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]\leq 2.$$

Proof. Let $\mathcal{S}\in\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ then we fix all other directions x_j for all $j\neq i$ so that the expansion

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})\simeq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S})(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$$

where $(\mathcal{S})(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$ is a tuple of polynomials in the ring $\mathbb{R}[x_i]$. Appealing to Lemma 2.7 we obtain the inequality

$$\mathcal{L}\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S})(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)\bigg]\leq 2$$

and by appealing to Proposition 5.2 we recover the following inequality

$$\varrho[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]\leq 2.$$

□

Remark 5.5. Next we show that the unionization stage of a typical expansion cannot be too big. In other words it cannot possibly be the case that the totient of an expansion in a specific direction coincides with the unionization stage. We show that it can in fact be a lot smaller than the expected value in the following result.

Theorem 5.6 (Analytic range)**.** *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ be an expansion. Then the expansion is analytic in the range*

$$j\geq\bigg\lfloor\frac{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]}{2}\bigg\rfloor.$$

Proof. Every expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is always analytic at the unionization stage so that by appealing to Proposition 5.1, we note that

$$j \geq \left\lfloor \frac{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]}{\varrho[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]} \right\rfloor$$
$$\geq \left\lfloor \frac{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]}{2} \right\rfloor$$

by appealing to Proposition 5.3 so that the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{j}_{[x_i]}(\mathcal{S})$ is analytic in the range

$$j \geq \left\lfloor \frac{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]}{2} \right\rfloor.$$

□

6. An application to function theory

In this section we illustrate how these notion could be used to study the certain statistics about functions. We show that we can use these notions to study the existence of singularities of certain multivariate functions which are solutions to certain polynomial equations.

Corollary 6.1. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{j}_{[x_i]}(\mathcal{S})$ be an expansion. Then for

$$j < \left\lfloor \frac{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]}{2} \right\rfloor$$

there exist some $(f_1, f_2, \ldots, f_l) \in \mathrm{Ker}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{j}_{[x_i]}(\mathcal{S})]$ and

$$(a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \in \mathbb{C}^{n-1}$$

such that

$$f_i[(a_1, \ldots a_{i-1}, a_{i+1}, \ldots, a_n)] = \infty$$

for some $1 \leq i \leq l$.

[1].

References

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