AN EXTENSION OF THE ERDŐS-TURÁN ADDITIVE BASE CONJECTURE VIA GENERALIZED CIRCLES OF PARTITION

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Abstract. In this paper we continue the development of the method of Circles of Partition by introducing the notion of generalized circles of partition. This is an extension program of the notion of circle of partition developed in our first paper [1]. As an application we prove an analogue of the Erdős-Turán additive base conjecture.

1. **Introduction**

In [1] we have developed a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of $\mathbb{N}$. The method is very elementary in nature and has parallels with configurations of points on the geometric circle.

Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u+v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then the new method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as the results obtained in this setting are then transformed back to results concerning the partition of integers.

Notations. We denote by $\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\}$ the **sequence** of the first n natural numbers and by $\mathbb{M}_{a,d}$ the set

$$\mathbb{M}_{a,d} := \{x \in \mathbb{N} \mid x \equiv a \,(\operatorname{mod} d)\,, d \in \mathbb{N}\}. \tag{1.1}$$

Then by virtue of our earlier studies

$$\mathcal{C}(n, \mathbb{M}_{a,d}) = \{[x] \mid x \in \mathbb{M}_{a,d}, x \leq n-a\}, n \in \mathbb{M}_{2a,d}.$$

For any subset $\mathbb{A} \subset \mathbb{N}$ and $t \in \mathbb{N}$, we will denote with $\mathbb{A}^t$ the set

$$\mathbb{A}^t := \left\{\prod_t a \middle|\ a \in \mathbb{A}\right\}$$

as the t-fold prod-set of the set $\mathbb{A}$ and $\prod_t$ denotes the product of t elements - not necessarily distinct - of the set $\mathbb{A}$. Additionally for a CoP with generators belonging to a special class of integers, say $\mathbb{G}$, and base set $\mathbb{M}$ we will write the corresponding CoP simply as $\mathcal{C}(\mathbb{G}(n), \mathbb{M})$. As is customary we will write $f(n) \gg g(n)$ for any two arithmetic functions if there exists a constant $c > 0$ such that $|f(n)| \geq cg(n)$. If the constant depends on another variable say k, then we will denote the relation

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as $f(n) \gg_k g(n)$. For any subset $\mathbb{A} \subset \mathbb{N}$, we still reserve the quantity $\mathcal{D}(\mathbb{A})$ for the density of the set $\mathbb{A}$ relative to the set of all integers $\mathbb{N}$.

2. **Generalized circles of partition**

In this section we introduce and study a generalization of circles of partitions. We launch the following language.

Definition 2.1. Let $\mathbb{G}, \mathbb{A}, \mathbb{M} \subseteq \mathbb{N}$ with $s, t, u \in \mathbb{N}$. Then we denote with

$$\mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u) = \left\{[x], [y] \mid x \in \mathbb{A}^t, y \in \mathbb{M}^u,\ n = x + y,\ n \in \mathbb{G}^s\right\}$$

the **generalized** circle of partition generated by $n \in \mathbb{G}^s$ with base **regulators** $\mathbb{A}^t, \mathbb{M}^u$ with the generator **house** $\mathbb{G}^s$ as the t, u and s-fold prod-set of the sets $\mathbb{A}, \mathbb{M}$ and $\mathbb{G}$, respectively. We call members of the generalized CoP generalized points. We call s, t, u the degrees of the generator house and the regulators, respectively, and we call the sum $u+t$ the **potential** of the generalized CoP, denoted $\ulcorner \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u) \urcorner$.

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}^{u,t,s}$ joining the point $[x]$ and $[y]$ as an **axis** of the generalized CoP $\mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)$ if and only if $[x], [y] \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)$ and $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}^{u,t,s}$ and $\mathbb{L}_{[y],[x]}^{u,t,s}$, since it is essentially the the same axis. In special cases where the points $[x] \in \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)$ such that $2x = n$ then $[x]$ is the **center** of the generalized CoP and $x \in \mathbb{A}^t \cap \mathbb{M}^u$. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}^{u,t,s}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}^{u,t,s}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}^{u,t,s}$ to the generalized CoP $\mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)$ as

$$\mathbb{L}_{[x],[y]}^{u,t,s} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u) \text{ which means } [x], [y] \in \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u) \text{ with } x + y = n$$

for a fixed $n \in \mathbb{G}^s$ with $x \in \mathbb{A}^t$ and $y \in \mathbb{M}^u$ or vice versa and the number of real axes of the generalized CoP as

$$\nu(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u) := \#\{\mathbb{L}_{[x],[y]}^{u,t,s} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u) \mid x < y\}.$$

The line $\mathcal{L}_{[x],[y]}^{u,t,s}$ joining any two arbitrary points $[x], [y] \in \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)$ which are not axes partners on the generalized CoP will be referred to as a **chord** of the generalized CoP.

Throughout this paper we will denote for simplicity the generalized circle of partition in simple wording as g-CoP. Also, it is worth pointing out that various basic features that holds for CoPs does hold for generalized CoPs, except for previously technical results that needs investigating. Next we introduce the notion of the concentration of g-CoPs.

3. **Concentration of the generalized circle of partition**

In this section we introduce the notion of the **concentration** of a g-CoP. This notion of concentration of a g-CoP can be thought of as a generalization of density of points on a CoP. We launch the following language.

Definition 3.1 (Concentration)**.** Let $\mathbb{A},\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t,\mathbb{M}^u)$ be g-CoP. Then by the concentration of the generalized CoP, denoted $\Lambda(\mathcal{C}(\mathbb{G}^s(\infty),\mathbb{A}^t,\mathbb{M}^u))$, we mean

$$\Lambda(\mathcal{C}(\mathbb{G}^s(\infty),\mathbb{A}^t,\mathbb{M}^u)) = \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t,\mathbb{M}^u)\right\}}{\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t\cup\mathbb{M}^u)\right\}}$$

if the limit exists.

The above definition of the concentration of a g-CoP can be seen as a generalized version of density of points on a CoP. To wit, let us take $u=t=s=1$ and $\mathbb{G}=\mathbb{N}$ with $\mathbb{A}\subset\mathbb{M}$ then we can write

$$\begin{aligned}\Lambda(\mathcal{C}(\infty),\mathbb{A},\mathbb{M})) &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A},\mathbb{M})\right\}}{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A}\cup\mathbb{M})\right\}}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M})\mid\{x,y\}\cap\mathbb{A}\neq\emptyset\right\}}{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M})\right\}}\end{aligned}$$

and this is the density of the points $[x]$ with $x\in\mathbb{A}$ on the CoP $\mathcal{C}(n,\mathbb{M})$ introduced and studied in [1]. The primacy of the notion concentration of a g-CoP and its application are vast, so it is worthwhile to study it into detailed with the goal of exploiting some potential applications.

Proposition 3.2. *Let $\mathbb{A},\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t,\mathbb{M}^u)$ be g-CoP. Then the following properties of concentration of a g-CoP holds.*

(i) $0\leq\Lambda(\mathcal{C}(\mathbb{G}^s(\infty),\mathbb{A}^t,\mathbb{M}^u))\leq 1$.

(ii) *If $\mathbb{A}\subset\mathbb{M}$ and $u=t$, then*

$$\Lambda(\mathcal{C}(\infty,\mathbb{A}^t,\mathbb{M}^u)) = \mathcal{D}(\mathbb{A}^t_{\mathcal{C}(\infty,\mathbb{M}^u)}).$$

where $\mathcal{D}(\mathbb{A}^t_{\mathcal{C}(\infty,\mathbb{M}^u)})$ denotes the density of points $[x]$ with weight $x\in\mathbb{A}^t$ on the CoP $\mathcal{C}(n,\mathbb{M}^u)$.

Proof. The upper bound in Property (i) follows very easily by noting the inequality

$$\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t,\mathbb{M}^u)\right\}\leq\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t\cup\mathbb{M}^u)\right\}.$$

The lower bound follows if either $\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t,\mathbb{M}^u)\right\}=0$ or if

$$\lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t,\mathbb{M}^u)\right\}}{\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(\mathbb{G}^s(n),\mathbb{A}^t\cup\mathbb{M}^u)\right\}}=0.$$

For (ii) under the assumption $\mathbb{A}\subset\mathbb{M}$ and $u=t$, then we can write

$$\begin{aligned}\Lambda(\mathcal{C}(\infty,\mathbb{A}^t,\mathbb{M}^t)) &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}^{t}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A}^t,\mathbb{M}^t)\right\}}{\#\left\{\mathbb{L}^{t}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A}^t\cup\mathbb{M}^t)\right\}}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}^{t}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M}^t)\mid\{x,y\}\cap\mathbb{A}^t\neq\emptyset\right\}}{\#\left\{\mathbb{L}^{t}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{M}^t)\right\}}\\ &= \mathcal{D}(\mathbb{A}^t_{\mathcal{C}(\infty,\mathbb{M}^t)}).\end{aligned}$$

□

Remark 3.3. Next we show that the concentration will always exists and be a nullity for all subsets of the integers with zero density.

Proposition 3.4. *If $\mathbb{A} \subset \mathbb{N}$ for $t \geq 2$ with $u = t$ then*

$$\Lambda(\mathcal{C}(\infty, \mathbb{A}^t, \mathbb{N}^u)) = 0.$$

Proof. We note that under the condition $\mathbb{A} \subset \mathbb{N}$ and $u = t$ then by appealing to the second part of Proposition 3.2

$$\Lambda(\mathcal{C}(\infty, \mathbb{A}^t, \mathbb{N}^u)) = \mathcal{D}(\mathbb{A}^t_{\mathcal{C}(\infty, \mathbb{N}^u)})$$

so that we can write the inequality (See [1])

$$\mathcal{D}(\mathbb{A}^t) \leq \Lambda(\mathcal{C}(\infty, \mathbb{A}^t, \mathbb{N}^u)) \leq 2\mathcal{D}(\mathbb{A}^t)$$

and the concentration follows by noting that $\mathbb{N}^t_n = \mathbb{N}_n$ and $\mathcal{D}(\mathbb{A}^t) = 0$ for $t \geq 2$. □

Theorem 3.5. *If $\Lambda(\mathcal{C}(\mathbb{G}^s(\infty), \mathbb{A}^t, \mathbb{M}^u)) > 0$ for $u \neq t$, then the equation*

$$x^t + y^u = z^s$$

always has a solution for $z \in \mathbb{G}$, $x \in \mathbb{A}$ and $y \in \mathbb{M}$.

Proof. Let us suppose that $\Lambda(\mathcal{C}(\mathbb{G}^s(\infty), \mathbb{A}^t, \mathbb{M}^u)) > 0$, then there exists some constant $c > 0$ such that we can write

$$\#\left\{\mathbb{L}^{u,t,s}_{[q],[r]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)\right\} \sim c\#\left\{\mathbb{L}^{u,t,s}_{[q],[r]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t \cup \mathbb{M}^u)\right\} > 0.$$

The claim follows from this assertion by noting that

$$\left\{\mathbb{L}^{u,t,s}_{[q],[r]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)\right\} = \left\{q + r \mid q + r = n, q \in \mathbb{A}^t, r \in \mathbb{M}^u, n \in \mathbb{G}^s\right\}.$$

□

4. Axial potential of generalized circles of partition

In this section we introduce and study the notion of the **axial potential** of a g-CoP. We launch the following language.

Definition 4.1. Let $\mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)$ be a g-CoP. Then by the k th **axial potential** denoted, $\lfloor \mathcal{C}(\mathbb{G}^s(\infty), \mathbb{A}^t, \mathbb{M}^u)\rfloor^k$, we mean the infinite sum

$$\lfloor \mathcal{C}(\mathbb{G}^s(\infty), \mathbb{A}^t, \mathbb{M}^u)\rfloor^k = \sum_{n=1}^{\infty} \frac{\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)\right\}^k}{\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t \cup \mathbb{M}^u)\right\}^k}.$$

We say the k th axial potential is finite if the series converges; otherwise, we say it diverges.

It is worth pointing out that in the case $\mathbb{G} = \mathbb{M} = \mathbb{N}$ then we have the collapsing of the quantity

$$\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t \cup \mathbb{M}^u)\right\} = \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N})\right\}$$

the number of axes of CoPs, since $\mathbb{N}^s_n = \mathbb{N}_n$ for any $s \in \mathbb{N}$.

Theorem 4.2. *Let* $\mathbb{A} \subset \mathbb{M}$ *and suppose* $\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^t)\right\} > 0$ *for all sufficiently large values of* n*. If* $\mathbb{M} = \mathbb{G} = \mathbb{N}$ *with* $u = t$ *for* $s \neq t$ *and* $|\mathbb{A}^t \cap \mathbb{N}_n| \geq n^{1-\epsilon}$ *for any* $0 < \epsilon \leq \frac{1}{2}$ *then*

$$\lim_{n \longrightarrow \infty} \#\left\{q + r \mid q + r = n, q \in \mathbb{A}^t, r \in \mathbb{M}^u, n \in \mathbb{G}^s\right\} = \infty.$$

Proof. Under the requirement $\mathbb{A} \subset \mathbb{M}$ and $\mathbb{M} = \mathbb{G} = \mathbb{N}$ with $u = t$ for $s \neq t$ then we must have

$$\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)\right\} = \#\left\{\mathbb{L}^{t}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N}) \mid \{x, y\} \cap \mathbb{A}^t \neq \emptyset\right\}$$

for any $t \in \mathbb{N}$. We can now evaluate a truncated form of the 2^{nd} axial potential so that under the requirement $\#\left\{\mathbb{L}^{u,t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{M}^u)\right\} > 0$ for all sufficiently large n there exists some constant $\mathcal{P} := \mathcal{P}(k) > 0$ such that

$$\begin{aligned}\sum_{n=1}^{k} \frac{\#\left\{\mathbb{L}^{t}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{A}^t, \mathbb{N})\right\}^2}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N})\right\}^2} &\geq \mathcal{P} \sum_{n=1}^{k} \frac{\lfloor \frac{n^{1-\epsilon}-1}{2} \rfloor^2}{\lfloor \frac{n-1}{2} \rfloor^2} \\ &\gg_k \sum_{n=1}^{k} \frac{\frac{n^{2-2\epsilon}}{4}}{\lfloor \frac{n-1}{2} \rfloor^2}\end{aligned}$$

since $\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N})\right\} = \lfloor \frac{n-1}{2} \rfloor$, so that we can compute the 2^{nd} axial potential

$$\begin{aligned}\lfloor \mathcal{C}(\mathbb{G}^s(\infty), \mathbb{A}^t, \mathbb{M}^u) \rfloor^2 &= \sum_{n=1}^{\infty} \frac{\#\left\{\mathbb{L}^{t}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{A}^t, \mathbb{N})\right\}^2}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N})\right\}^2} \\ &\gg \sum_{n=1}^{\infty} \frac{\frac{n^{2-2\epsilon}}{4}}{\lfloor \frac{n-1}{2} \rfloor^2} \\ &\gg \sum_{n=1}^{\infty} \frac{1}{n^{2\epsilon}} = \infty\end{aligned}$$

since $0 \leq \epsilon \leq \frac{1}{2}$. It follows that

$$\lim_{n \longrightarrow \infty} \#\left\{\mathbb{L}^{t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{N})\right\} = \infty$$

and the claim follow immediately since

$$\#\left\{\mathbb{L}^{t,s}_{[x],[y]} \hat{\in} \mathcal{C}(\mathbb{G}^s(n), \mathbb{A}^t, \mathbb{N})\right\} = \#\left\{q + r \mid q + r = n, q \in \mathbb{A}^t, r \in \mathbb{M}^t, n \in \mathbb{G}^s\right\}.$$

□

It is important to recognize Theorem 4.2 is in many ways an extension of the Erdős-Turán additive base conjecture. To see that, we first note that under the same assumption of the main theorem, we can write the following decomposition

$$\begin{aligned}\#\left\{\mathbb{L}^{t}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N}) \mid \{x, y\} \cap \mathbb{A}^t \neq \emptyset\right\} &= \#\left\{\mathbb{L}^{t}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{A})\right\} + \\ &\quad \#\left\{\mathbb{L}^{t}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{N}) \mid x \in \mathbb{A}^t,\ y \in \mathbb{N} \setminus \mathbb{A}^t\right\}\end{aligned}$$

so that we take $t=1$ and we have

$$\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\mid\{x,y\}\cap\mathbb{A}\neq\emptyset\right\}=\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}+\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\mid x\in\mathbb{A},\ y\in\mathbb{N}\setminus\mathbb{A}\right\}$$

It is known that for any additive base $\mathbb{A}$ the quantity

$$\#\left\{n\leq x\mid n\in\mathbb{A}\right\}\geq\sqrt{x}.$$

Using this fact we then obtain a proof of the Erdős-Turán additive base conjecture in the form below

Corollary 4.3. *Let $\mathbb{A}\subset\mathbb{M}$ and suppose $\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}>0$ for all sufficiently large values of n. If $\mathbb{M}=\mathbb{G}=\mathbb{N}$ and $|\mathbb{A}\cap\mathbb{N}_n|\geq n^{1-\epsilon}$ for any $0<\epsilon\leq\frac{1}{2}$ then*

$$\lim_{n\longrightarrow\infty}\#\left\{q+r\mid q+r=n,\ q,r\in\mathbb{A}\right\}=\infty.$$

Proof. Under the requirement $\mathbb{A}\subset\mathbb{M}$ and suppose $\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}>0$ for all sufficiently large values of n tied with the decomposition

$$\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\mid\{x,y\}\cap\mathbb{A}\neq\emptyset\right\}=\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}+\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\mid x\in\mathbb{A},\ y\in\mathbb{N}\setminus\mathbb{A}\right\}$$

then there exists some constant $\mathcal{P}:=\mathcal{P}(k)>0$ such that we can write

$$\begin{aligned}\sum_{n=1}^{k}\frac{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A},\mathbb{N})\right\}^2}{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\right\}^2}&\geq\sum_{n=1}^{k}\frac{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}^2}{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\right\}^2}\\&\geq\mathcal{P}\sum_{n=1}^{k}\frac{\lfloor\frac{n^{1-\epsilon}-1}{2}\rfloor^2}{\lfloor\frac{n-1}{2}\rfloor^2}\\&\gg_k\sum_{n=1}^{k}\frac{\frac{n^{2-2\epsilon}}{4}}{\lfloor\frac{n-1}{2}\rfloor^2}\end{aligned}$$

since $\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\right\}=\lfloor\frac{n-1}{2}\rfloor$, so that we can compute the 2^{nd} axial potential

$$\begin{aligned}\lfloor\mathcal{C}(\mathbb{G}(\infty),\mathbb{A},\mathbb{M})\rfloor^2&\geq\sum_{n=1}^{\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}^2}{\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{N})\right\}^2}\\&\gg\sum_{n=1}^{\infty}\frac{\frac{n^{2-2\epsilon}}{4}}{\lfloor\frac{n-1}{2}\rfloor^2}\\&\gg\sum_{n=1}^{\infty}\frac{1}{n^{2\epsilon}}=\infty\end{aligned}$$

since $0\leq\epsilon\leq\frac{1}{2}$. It follows that

$$\lim_{n\longrightarrow\infty}\#\left\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n,\mathbb{A})\right\}=\infty$$

and it implies that

$$\lim_{n\longrightarrow\infty}\#\left\{q+r\mid q+r=n,\ q,r\in\mathbb{A}\right\}=\infty.$$

□

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