ON A COVERING METHOD AND APPLICATIONS

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Abstract. In this paper we introduce and develop a method for studying problems concerning packing and covering dilemmas and explore some potential applications.

1. Introduction

Problems concerning packing or covering are of much centrality, whose perceived solutions will have much importance in various applied areas. There abounds a good number of problems in this area of study, where the primary motive is to optimally pack similar copies of a given shape into another shape. Conversely, the problem often seeks for the smallest cover of similar copies of a shape so that the wasted space is as small as possible. One typical example is the square packing problem (See [1]), where the objective is to find the optimal number of packing unit squares into squares of side $a > 1$. In other words, it seeks to minimize the wasted space of such a packing. Another problem of similar flavor is the circle packing in a square (See [2]), where the goal is to pack n circles into the smallest possible square. There are a whole host of other packing problems not least of which is the rectangle packing problem, where the objective is the pack rectangles in the smallest polygon such no two of them overlaps [3].
The current paper introduces and develops a method for finding possible covers of geometric shapes in the plane.

2. The notion of covering and covering capacity

In this section we introduce the notion of **covering** and the **covering capacity**. We study this notion in depth and explore some potential application in relation to problems concerning packing dilemmas.

Definition 2.1. Let $\mathbb{D} \subseteq \mathbb{R}^2$ and $\{\mathcal{S}_i\}$ and $\{\mathcal{T}_i\}$ be a class of shapes in $\mathbb{D}$. Then we say $\mathcal{S}_i \in \{\mathcal{S}_i\}$ admits a **cover** in $\{\mathcal{T}_i\}$ if there exists a self transformation $\mathbb{F}$ such that $\mathbb{F} : \mathcal{S}_i \longrightarrow \mathcal{S}_j$ for some j such that $\mathcal{S}_j$ is embedded in $\mathcal{T}_i$. We denote the embedding as $\mathcal{S}_j \prec \mathcal{T}_i$ and we call $\mathcal{T}_i$ the **cover** of $\mathcal{S}_i$.

Proposition 2.1. *Let $\mathbb{D} \subseteq \mathbb{R}^2$. Then every shape $\mathcal{S}$ in $\mathbb{D}$ admits a cover.*

Proof. Let $\mathcal{S}$ be a fixed shape in $\mathbb{D} \subseteq \mathbb{R}^2$. Let us apply a dilation $\mathbb{M}_t : \mathcal{S} \to \mathcal{S}'$ with scale factor $t > 1$ such that $\mathcal{S}'$ is a shape in $\mathbb{D}$. It follows that $\mathcal{S} \prec \mathcal{S}'$ and since $\mathbb{M}_t$ is a transformation we can choose $\mathbb{F} := \mathbb{M}_1$. □

Definition 2.2. Let $\mathcal{S}$ and $\mathcal{T}$ be any two shapes in $\mathbb{D} \subseteq \mathbb{R}^2$. Then we say $\mathcal{S}$ and $\mathcal{T}$ are **congruent** if and only if $\mathbf{Area}_{\mathcal{T}} = \mathbf{Area}_{\mathcal{S}}$.

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Definition 2.3. Let $\mathcal{S}$ and $\mathcal{T}$ be any two shapes in $\mathbb{D} \subseteq \mathbb{R}^2$ such that $\mathcal{T}$ is a cover of $\mathcal{S}$. Then by the **covering capacity** of $\mathcal{S}$ relative to the cover $\mathcal{T}$ we mean the **maximum** number of copies of shapes congruent to $\mathcal{S}$ that can be embedded into the shape $\mathcal{T}$. We denote the covering capacity by

$$\Omega_{\mathcal{T}}(\mathcal{S}) = \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} \right\rfloor$$

where $\mathbf{Area}_{\mathcal{T}}$ and $\mathbf{Area}_{\mathcal{S}}$ denotes the area occupied by $\mathcal{T}$ and $\mathcal{S}$, respectively, and $\lfloor \cdot \rfloor$ is the floor function.

Proposition 2.2. *Let $\mathcal{T}'$ and $\mathcal{T}$ be a unit square and a square of side a units, respectively. If $\mathcal{T}$ admits n optimal packing of similar copies of $\mathcal{T}'$ and a is an integer, then $a = \sqrt{n}$.*

Proof. Under the assumption $\mathcal{T}$ admits n optimal packing of similar copies of $\mathcal{T}'$, then we have

$$\begin{aligned}\Omega_{\mathcal{T}}(\mathcal{T}') &= \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{T}'}} \right\rfloor \\ &= \lfloor a^2 \rfloor \\ &= a^2.\end{aligned}$$

Since a is an integer, $\Omega_{\mathcal{T}}(\mathcal{T}')$ is the number of optimal packing and it follows that

$$n = a^2$$

and the equality follows. $\square$

Definition 2.4. Let $\mathcal{S}$ and $\mathcal{T}$ be any two shapes in $\mathbb{D} \subseteq \mathbb{R}^2$ such that $\mathcal{T}$ is a cover of $\mathcal{S}$. Let $\rho_{\mathcal{T}}(\mathcal{S})$ denotes a packing of $\mathcal{S}$ into $\mathcal{T}$ and $\chi[\rho_{\mathcal{T}}(\mathcal{S})]$ denotes the number of such packing. Then we write the packing **inefficiency** of $\rho_{\mathcal{T}}(\mathcal{S})$ as the discrepancy

$$\text{Inefficiency}[\rho_{\mathcal{T}}(\mathcal{S})] = \Omega_{\mathcal{T}}(\mathcal{S}) - \chi[\rho_{\mathcal{T}}(\mathcal{S})].$$

We call the numbers $\min \chi[\rho_{\mathcal{T}}(\mathcal{S})]$ and $\max \chi[\rho_{\mathcal{T}}(\mathcal{S})]$ the least and the worst **complexity** of the packing $\rho_{\mathcal{T}}(\mathcal{S})$.

Remark 2.5. The notion of the packing inefficiency can intuitively be thought of as a numerical measure of the inefficiency of any sort of packing of a shape into its smallest cover. Indeed, if the inefficiency of any such packing is minimal then the packing must be an optimal packing. Conversely the packing will nonetheless fail to be optimal if the inefficiency is too big.

It is a well-known open problem to determine an asymptotic for the wasted space of any optimal packing of units squares in a square [1]. In the following Proposition we obtain a crude estimate for the wasted space of any packing of a square in a square of non-integer length.

Proposition 2.3. *Let $\mathcal{T}'$ and $\mathcal{T}$ be a square of side b and a units, respectively. If $\mathcal{T}$ admits packing of similar copies of $\mathcal{T}'$ and a is a non-integer with b an integer, then there exists a packing for which the wasted space is*

$$\frac{2a}{b}\left\{\frac{a}{b}\right\} - \left\{\frac{a}{b}\right\}^2,$$

where $\{\cdot\}$ denotes the fractional part of any real number.

Proof. Let us choose $\rho_{\mathcal{T}}(\mathcal{T}')$ to be the packing of similar copies of $\mathcal{T}'$ along $\lfloor \frac{a}{b} \rfloor \times \lfloor \frac{a}{b} \rfloor$ grid of axis-aligned b units squares within the square $\mathcal{T}$. Then the wasted space of packing congruent copies of $\mathcal{T}'$ into $\mathcal{T}$ is given by

$$\begin{aligned}\left(\frac{a}{b}\right)^2 - \Omega_{\mathcal{T}}(\mathcal{T}') &= \left(\frac{a}{b}\right)^2 - \left\lfloor \left(\frac{a}{b}\right)^2 \right\rfloor \\ &= \left\{ \left(\frac{a}{b}\right)^2 \right\}\end{aligned}$$

so that the wasted space of the packing of squares along the square grid is

$$\begin{aligned}\left\lfloor \left(\frac{a}{b}\right)^2 \right\rfloor - \chi[\rho_{\mathcal{T}}(\mathcal{T}')] + \left\{ \left(\frac{a}{b}\right)^2 \right\} &= \left\lfloor \left(\frac{a}{b}\right)^2 \right\rfloor - \left\lfloor \frac{a}{b} \right\rfloor^2 + \left\{ \left(\frac{a}{b}\right)^2 \right\} \\ &= \frac{2a}{b}\left\{\frac{a}{b}\right\} - \left\{\frac{a}{b}\right\}^2.\end{aligned}$$

□

It follows from Proposition 2.3 any improvement on the current upper and lower bounds of the wasted space of any packing can be obtained by improving on upper or lower bounds of the packing complexity of any such packing into a square. It turns out that we can dramatically cut down on the wasted space of some particular type of packing with certain complexity. These ideas are espoused in the following proposition.

Theorem 2.6. *Let $\mathcal{T}'$ and $\mathcal{T}$ be a square of side b and a units, respectively. If $\mathcal{T}$ admits packing of similar copies of $\mathcal{T}'$ and a is a non-integer with b an integer, then there exists a packing for which the wasted space is*

$$\left\{ \left(\frac{a}{b}\right)^2 \right\}$$

provided $\sqrt{\Omega_{\mathcal{T}}(\mathcal{T}')}$ is an integer.

Proof. Under the assumption $\sqrt{\Omega_{\mathcal{T}}(\mathcal{T}')}$ is an integer, let us choose $\rho_{\mathcal{T}}(\mathcal{T}')$ to be the packing of similar copies of $\mathcal{T}'$ along $\sqrt{\left\lfloor \left(\frac{a}{b}\right)^2 \right\rfloor} \times \sqrt{\left\lfloor \left(\frac{a}{b}\right)^2 \right\rfloor}$ grid of axis-aligned b units squares within the square $\mathcal{T}$. Then the wasted space of packing congruent copies of $\mathcal{T}'$ into $\mathcal{T}$ is given by

$$\begin{aligned}\left(\frac{a}{b}\right)^2 - \Omega_{\mathcal{T}}(\mathcal{T}') &= \left(\frac{a}{b}\right)^2 - \left\lfloor \left(\frac{a}{b}\right)^2 \right\rfloor \\ &= \left\{ \left(\frac{a}{b}\right)^2 \right\}.\end{aligned}$$

Since $\sqrt{\Omega_{\mathcal{T}}(\mathcal{T}')}$ is an integer, congruent copies of $\mathcal{T}'$ packed into $\mathcal{T}$ are all squares so that the wasted space of the packing of squares along the square grid is also

$$\left\{ \left(\frac{a}{b}\right)^2 \right\}.$$

□

Corollary 2.1. Let $\mathcal{T}'$ be a unit square and $\mathcal{T}$ be a square of sides $a > 1$ units. If a is a non-integer, then there exists a packing of copies of $\mathcal{T}'$ into $\mathcal{T}$ for which the wasted space is

$$\{a^2\}$$

provided $\sqrt{\Omega_{\mathcal{T}}(\mathcal{T}')}$ is an integer.

Proof. This is a consequence of Theorem 2.6. □

Proposition 2.4. *Let $\mathcal{S}, \mathcal{T}'$ and $\mathcal{T}$ be shapes in $\mathbb{D} \subseteq \mathbb{R}^2$ such that $\mathcal{T}$ and $\mathcal{T}'$ are covers of $\mathcal{S}$. If $\mathcal{T} \prec \mathcal{T}'$ then $\Omega_{\mathcal{T}}(\mathcal{S}) \leq \Omega_{\mathcal{T}'}(\mathcal{S})$.*

Proof. Let us suppose $\mathcal{T} \prec \mathcal{T}'$, then $\mathbf{Area}_{\mathcal{T}} < \mathbf{Area}_{\mathcal{T}'}$. It follows that

$$\left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} \right\rfloor \leq \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}'}}{\mathbf{Area}_{\mathcal{S}}} \right\rfloor$$

so that $\Omega_{\mathcal{T}}(\mathcal{S}) \leq \Omega_{\mathcal{T}'}(\mathcal{S})$, since $\mathcal{S} \prec \mathcal{T} \prec \mathcal{T}'$. □

Remark 2.7. The next result introduces a criterion for determining the plausibility of a predetermined cover admitting packing of the same number of congruent copies of two distinct shapes in the plane $\mathbb{R}^2$.

Proposition 2.5. *Let $\mathcal{S}, \mathcal{S}'$ and $\mathcal{T}$ be shapes in $\mathbb{D} \subseteq \mathbb{R}^2$ such that $\mathcal{T}$ is a cover of $\mathcal{S}$ and $\mathcal{S}'$. If $\Omega_{\mathcal{T}}(\mathcal{S}) = \Omega_{\mathcal{T}}(\mathcal{S}')$ then*

$$|\mathbf{Area}_{\mathcal{S}} - \mathbf{Area}_{\mathcal{S}'}| < \frac{\mathbf{Area}_{\mathcal{S}'}\mathbf{Area}_{\mathcal{S}}}{\mathbf{Area}_{\mathcal{T}}}.$$

Moreover, If $\Omega_{\mathcal{T}}(\mathcal{S}) = \Omega_{\mathcal{T}'}(\mathcal{S})$ then $|\mathbf{Area}_{\mathcal{T}} - \mathbf{Area}_{\mathcal{T}'}| < \mathbf{Area}_{\mathcal{S}}$.

Proof. Suppose $\Omega_{\mathcal{T}}(\mathcal{S}) = \Omega_{\mathcal{T}}(\mathcal{S}')$ then we can write

$$\left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} \right\rfloor = \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}'}} \right\rfloor$$

so that

$$\frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} - \left\{ \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} \right\} = \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}'}} - \left\{ \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}'}} \right\}.$$

It follows that we can write the discrepancy

$$\mathbf{Area}_{\mathcal{S}'} + \frac{\mathbf{Area}_{\mathcal{S}'}\mathbf{Area}_{\mathcal{S}}\left\{\frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}}\right\}}{\mathbf{Area}_{\mathcal{T}}} = \mathbf{Area}_{\mathcal{S}} + \frac{\mathbf{Area}_{\mathcal{S}}\mathbf{Area}_{\mathcal{S}'}\left\{\frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}'}}\right\}}{\mathbf{Area}_{\mathcal{T}}}$$

so that

$$\begin{aligned} |\mathbf{Area}_{\mathcal{S}} - \mathbf{Area}_{\mathcal{S}'}| &= \left| \frac{\mathbf{Area}_{\mathcal{S}}\mathbf{Area}_{\mathcal{S}'}\left\{\frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}}\right\}}{\mathbf{Area}_{\mathcal{T}}} - \frac{\mathbf{Area}_{\mathcal{S}}\mathbf{Area}_{\mathcal{S}'}\left\{\frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}'}}\right\}}{\mathbf{Area}_{\mathcal{T}}} \right| \\ &< \frac{\mathbf{Area}_{\mathcal{S}'}\mathbf{Area}_{\mathcal{S}}}{\mathbf{Area}_{\mathcal{T}}} \end{aligned}$$

since $\mathbf{Area}_{\mathcal{T}} > \mathbf{Area}_{\mathcal{S}}, \mathbf{Area}_{\mathcal{S}'}$ and where $\{\cdot\}$ denotes the fractional part of any decimal representation.
Similarly, we can write the equality as $\Omega_{\mathcal{T}}(\mathcal{S}) = \Omega_{\mathcal{T}'}(\mathcal{S})$

$$\left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} \right\rfloor = \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}'}}{\mathbf{Area}_{\mathcal{S}}} \right\rfloor$$

which is equivalent to

$$\begin{aligned} |\mathbf{Area}_{\mathcal{T}'} - \mathbf{Area}_{\mathcal{T}}| &< \mathbf{Area}_{\mathcal{S}} \left| \left\{ \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}}} \right\} - \left\{ \frac{\mathbf{Area}_{\mathcal{T}'}}{\mathbf{Area}_{\mathcal{S}}} \right\} \right| \\ &< \mathbf{Area}_{\mathcal{S}}. \end{aligned}$$

□

The first part of Proposition 2.5 is quite suggestive. It follows that for any two shapes $\mathcal{S}, \mathcal{S}'$ admitting a cover $\mathcal{T}$ in the plane $\mathbb{R}^2$, If on the contrary

$$|\mathbf{Area}_{\mathcal{S}} - \mathbf{Area}_{\mathcal{S}'}| \geq \frac{\mathbf{Area}_{\mathcal{S}'}\mathbf{Area}_{\mathcal{S}}}{\mathbf{Area}_{\mathcal{T}}}$$

then $\mathcal{T}$ cannot admit the same maximal packing of congruent copies of $\mathcal{S}$ and $\mathcal{S}'$.

Theorem 2.8. *Let $\mathcal{S}, \mathcal{S}'$ be any two shapes admitting a cover $\mathcal{T}$ in the plane $\mathbb{R}^2$. If*

$$|\mathbf{Area}_{\mathcal{S}} - \mathbf{Area}_{\mathcal{S}'}| \geq \frac{\mathbf{Area}_{\mathcal{S}'}\mathbf{Area}_{\mathcal{S}}}{\mathbf{Area}_{\mathcal{T}}}$$

then $\mathcal{T}$ cannot admit the same maximal packing of congruent copies of $\mathcal{S}$ and $\mathcal{S}'$. Similarly, If

$$|\mathbf{Area}_{\mathcal{T}} - \mathbf{Area}_{\mathcal{T}'}| \geq \mathbf{Area}_{\mathcal{S}}$$

then $\mathcal{T}$ and $\mathcal{T}'$ do not admit the same maximal packing of congruent copies of $\mathcal{S}$.

Proof. This is a contrapositive of proposition 2.5. □

3. Covering potential

In this section we introduce the notion of the **covering potential**. We launch the following languages.

Definition 3.1. Let $\Im = \{\mathcal{S}_j : 1 \leq j \leq n\}$ be a finite class of shapes of varying sizes in $\mathbb{R}^2$. Then by the **covering potential** of the class $\Im$ relative to the cover $\mathcal{T}$, denoted $\Gamma_{\mathcal{T}}(\Im)$, we mean the finite sum

$$\Gamma_{\mathcal{T}}(\Im) = \sum_{1 \leq j \leq n} \Omega_{\mathcal{T}}(\mathcal{S}_j).$$

Proposition 3.1. *Let $\Im = \{\mathcal{S}_j : 1 \leq j \leq n\}$ be a finite class of shapes of varying sizes in $\mathbb{R}^2$ and $\Im_c$ be the set of dilation of the shapes in $\Im$ for $c > 0$. Then the inequality holds*

$$c\Gamma_{\mathcal{T}}(\Im_c) \leq \Gamma_{\mathcal{T}}(\Im).$$

Proof. We note that

$$\begin{aligned}\Gamma_{\mathcal{T}}(\Im_c) &= \sum_{\mathcal{S}_j \in \Im_c} \Omega_{\mathcal{T}}(\mathcal{S}_j)\\ &= \sum_{\mathcal{S}_j \in \Im_c} \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}_j}} \right\rfloor\\ &= \sum_{\mathcal{S}_j \in \Im} \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{c\mathbf{Area}_{\mathcal{S}_j}} \right\rfloor\\ &\leq \frac{1}{c} \sum_{\mathcal{S}_j \in \Im} \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}_j}} \right\rfloor\\ &= \frac{1}{c}\Gamma_{\mathcal{T}}(\Im)\end{aligned}$$

and the inequality follows as a consequence. ☐

Proposition 3.2. *Let $\Im_1 = \{\mathcal{S}_j : \ 1 \leq j \leq n\}$ and $\Im_2 = \{\mathcal{S}'_j : \ 1 \leq j \leq m\}$ be any two finite class of shapes of varying sizes in $\mathbb{R}^2$. Then we have*

$$\Gamma_{\mathcal{T}}(\Im_1 \cup \Im_2) = \Gamma_{\mathcal{T}}(\Im_1) + \Gamma_{\mathcal{T}}(\Im_2) - \Gamma_{\mathcal{T}}(\Im_1 \cap \Im_2).$$

Proof. We note that

$$\begin{aligned}\Gamma_{\mathcal{T}}(\Im_1 \cup \Im_2) &= \sum_{\mathcal{S}_j \in \Im_1 \cup \Im_2} \Omega_{\mathcal{T}}(\mathcal{S}_j)\\ &= \sum_{\mathcal{S}_j \in \Im_1} \Omega_{\mathcal{T}}(\mathcal{S}_j) + \sum_{\mathcal{S}_j \in \Im_2} \Omega_{\mathcal{T}}(\mathcal{S}_j) - \sum_{\mathcal{S}_j \in \Im_1 \cap \Im_2} \Omega_{\mathcal{T}}(\mathcal{S}_j)\\ &= \Gamma_{\mathcal{T}}(\Im_1) + \Gamma_{\mathcal{T}}(\Im_2) - \Gamma_{\mathcal{T}}(\Im_1 \cap \Im_2).\end{aligned}$$

☐

Proposition 3.3. *If $\mathcal{T} \prec \mathcal{T}'$, then $\Gamma_{\mathcal{T}}(\Im) \leq \Gamma_{\mathcal{T}'}(\Im)$.*

Proof. Let us suppose $\mathcal{T} \prec \mathcal{T}'$ then appealing to Proposition 2.4 we can write $\Omega_{\mathcal{T}}(\mathcal{S}) \leq \Omega_{\mathcal{T}'}(\mathcal{S})$ so that we have

$$\sum_{1 \leq j \leq n} \Omega_{\mathcal{T}}(\mathcal{S}_j) \leq \sum_{1 \leq j \leq n} \Omega_{\mathcal{T}'}(\mathcal{S}_j)$$

and the inequality follows. ☐

Proposition 3.4. *Let $\Im = \{\mathcal{S}_j : \ 1 \leq j \leq n\}$ be a finite class of shapes of varying sizes in $\mathbb{R}^2$. If $\Gamma_{\mathcal{T}}(\Im) > 0$ then there exist at least a shape $\mathcal{S}_l \in \Im$ admitting the cover $\mathcal{T}$.*

Proof. Under the requirement $\Gamma_{\mathcal{T}}(\Im) > 0$, it follows that

$$\sum_{1 \leq j \leq n} \Omega_{\mathcal{T}}(\mathcal{S}_j) > 0$$

and since each $\Omega_{\mathcal{T}}(\mathcal{S}_j) > 0$, there must exists some $\mathcal{S}_l$ for $1 \leq l \leq n$ such that $\Omega_{\mathcal{T}}(\mathcal{S}_l) > 0$. ☐

Remark 3.2. Next we upper and lower bound the covering potential of a given finite class of shapes with a universal cover.

Proposition 3.5. *Let $\alpha_{\mathcal{T}} = \{\mathcal{S}_j : \mathcal{S}_j \prec \mathcal{T},\ 1 \leq j \leq n\}$ be a finite class of shapes of varying sizes in $\mathbb{R}^2$ with a cover. Then we have the inequality*

$$\frac{n\mathbf{Area}_{\mathcal{T}}}{\max\{\mathbf{Area}_{\mathcal{S}_j}\}_{j=1}^{n}} + O(n) \leq \Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}}) \leq \frac{n\mathbf{Area}_{\mathcal{T}}}{\min\{\mathbf{Area}_{\mathcal{S}_j}\}_{j=1}^{n}} + O(n).$$

Proof. First we note that we can write

$$\begin{aligned}
\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}}) &= \sum_{1\leq j\leq n} \Omega_{\mathcal{T}}(\mathcal{S}_j)\\
&= \sum_{1\leq j\leq n} \left\lfloor \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}_j}} \right\rfloor\\
&= \sum_{1\leq j\leq n} \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}_j}} - \sum_{1\leq j\leq n} \left\{ \frac{\mathbf{Area}_{\mathcal{T}}}{\mathbf{Area}_{\mathcal{S}_j}} \right\}\\
&= \mathbf{Area}_{\mathcal{T}} \sum_{1\leq j\leq n} \frac{1}{\mathbf{Area}_{\mathcal{S}_j}} + O\bigg(\sum_{1\leq j\leq n} 1 \bigg)\\
&= \mathbf{Area}_{\mathcal{T}} \sum_{1\leq j\leq n} \frac{1}{\mathbf{Area}_{\mathcal{S}_j}} + O(n)
\end{aligned}$$

and the upper and the lower bound follows by controlling the main term in the sum by the area of the minimal and the maximal shape in the class. ☐

4. The Density of Optimal Packing

In this section we introduce and develop the notion of the **density** of maximal packing of congruent shapes in a given cover. We launch the following languages and exploit some applications.

Definition 4.1. Let $\alpha_{\mathcal{T}} = \{\mathcal{S}_j : \mathcal{S}_j \prec \mathcal{T},\ 1 \leq j \leq n\}$ be a class of shapes of varying sizes in $\mathbb{R}^2$. Then we denote the **density** relative to the class $\alpha_{\mathcal{T}}$ with a cover of the maximal packing of shapes congruent to $\mathcal{S}$ as the limit

$$\mathcal{D}(\mathcal{S}) := \lim_{n\longrightarrow\infty} \frac{\Omega_{\mathcal{T}}(\mathcal{S})}{\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}})}$$

if it exists.

Remark 4.2. The notion of density of maximal packing of a given shape in a plane relative to the class $\alpha_{\mathcal{T}}$ with a cover does not necessarily confine the underlying shape into the same class. It may turn out that the shape does not belong to the class, in which case chances are the covering capacity of this shape relative to the class is a nullity and so the density of the maximal packing would be nullity as well in this setting. This possibility confirms the first property espoused in the following proposition.

Proposition 4.1. *Let $\alpha_{\mathcal{T}} = \{\mathcal{S}_j : \mathcal{S}_j \prec \mathcal{T},\ 1 \leq j \leq n\}$ be a class of shapes of varying sizes in $\mathbb{R}^2$ with a cover. Then the following properties of density of the maximal packing of congruent shapes relative to the class $\alpha_{\mathcal{T}} = \{\mathcal{S}_j : \mathcal{S}_j \prec \mathcal{T},\ 1 \leq j \leq n\}$ with a cover holds*

(i) $0 \leq \mathcal{D}(\mathcal{S}_k) \leq 1$.
(ii) $\mathcal{D}(\mathcal{S}_k) \leq \mathcal{D}(\mathcal{S}_l)$ *for a fixed k,l if $\mathcal{S}_l \prec \mathcal{S}_k$.*

(iii) $\sum\limits_{\mathcal{S}_j\in \alpha_{\mathcal{T}}}\mathcal{D}(\mathcal{S}_j)=1$

Proof. Part (i) of Proposition 4.1 is an easy consequence of the Definition 4.1. For (ii) if $\mathcal{S}_l\prec \mathcal{S}_k$ for $\mathcal{S}_k,\mathcal{S}_l\in \Im_{\mathcal{T}}$ then $\mathbf{Area}_{\mathcal{S}_l}\leq \mathbf{Area}_{\mathcal{S}_k}$ so that $\Omega_{\mathcal{T}}(\mathcal{S}_k)\leq \Omega_{\mathcal{T}}(\mathcal{S}_l)$. It follows that $\mathcal{D}(\mathcal{S}_k)\leq \mathcal{D}(\mathcal{S}_l)$.
For property (iii) we can write

$$\begin{aligned}\sum_{\mathcal{S}_j\in \alpha_{\mathcal{T}}}\mathcal{D}(\mathcal{S}_j)&=\sum_{\mathcal{S}_j\in \alpha_{\mathcal{T}}}\lim_{n\longrightarrow \infty}\frac{\Omega_{\mathcal{T}}(\mathcal{S}_j)}{\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}})}\\&=\lim_{n\longrightarrow \infty}\sum_{\mathcal{S}_j\in \alpha_{\mathcal{T}}}\frac{\Omega_{\mathcal{T}}(\mathcal{S}_j)}{\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}})}\\&=\lim_{n\longrightarrow \infty}\frac{\sum\limits_{\mathcal{S}_j\in \alpha_{\mathcal{T}}}\Omega_{\mathcal{T}}(\mathcal{S}_j)}{\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}})}=1\end{aligned}$$

since the limit exists for each $j\geq 1$ by virtue of Definition 4.1. □

Remark 4.3. The notion of the density relative to a class with a cover can be a useful tool in practice; indeed, it can help to determine if a fixed preassigned shape can admit congruent copies of a shape in the plane. The converse may not necessarily be the true. The very notion the packing density of a shape relative to a preassigned given class with a cover is zero would not necessarily mean the underlying cover does not admit packing of congruent copies.

Proposition 4.2. *Let $\alpha_{\mathcal{T}}=\{\mathcal{S}_j:\mathcal{S}_j\prec \mathcal{T},\ 1\leq j\leq n\}$ be a class of shapes of varying sizes in $\mathbb{R}^2$ with a cover. If $\mathcal{D}(\mathcal{S})>0$ relative to the class $\alpha_{\mathcal{T}}$, then $\mathcal{T}$ admits packing of congruent copies of $\mathcal{S}$.*

Proof. Under the assumption that $\mathcal{D}(\mathcal{S})>0$ relative to the class $\alpha_{\mathcal{T}}$ with a cover, it follows that $\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}})>0$ and there exists some constant $c>0$ such that we can write

$$\Omega_{\mathcal{T}}(\mathcal{S})\sim c\Gamma_{\mathcal{T}}(\alpha_{\mathcal{T}})>0$$

and $\mathcal{T}$ admits packing of congruent copies of $\mathcal{S}$. □

Proposition 4.3. *Let $\alpha_{\mathcal{T}}=\{\mathcal{S}_j:\mathcal{S}_j\prec \mathcal{T},\ 1\leq j\leq n\}$ be a class of shapes of varying sizes in $\mathbb{R}^2$ with a cover. Then the density $\mathcal{D}(\mathcal{S})$ relative to the class $\alpha_{\mathcal{T}}=\{\mathcal{S}_j:\mathcal{S}_j\prec \mathcal{T},\ 1\leq j\leq n\}$ with a cover satisfies the asymptotic*

$$\mathcal{D}(\mathcal{S})\sim \frac{\frac{1}{\mathbf{Area}_{\mathcal{S}}}}{\sum\limits_{1\leq j\leq n}\frac{1}{\mathbf{Area}_{\mathcal{S}_j}}}.$$

Proof. This asymptotic is a consequence of the estimate in Proposition 3.5. □

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