A Rotor Problem from Professor Miroslav Josipović

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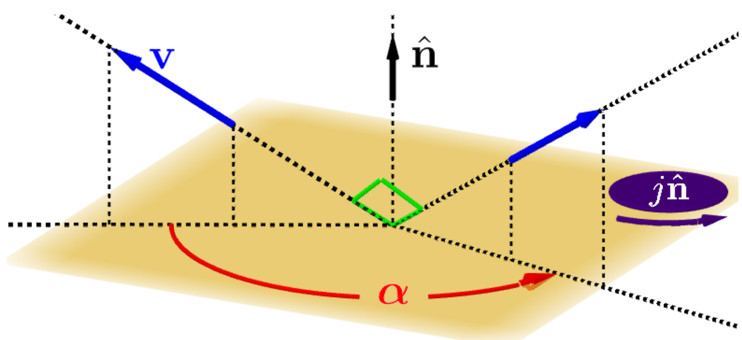
Abstract

We present two Geometric-Algebra (GA) solutions to a vector-rotation problem posed by Professor Miroslav Josipović. We follow the sort of solution process that might be useful to students. First, we review concepts from GA and classical geometry that may prove useful. Then, we formulate and carry-out two solution strategies. After testing the resulting solutions, we propose an extension to the original problem.



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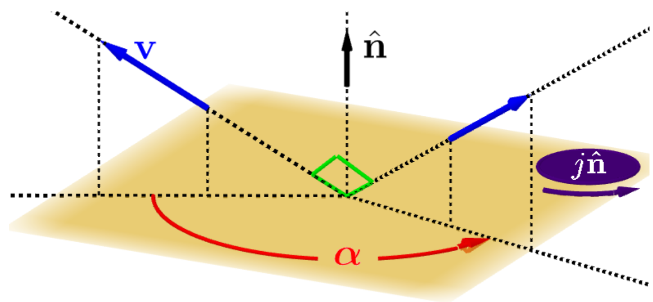

Figure 1: For a given vector $\mathbf{v}$ and the unit vector $\hat{\mathbf{n}}$, find a rotor $R = \exp(-\alpha j\hat{\mathbf{n}}/2)$ such that $\mathbf{v} \cdot \left(R\mathbf{v}R^{\dagger}\right) = 0$ where $j = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

1 Introduction

Professor Miroslav Josipović presented the following problem in the LinkedIn group "Pre-University Geometric Algebra":

> For a given vector $\mathbf{v}$ and the unit vector $\hat{\mathbf{n}}$, find a rotor $R = \exp(-\alpha j\hat{\mathbf{n}}/2)$ such that $\mathbf{v} \cdot \left(R\mathbf{v}R^{\dagger}\right) = 0$ where $j = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

The two solutions given herein may not be the most efficient, but the discussions of various aspects and stratagems may be useful to readers who are newcomers to GA.

2 Observations and Ideas

We'll begin with some thoughts that might prove useful.

- Initial GA observations
  - $j\hat{\mathbf{n}}$ is a bivector.
  - $R = \exp(-\alpha j\hat{\mathbf{n}}/2) = \cos\frac{\alpha}{2} - j\hat{\mathbf{n}}\sin\frac{\alpha}{2}$, and $R^{\dagger} = \exp(+\alpha j\hat{\mathbf{n}}/2) = \cos\frac{\alpha}{2} + j\hat{\mathbf{n}}\sin\frac{\alpha}{2}$. Therefore, to find R, all we need to do is identify α.
  - $R\mathbf{v}R^{\dagger}$ is a vector; specifically, the rotation of $\mathbf{v}$ around $\hat{\mathbf{n}}$ by α radians in the right-hand sense.
  - The condition $\mathbf{v} \cdot \left(R\mathbf{v}R^{\dagger}\right) = 0$ means that the vector $R\mathbf{v}R^{\dagger}$ is perpendicular to $\mathbf{v}$.
  - For any two vectors $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}\mathbf{v} \rangle_0$ ([2], p. p. 101).
- Geometric observations

- The rotation of $\mathbf{v}$ about $\hat{\mathbf{n}}$ generates a cone whose apex angle is twice as large as the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$. Therefore, the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ must be $\geq 45°$; otherwise, no rotation of $\mathbf{v}$ about $\hat{\mathbf{n}}$ will be perpendicular to $\mathbf{v}$.
  - If the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ is $45°$, then $\alpha = \pi$.
  - If $\mathbf{v}$ is parallel to the bivector $j\hat{\mathbf{n}}$ (and therefore perpendicular to $\hat{\mathbf{n}}$) , then $\alpha = \pi/2$.

- Further GA observations and ideas
  - $\mathbf{v} \cdot \left(R\mathbf{v}R^\dagger\right) = \langle \mathbf{v}R\mathbf{v}R^\dagger \rangle_0$.
  - In the nomenclature of [2] (p. 105), j is the unit pseudoscalar $\mathbf{I}$for $\mathbb{G}^3$ (that is, of 3-D GA). The properties of $\mathbf{I}$ for a given $\mathbb{G}^n$ include
    * $\mathbf{I} = (-1)^{n(n-1)/2}$; $\therefore$ $j^2 = -1$.
    * For any k-vector B, $B\mathbf{I} = \left[(-1)^{k(n-1)}\right]\mathbf{I}B$; $\therefore$ $\mathbf{uI} = \mathbf{Iu}$ for any vector $\mathbf{u}$ in $\mathbb{G}^3$.

A direct approach that is based upon these ideas might expand $\mathbf{v}R\mathbf{v}R^\dagger$, then use the properties of j to simplify the job of finding the $\langle\rangle_0$ of that expansion. Because, $RR^\dagger = 1$, a less-direct approach that might save ourselves some work would start by reversing the order of the product $\mathbf{v}R$.

We'll use both approaches here.

3 Solutions

3.1 A Direct Solution, Using the Properties of j

Based upon previous experience, we'll divide both sides of $\mathbf{v}R\mathbf{v}R^\dagger = 0$ by v^2 to give $\hat{\mathbf{v}}R\hat{\mathbf{v}}R^\dagger = 0$. Then, we proceed with the expansion.

$$\begin{aligned}
\hat{\mathbf{v}}R\hat{\mathbf{v}}R^\dagger &= \mathbf{v}\underbrace{\left[\cos\frac{\alpha}{2} - (j\hat{\mathbf{n}})\sin\frac{\alpha}{2}\right]}_{R}\hat{\mathbf{v}}\underbrace{\left[\cos\frac{\alpha}{2} + (j\hat{\mathbf{n}})\sin\frac{\alpha}{2}\right]}_{R^\dagger} \\
&= \hat{\mathbf{v}}\hat{\mathbf{v}}\cos^2\frac{\alpha}{2} + [\hat{\mathbf{v}}\hat{\mathbf{v}}j\hat{\mathbf{n}}]\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\
&\quad - [\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}]\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - [\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}j\hat{\mathbf{n}}]\sin^2\frac{\alpha}{2}.
\end{aligned}$$

Using $\hat{\mathbf{v}}\hat{\mathbf{v}} = 1$ and the properties of $\langle\rangle_0$, $\langle\hat{\mathbf{v}}R\hat{\mathbf{v}}R^\dagger\rangle_0 = 0$ the previous equation becomes

$$\cos^2\frac{\alpha}{2} - \langle j\hat{\mathbf{n}}\rangle_0 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \langle\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}\rangle_0 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \langle\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}j\hat{\mathbf{n}}\rangle_0 \sin^2\frac{\alpha}{2} = 0. \tag{3.1}$$

Now, let's examine each of the $\langle\ \rangle_0$ factors.

- $j\hat{\mathbf{n}}$ is a bivector, so $\langle j\hat{\mathbf{n}}\rangle_0 = 0$.

- Using the properties of j (Section 2) and the associative properties of the geometric product, we find that $\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}} = \hat{\mathbf{v}}\hat{\mathbf{n}}j\hat{\mathbf{v}} = \hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}}j = [\hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}}]\,j$. In addition, for any two vectors $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{uw} = 2\mathbf{u}\cdot\mathbf{w} - \mathbf{wu}$ ([1], p. 32). Therefore, $\hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}} = \hat{\mathbf{v}}\,[2\hat{\mathbf{n}}\cdot\hat{\mathbf{v}} - \hat{\mathbf{v}}\hat{\mathbf{n}}] = 2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} - \hat{\mathbf{n}}$, which is a vector. In $\mathbb{G}^3$, the product of any vector with j is a bivector. Therefore, $[\hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}}]\,j$ is a bivector, so $\langle\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}\rangle_0 = 0$.

- Again using the properties of j, $\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}j\hat{\mathbf{n}} = \hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}}\hat{\mathbf{n}}jj = -\hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}}\hat{\mathbf{n}}$. When we analyzed $\hat{\mathbf{v}}j\hat{\mathbf{n}}\hat{\mathbf{v}}$, we found that $\hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}} = 2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} - \hat{\mathbf{n}}$. Thus, $-\hat{\mathbf{v}}\hat{\mathbf{n}}\hat{\mathbf{v}}\hat{\mathbf{n}} = -\left[2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} - \hat{\mathbf{n}}\right]\hat{\mathbf{n}} = -2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}}\hat{\mathbf{n}} + 1 = -2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}} + \underbrace{\hat{\mathbf{n}}\wedge\hat{\mathbf{v}}}_{bivector}\right) + 1$. The $\langle\ \rangle_0$ of that result is $-2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2 + 1$.

Substituting these $\langle\rangle_0$'s into Eq. (3.1),

$$\cos^2\frac{\alpha}{2} - \left[-2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2 + 1\right]\sin^2\frac{\alpha}{2} = 0 \tag{3.2}$$

$$1 - \sin^2\frac{\alpha}{2} + \left[2\,(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2 - 1\right]\sin^2\frac{\alpha}{2} = 0,$$

and finally

$$\alpha = \pm 2\arcsin\left\{\sqrt{\frac{1}{2\left[1 - (\hat{\mathbf{v}}\cdot\hat{\mathbf{n}})^2\right]}}\right\}. \tag{3.3}$$

3.2 A Solution that Reverses $\mathbf{v}R$

This solution strategy (at least my execution of it) is much less efficient than the first, but some of the ideas involved might interest newcomers to GA. To reverse the product $\mathbf{v}R$, we'll express it in terms of $R\mathbf{v}$. First, we note that

$$\begin{aligned}\mathbf{v}R &= \mathbf{v}\left\{\cos\frac{\alpha}{2} - j\hat{\mathbf{n}}\sin\frac{\alpha}{2}\right\}\\ &= \mathbf{v}\cos\frac{\alpha}{2} - [\mathbf{v}\,(j\hat{\mathbf{n}})]\sin\frac{\alpha}{2};\ \text{in contrast,}\\ R\mathbf{v} &= \left\{\cos\frac{\alpha}{2} - j\hat{\mathbf{n}}\sin\frac{\alpha}{2}\right\}\mathbf{v}\\ &= \mathbf{v}\cos\frac{\alpha}{2} - [(j\hat{\mathbf{n}})\,\mathbf{v}]\sin\frac{\alpha}{2}.\end{aligned}$$

Therefore,

$$\mathbf{v}R = R\mathbf{v} - [\mathbf{v}\,(j\hat{\mathbf{n}}) - (j\hat{\mathbf{n}}\mathbf{v})]\sin\frac{\alpha}{2}.$$

Next, by using the identity that for any vector $\mathbf{u}$ and any bivector $\mathbf{B}$, $\mathbf{u}\cdot\mathbf{B} = \frac{1}{2}\left[\mathbf{uB} - \mathbf{Bu}\right]$ ([1]. p. 32), we obtain

$$\mathbf{v}R = R\mathbf{v} - 2\,[\mathbf{v}\cdot(j\hat{\mathbf{n}})]\sin\frac{\alpha}{2}.$$

Making this substitution, and using $R^\dagger = \cos\frac{\alpha}{2} + j\hat{\mathbf{n}}\sin\frac{\alpha}{2}$, our equation $\langle \mathbf{v}R\mathbf{v}R^\dagger\rangle_0 = 0$ becomes, progressively,

$$\begin{aligned}
\langle \left\{ R\mathbf{v} - 2\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\sin\frac{\alpha}{2} \right\} \mathbf{v}R^\dagger\rangle_0 &= 0, \\
\langle R\mathbf{v}\mathbf{v}R^\dagger - 2\left\{ \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\sin\frac{\alpha}{2} \right\} \mathbf{v}R^\dagger\rangle_0 &= 0, \\
\langle v^2 - 2\left\{ \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\sin\frac{\alpha}{2} \right\} \left\{ \mathbf{v}\underbrace{\left[\cos\frac{\alpha}{2} + j\hat{\mathbf{n}}\sin\frac{\alpha}{2}\right]}_{=R^\dagger} \right\}\rangle_0 &= 0, \\
\langle v^2 - \left[2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]\left\{\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\mathbf{v}\right\} - \left[2\sin^2\frac{\alpha}{2}\right]\left\{\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\,(j\hat{\mathbf{n}})\right]\right\}\rangle_0 &= 0, \\
v^2 - \left[2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]\underbrace{\langle\left\{\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\mathbf{v}\right\}\rangle_0}_{\text{(I)}} - \left[2\sin^2\frac{\alpha}{2}\right]\underbrace{\langle\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\,(j\hat{\mathbf{n}})\right]\rangle_0}_{\text{(II)}} &= 0.
\end{aligned} \tag{3.4}$$

Now, let's discuss the factors (I) and (II). To evaluate factor (I), we'll write $\mathbf{v}$ as the sum of its components parallel and perpendicular to the bivector $j\hat{\mathbf{n}}$. To do this, we make use of the identity that for any vector u and any bivector $\mathbf{B}$. the component of $\mathbf{u}$ parallel to $\mathbf{B}$ is $(\mathbf{u}\cdot\mathbf{B})\,\mathbf{B}^{-1}$, and the component perpendicular to $\mathbf{B}$ is $(\mathbf{u}\wedge\mathbf{B})\,\mathbf{B}^{-1}$ ([1], p. 65). (Of course, $\mathbf{B}^{-1}$ is itself a bivector.) Thus, we expand $\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\mathbf{v}$ as

$$\begin{aligned}
\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\mathbf{v} &= \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left\{ \underbrace{\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right](j\hat{\mathbf{n}})^{-1}}_{\mathbf{v}_\parallel} + \underbrace{\left[\mathbf{v}\wedge(j\hat{\mathbf{n}})\right](j\hat{\mathbf{n}})^{-1}}_{\mathbf{v}_\perp} \right\} \\
&= \underbrace{\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]^2}_{scalar}(j\hat{\mathbf{n}})^{-1} + \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\wedge(j\hat{\mathbf{n}})\right](j\hat{\mathbf{n}})^{-1}
\end{aligned}$$

The first term on the right-hand side of that result is a bivector. The second term is a bivector, too. We can see this by recalling that $\mathbf{v}\cdot(j\hat{\mathbf{n}})$ is a vector that's parallel to the bivector $j\hat{\mathbf{n}}$. (Actually, $\mathbf{v}\cdot(j\hat{\mathbf{n}})$ is a 90° rotation, in the plane of $j\hat{\mathbf{n}}$, of the vector $\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right](j\hat{\mathbf{n}})^{-1}$), which is $\mathbf{v}_\parallel$.) Also, $\left[\mathbf{v}\wedge(j\hat{\mathbf{n}})\right](j\hat{\mathbf{n}})^{-1}$ is $\mathbf{v}_\perp$. Therefore, $\mathbf{v}\cdot(j\hat{\mathbf{n}})$ and $\left[\mathbf{v}\wedge(j\hat{\mathbf{n}})\right](j\hat{\mathbf{n}})^{-1}$ are perpendicular to each other. As a result, their product is a bivector.

Because both terms on the right-hand side of the expansion of $\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\mathbf{v}$ are bivectors, so is their sum. Thus, $\langle\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\mathbf{v}\rangle_0 = 0$.

To evaluate $\langle\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\,(j\hat{\mathbf{n}})\right]\rangle_0$, we will use similar reasoning. We start by expanding the geometric product $\mathbf{v}\,(j\hat{\mathbf{n}})$ as $\mathbf{v}\,(j\hat{\mathbf{n}}) = \mathbf{v}\cdot(j\hat{\mathbf{n}}) + \mathbf{v}\wedge(j\hat{\mathbf{n}})$. Therefore,

$$\begin{aligned}
\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\,(j\hat{\mathbf{n}})\right] &= \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\cdot(j\hat{\mathbf{n}}) + \mathbf{v}\wedge(j\hat{\mathbf{n}})\right] \\
&= \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]^2 + \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\wedge(j\hat{\mathbf{n}})\right].
\end{aligned}$$

The first term on the right-hand side is a scalar, and the second is a trivector. Therefore, $\langle\left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]\left[\mathbf{v}\,(j\hat{\mathbf{n}})\right]\rangle_0 = \left[\mathbf{v}\cdot(j\hat{\mathbf{n}})\right]^2$. Because (I) $= 0$ and (II) $=$

$[\mathbf{v} \cdot (j\hat{\mathbf{n}})]^2$, Eq. (3.4) becomes

$$v^2 - \left[2\sin^2\frac{\alpha}{2}\right][\mathbf{v} \cdot (j\hat{\mathbf{n}})]^2 = 0.$$

Before proceeding further, we'll divide through by v^2 to obtain

$$1 - \left[2\sin^2\frac{\alpha}{2}\right][\hat{\mathbf{v}} \cdot (j\hat{\mathbf{n}})]^2 = 0.$$

We'll also recognize that $[\hat{\mathbf{v}} \cdot (j\hat{\mathbf{n}})]^2 = 1 - (\hat{\mathbf{v}} \cdot \hat{\mathbf{n}})^2$. Solving for α,

$$\alpha = \pm 2\arcsin\left\{\sqrt{\frac{1}{2\left[1 - (\hat{\mathbf{v}} \cdot \hat{\mathbf{n}})^2\right]}}\right\}, \tag{3.5}$$

which is Eq. (3.3).

4 Testing the Solution

4.1 Does the solution make sense?

In Eq. 3.5, the first thing that we might question is whether the "$\pm$" makes sense. It does indeed make sense, because either direction of rotation will produce a vector that's perpendicular to $\mathbf{v}$.

The geometric observations listed in Section 2 provide bases for testing other aspects of our solution. The first observation was that the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ must be $\geq 45°$; otherwise, no rotation of $\mathbf{v}$ about $\hat{\mathbf{n}}$ will be perpendicular to $\mathbf{v}$. Eq. (3.5) is consistent with that observation: if said angle is less than $45°$, then $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}} > 1/\sqrt{2}$, and $1/\sqrt{2\left[1 - (\hat{\mathbf{v}} \cdot \hat{\mathbf{n}})^2\right]} > 1$. The value of the sine function never exceeds 1, so no solution exists when the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ is $<45°$.

The second geometrical observation was that if the angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ is $45°$, then $\alpha = \pi$. In this case, $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}} = 1/\sqrt{2}$, so $1/\sqrt{2\left[1 - (\hat{\mathbf{v}} \cdot \hat{\mathbf{n}})^2\right]} = 1$, and $\alpha = \pi$.

The third observation was that if $\mathbf{v}$ is perpendicular to $\hat{\mathbf{n}}$, then $\alpha = \pi/2$. Here, $\hat{\mathbf{v}} \cdot \hat{\mathbf{n}} = 0$, so $1/\sqrt{2\left[1 - (\hat{\mathbf{v}} \cdot \hat{\mathbf{n}})^2\right]} = 1/\sqrt{2}$, and $\alpha = \pi/2$.

4.2 Analytical Test

This test consists of substituting the two solutions (the "-" as well as the "+") in Eq. (3.2). Note that we needn't go all the way back to $\mathbf{v}R\mathbf{v}R^\dagger$, because $\langle\mathbf{v}R\mathbf{v}R^\dagger\rangle_0$ simplifies to $\cos^2\frac{\alpha}{2} - \left[1 - 2(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}})^2\right]\sin^2\frac{\alpha}{2}$ for all values of α.

Here, we'll test only the "+" solution from Eq. (3.5). From said solution,

$$\sin\frac{\alpha}{2} = \sin\left\{\frac{1}{2}\left[2\arcsin\left\{\sqrt{\frac{1}{2\left[1-(\hat{\mathbf{v}}\cdot\hat{\mathbf{n}})^2\right]}}\right\}\right]\right\}$$
$$= \sqrt{\frac{1}{2\left[1-(\hat{\mathbf{v}}\cdot\hat{\mathbf{n}})^2\right]}} .$$

Substituting this expression in Eq. (3.2), and using $\cos^2\frac{\alpha}{2} = 1 - \sin^2\frac{\alpha}{2}$, we find that the "+" solution in Eq. (3.5) does work.

4.3 Numerical Test

We can test solution quantitatively via an interactive construction. For example, one made with GeoGebra.

5 Extending the Problem

Suppose the problem were

> For a given vector $\mathbf{v}$ and the unit vector $\hat{\mathbf{n}}$, find a rotor $R = \exp\left(-\alpha j\hat{\mathbf{n}}/2\right)$ such that the angle between $\mathbf{v}$ and $R\mathbf{v}R^\dagger$ is a given angle μ . What must the angle α be?

As a hint, what is the relation between $\langle R\mathbf{v}R^\dagger\rangle_0$, $\langle R\mathbf{v}R^\dagger\rangle_2$, and the trigonometric functions of μ?

References

[1] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[2] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).