HCR's Theorem

Rotation of two coplanar planes, meeting at angle-bisector, about their intersecting edges

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Introduction: This theorem says that if two coplanar planes (i.e. lying in the same plane), meeting each other at a straight edge which is bisector of angle between their intersecting straight edges, are to be rotated through same angle about their intersecting straight edges then it is first required to cut remove V-shaped plane symmetrically about their common straight edge & then planes are rotated about their intersecting straight edges through a desired angle. But if these two coplanar planes have to be rotated through a desired angle about their intersecting straight edges such that their new edges (generated after removing V-shaped planar region) coincide each other then we require a specific angle (i.e. **V-cut angle**) to cut remove V-shaped plane to allow rotation of the co-planar planes meeting at a common edge.

In this theorem, we have to derive a mathematical expression to analytically compute the V-cut angle (δ) required for rotating through the same angle (θ) the two co-planar planes, initially meeting at a common edge bisecting the angle (α) between their intersecting straight edges, about their intersecting straight edges until their new straight edges (generated after removing V-shaped planar region) coincide. As a result, we get a point (apex) where three planes intersect one another out of which two are original planes (rotated) & third one is their co-plane (fixed).

This theorem is very important for creating pyramidal flat containers with polygonal (regular or irregular) base, closed right pyramids & polyhedrons having two regular n-gonal & 2n congruent trapezoidal faces.

HCR's Theorem: If two co-planar planes initially meet or intersect each other at a straight line (edge) which bisects the angle α ($\forall\ \alpha < \pi$) between two intersecting straight edges of the planes then V-cut angle δ, required to cut remove V-shaped plane bisected by the common edge so that two planes (after cutting V-plane) are rotated through the same angle θ about their intersecting straight edges until their new edges (i.e. generated after cut-removing V-plane) coincide, is given by following formula

$$\boldsymbol{\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) - \alpha = 2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right) - \alpha}$$

Where, $\boldsymbol{\theta_d}$ is the dihedral angle between rotated cut planes when their new edges coincide such that

$$\boldsymbol{\theta_d \geq \alpha\ \ \&\ \ 0 < \alpha, \theta, \theta_d < \pi}$$

Proof: Consider two planes 1 & 2 initially lying in the same plane (i.e. plane of paper) such that they meet or intersect each other at a common straight edge AB which bisects angle α ($\forall\ \alpha < \pi$) between straight edges BC & BD intersecting each other at point B (as shown in the figure-1). It is to cut remove V-plane to allow rotation

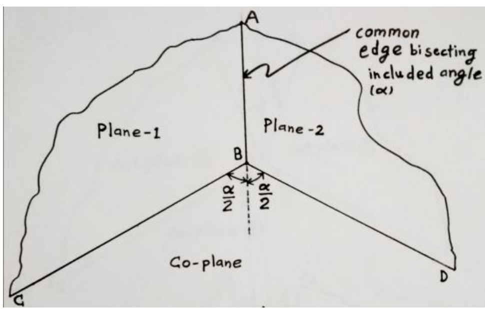


Figure 1: Two co-planar planes meet at edge AB & their edges BC & BD intersect each other at angle α which is bisected by common edge AB. The plane of paper is taken as co-plane in which two planes 1 & 2 initially lie

Now, in order to cut remove V-shaped plane equally divided by common edge AB, we make V-cut angle δ bisected by common edge AB. Mark the V-shaped planar region (as shaded) which is to be cut removed so as to rotate the planes 1 & 2 through the same angle until their new edges coincide (See fig-2 below)

After cut removing the V-shaped planar region, we get two cut planes 1 & 2 with new straight edges A_1B & A_2B meeting at angle δ (As shown in the fig-3 below)

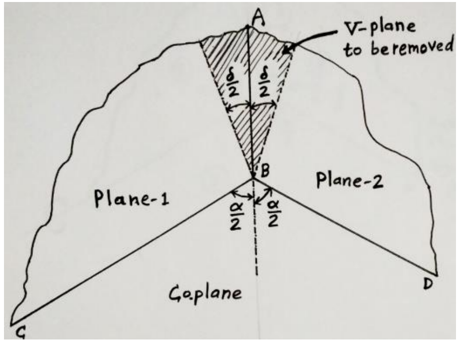


Figure 2: V-shape planar region (i.e. shaded) is equally divided by common edge AB

By symmetry (See figures 3 & 4 below), the following angles are given as

$$\angle A_1BC = \angle A_2BD = \pi - \frac{\alpha}{2} - \frac{\delta}{2}$$

Now, two cut planes 1 & 2 are rotated through the same angle θ about their intersecting edges BC & BD respectively (As shown in fig-4 below) until their new (generated) edges A_1B & A_2B coincide each other thus three planes 1, 2 & co-plane 3 intersect one another at the point B & three straight lines (edges) A'B, BC & BD (As shown in fig-5 below). It's worth noticing that during rotation of cut-planes 1 & 2 (see fig-4 below), the angles $\angle A_1BC$ & $\angle A_2BD$ remain equal & unchanged.

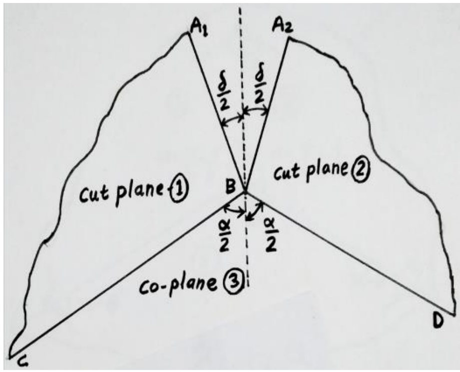


Figure 3: Cut planes 1 & 2 after removing V-shaped planar region to allow rotation

From figure-5,

$$\angle A'BC = \angle A'BD = \pi - \frac{\alpha}{2} - \frac{\delta}{2} \quad \& \quad \angle CBD = \alpha$$

Now, $\pi - \theta$ is dihedral angle between the planes 1 & 3 or 2 & 3, and θ_d is dihedral angle between the planes 1 & 2 (See fig-5 below).

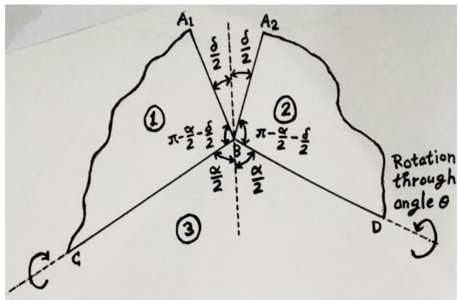


Figure 4: The angles α & $\pi - \frac{\alpha}{2} - \frac{\delta}{2}$ do not change due to rotation of cut planes 1 & 2

In general, if three planes intersect one another at a single point in 3-D space such that the angle between consecutive lines of intersection are φ, β & γ then the dihedral angle θ_φ, between two intersecting planes, opposite to the angle φ is given by **HCR's Inverse Cosine Formula (**as derived earlier in author's paper**)** as follows

$$\theta_\varphi = \cos^{-1}\left(\frac{\cos\varphi - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)$$

$$or \quad \cos\theta_\varphi = \frac{\cos\varphi - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

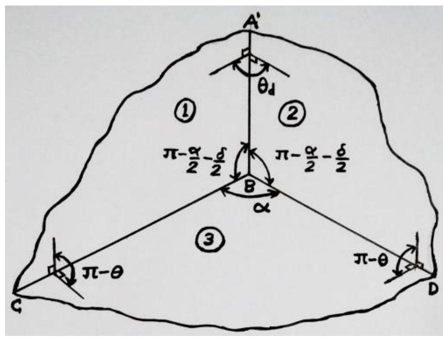


Figure 5: Three planes 1, 2 & 3 intersect one another at point B & lines A'B, BC & BD

Now, applying above inverse cosine formula & substituting the corresponding values of dihedral angle $\theta_\varphi = \pi - \theta$ (between planes 1 & 3) opposite to the angle $\varphi = \pi - \frac{\alpha}{2} - \frac{\delta}{2}$ (between lines A'B & BD), $\beta = \pi - \frac{\alpha}{2} - \frac{\delta}{2}$ (between lines A'B & BC)& $\gamma = \alpha$ (between lines BC & BD) (As shown in fig-5), we get

$$\cos(\pi - \theta) = \frac{\cos\left(\pi - \frac{\alpha}{2} - \frac{\delta}{2}\right) - \cos\left(\pi - \frac{\alpha}{2} - \frac{\delta}{2}\right)\cos\alpha}{\sin\left(\pi - \frac{\alpha}{2} - \frac{\delta}{2}\right)\sin\alpha}$$

$$-\cos\theta = \frac{-\cos\left(\frac{\alpha+\delta}{2}\right) - \left(-\cos\left(\frac{\alpha+\delta}{2}\right)\right)\cos\alpha}{\sin\left(\frac{\alpha+\delta}{2}\right)\sin\alpha}$$

$$\cos\theta = \frac{\cos\left(\frac{\alpha+\delta}{2}\right) - \cos\left(\frac{\alpha+\delta}{2}\right)\cos\alpha}{\sin\left(\frac{\alpha+\delta}{2}\right)\sin\alpha}$$

$$\cos\theta = \frac{\cos\left(\frac{\alpha+\delta}{2}\right)(1-\cos\alpha)}{\sin\left(\frac{\alpha+\delta}{2}\right)\sin\alpha}$$

$$\cos\theta = \frac{(1-\cos\alpha)}{\tan\left(\frac{\alpha+\delta}{2}\right)\sin\alpha}$$

$$\tan\left(\frac{\alpha+\delta}{2}\right) = \frac{(1-\cos\alpha)}{\sin\alpha\cos\theta}$$

$$\tan\left(\frac{\alpha+\delta}{2}\right) = \frac{\left(1-1+2\sin^2\frac{\alpha}{2}\right)\sec\theta}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$\tan\left(\frac{\alpha+\delta}{2}\right) = \frac{\sin\frac{\alpha}{2}\sec\theta}{\cos\frac{\alpha}{2}}$$

$$\tan\left(\frac{\alpha+\delta}{2}\right) = \tan\frac{\alpha}{2}\sec\theta$$

$$\frac{\alpha+\delta}{2} = \tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)$$

$$\alpha+\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)$$

$$\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)-\alpha \qquad \dots\dots\dots\dots(1)$$

Similarly, applying above inverse cosine formula & substituting the corresponding values of dihedral angle $\theta_\varphi = \theta_d$(between planes 1 & 2 after rotation) opposite to the angle $\varphi = \alpha$ (between lines BC & BD), $\beta = \pi - \frac{\alpha}{2} - \frac{\delta}{2}$ (between lines A'B & BC) & $\gamma = \pi - \frac{\alpha}{2} - \frac{\delta}{2}$ (between lines A'B & BD) (See fig-5 above), we get

$$\cos\theta_d = \frac{\cos\alpha - \cos\left(\pi-\frac{\alpha}{2}-\frac{\delta}{2}\right)\cos\left(\pi-\frac{\alpha}{2}-\frac{\delta}{2}\right)}{\sin\left(\pi-\frac{\alpha}{2}-\frac{\delta}{2}\right)\sin\left(\pi-\frac{\alpha}{2}-\frac{\delta}{2}\right)}$$

$$\cos\theta_d = \frac{\cos\alpha - \cos\left(\frac{\alpha+\delta}{2}\right)\cos\left(\frac{\alpha+\delta}{2}\right)}{\sin\left(\frac{\alpha+\delta}{2}\right)\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$\cos\theta_d = \frac{\cos\alpha - \cos^2\left(\frac{\alpha+\delta}{2}\right)}{\sin^2\left(\frac{\alpha+\delta}{2}\right)}$$

$$\cos\theta_d\sin^2\left(\frac{\alpha+\delta}{2}\right) = \cos\alpha - \cos^2\left(\frac{\alpha+\delta}{2}\right)$$

$$\cos\theta_d\sin^2\left(\frac{\alpha+\delta}{2}\right) = \cos\alpha - \left(1-\sin^2\left(\frac{\alpha+\delta}{2}\right)\right)$$

$$\cos\theta_d\sin^2\left(\frac{\alpha+\delta}{2}\right) = \cos\alpha - 1 + \sin^2\left(\frac{\alpha+\delta}{2}\right)$$

$$\sin^2\left(\frac{\alpha+\delta}{2}\right)-\sin^2\left(\frac{\alpha+\delta}{2}\right)\cos\theta_d=1-\cos\alpha$$

$$\sin^2\left(\frac{\alpha+\delta}{2}\right)(1-\cos\theta_d)=1-\cos\alpha$$

$$\sin^2\left(\frac{\alpha+\delta}{2}\right)=\frac{1-\cos\alpha}{1-\cos\theta_d}$$

$$\sin^2\left(\frac{\alpha+\delta}{2}\right)=\frac{1-1+2\sin^2\frac{\alpha}{2}}{1-1+2\sin^2\frac{\theta_d}{2}}$$

$$\sin^2\left(\frac{\alpha+\delta}{2}\right)=\frac{\sin^2\frac{\alpha}{2}}{\sin^2\frac{\theta_d}{2}}$$

$$\sin\left(\frac{\alpha+\delta}{2}\right)=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}$$

$$\frac{\alpha+\delta}{2}=\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right)$$

$$\alpha+\delta=2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right)$$

$$\delta=2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right)-\alpha \qquad \dots\dots\dots\dots\dots (2)$$

Now, equating the results from (1) & (2), we get **V-cut angle $\boldsymbol{\delta}$** for rotation as follows

$$\boldsymbol{\delta=2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)-\alpha=2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right)-\alpha}$$

$$\forall \ \ \boldsymbol{0<\alpha,\theta,\theta_d<\pi \ \ \& \ \ \theta_d\geq\alpha}$$

Above is the generalized formula to compute V-cut angle δ for rotating two co-planar planes through the same angle θ about their edges intersecting each other at an angle α when

1) Angle of rotation θ of two co-planar planes is known or
2) Dihedral angle $\theta_d$between two rotated planes is known

HCR's Corollary: If two co-planar planes initially meet or intersect each other at a straight line (edge) which bisects the angle α between two intersecting straight edges of the planes & V-shaped plane bisected by the common edge is cut removed so that two cut planes (after cutting V-plane) are rotated through the same angle θ about their intersecting edges until their new edges (i.e. generated after cut-removing V-plane) coincide then the dihedral angle θ_d, between two rotated cut planes, is given by following formula

$$\boldsymbol{\cos\frac{\theta_d}{2}=\sin\theta\cos\frac{\alpha}{2}}$$

Where, $0 < \alpha, \theta,\ \theta_d < \pi\ \ \&\ \ \theta_d \geq \alpha$

Proof: Consider two planes 1 & 2 lying in the same plane (i.e. plane of paper) such that they meet each other at common straight edge AB which bisects the angle α ($\forall\ \alpha < \pi$) between straight edges BC & BD intersecting each other at point B (as shown in the figure-1 above).

Now, we need cut remove V-shaped planar region bisected by the common edge AB to rotate planes 1 & 2 through same angle θ until their new edges A_1B & A_2B coincide (See fig-5 above).

Using above theorem, V-cut angle δ in terms of angle of rotation θ & angle of intersection , is given as

$$\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) - \alpha$$

Similarly if θ_d is the dihedral angle between two planes after cutting & rotation then using above theorem, V-cut angle δ in terms of dihedral angle θ_d & angle of intersection α , is given as

$$\delta = 2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right) - \alpha$$

Now, equating both the values of V-cut angle δ, we get

$$2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) - \alpha = 2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right) - \alpha$$

$$\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) = \sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right)$$

$$\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) = \tan^{-1}\left(\frac{\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}}{\sqrt{1-\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right)^2}}\right) \qquad \left(\text{since,}\quad \sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)$$

$$\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) = \tan^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{\sin^2\frac{\theta_d}{2} - \sin^2\frac{\alpha}{2}}}\right)$$

$$\sec\theta\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\sqrt{\sin^2\frac{\theta_d}{2} - \sin^2\frac{\alpha}{2}}}$$

$$\sec^2\theta\tan^2\frac{\alpha}{2} = \frac{\sin^2\frac{\alpha}{2}}{\sin^2\frac{\theta_d}{2} - \sin^2\frac{\alpha}{2}}$$

$$\sin^2\frac{\theta_d}{2} - \sin^2\frac{\alpha}{2} = \frac{\sin^2\frac{\alpha}{2}}{\sec^2\theta\tan^2\frac{\alpha}{2}}$$

$$\sin^2\frac{\theta_d}{2}-\sin^2\frac{\alpha}{2}=\frac{\sin^2\frac{\alpha}{2}\cos^2\theta\cos^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}}$$

$$1-\cos^2\frac{\theta_d}{2}-\left(1-\cos^2\frac{\alpha}{2}\right)=\cos^2\theta\cos^2\frac{\alpha}{2}$$

$$-\cos^2\frac{\theta_d}{2}+\cos^2\frac{\alpha}{2}=\cos^2\theta\cos^2\frac{\alpha}{2}$$

$$\cos^2\frac{\theta_d}{2}=\cos^2\frac{\alpha}{2}-\cos^2\theta\cos^2\frac{\alpha}{2}$$

$$\cos^2\frac{\theta_d}{2}=\cos^2\frac{\alpha}{2}(1-\cos^2\theta)$$

$$\cos^2\frac{\theta_d}{2}=\cos^2\frac{\alpha}{2}(\sin^2\theta)$$

$$\cos^2\frac{\theta_d}{2}=\sin^2\theta\cos^2\frac{\alpha}{2}$$

Taking square roots on both sides,

$$\sqrt{\cos^2\frac{\theta_d}{2}}=\sqrt{\sin^2\theta\cos^2\frac{\alpha}{2}}$$

$$\left|\cos\frac{\theta_d}{2}\right|=\left|\sin\theta\cos\frac{\alpha}{2}\right|$$

$$\cos\frac{\theta_d}{2}=\sin\theta\cos\frac{\alpha}{2}\qquad\qquad(\text{since},\ \ 0<\alpha,\theta,\theta_d<\pi)$$

The above mathematical relation is very useful to compute dihedral angle $\theta_d$when angle of rotation θ is known & vice-versa when angle of intersection α is given. The above relation holds only if two co-planar planes with a common straight edge, are cut and rotated to coincide their new edges

Illustrative Numerical Problems based on HCR's Theorem and Corollary

Q1. Two co-planar planes initially meet at a straight line which bisects the angle 100^o between their straight edges intersecting each other. Compute V-cut angle required to cut remove V-plane symmetrically about the common edge so that the planes are rotated through the same angle $\theta=80^o$about their intersecting edges until their new edges coincide & also compute the dihedral angle between the rotated planes.

Sol. Given that angle between intersecting straight edges, $\alpha=100^o$,

Angle of rotation of co-planar planes, $\theta=80^o$, $\delta=?,\ \theta_d=?$

Now, using formula for V-cut angle δ in terms of θ & α as follows

$$\delta=2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right)-\alpha$$

Setting the corresponding values in above formula, we get

$$\delta = 2\tan^{-1}\left(\sec 80^o \tan\frac{100^o}{2}\right) - 100^o$$

$$\boldsymbol{\delta = 63.4197597^o}$$

Now, using formula of corollary to compute dihedral angle $\theta_d$between the rotated planes as follows

$$\cos\frac{\theta_d}{2} = \sin\theta\cos\frac{\alpha}{2}$$

Setting the corresponding values $\theta = 80^o$ & $\alpha = 100^o$, we get

$$\cos\frac{\theta_d}{2} = \sin 80^o \cos\frac{100^o}{2}$$

$$\theta_d = 101.4530996^o$$

Q2. Two co-planar planes initially meet at a straight line which bisects the angle 150^o between their straight edges intersecting each other. Compute V-cut angle required to cut remove V-plane symmetrically about the common edge so that the planes are rotated through the same angle until their new edges coincide & the dihedral angle between the rotated planes is found to be 160^o. Also compute the angle through which two planes are rotated about their intersecting straight edges.

Sol. Given that angle between intersecting straight edges, $\alpha = 150^o$,

Dihedral angle between two rotated planes, $\theta_d = 160^o$, $\delta =?$, $\theta =?$

Now, using formula for V-cut angle δ in terms of θ_d & α as follows

$$\delta = 2\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\theta_d}{2}}\right) - \alpha$$

Setting the corresponding values in above formula, we get

$$\delta = 2\sin^{-1}\left(\frac{\sin\frac{150^o}{2}}{\sin\frac{160^o}{2}}\right) - 150^o$$

$$\boldsymbol{\delta = 7.524394318^o}$$

Now, using formula of corollary to compute dihedral angle $\theta_d$between the rotated planes as follows

$$\cos\frac{\theta_d}{2} = \sin\theta\cos\frac{\alpha}{2}$$

$$\sin\theta = \frac{\cos\frac{\theta_d}{2}}{\cos\frac{\alpha}{2}}$$

Setting the corresponding values $\theta_d = 160^o$ & $\alpha = 150^o$, we get

$$\sin\theta = \frac{\cos\frac{160^o}{2}}{\cos\frac{150^o}{2}}$$

$$\theta = 42.13849988^o$$

It can proved that all the numerical values obtained above by these formula are correct to the best of author's knowledge & experience.

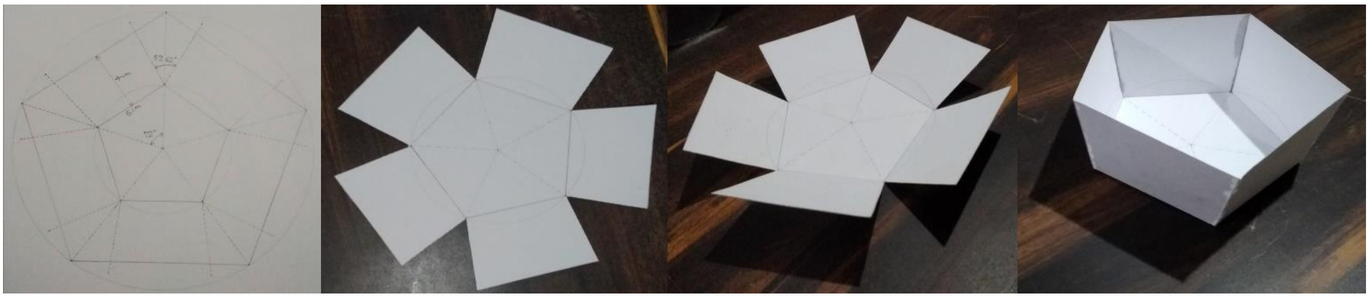
Applications of HCR's Theorem for making pyramidal flat containers with regular polygonal base

In order to make pyramidal flat container with regular n-gonal base, we need to

1) make drawing on sheet of paper, plastic or metal which can bent easily
2) Cut remove V-shaped planes from common edges & from the sides of regular polygon
3) Fold or rotate the trapezoidal (in this case) faces about the sides of regular polygonal base
4) Glue (in case of paper sheet) all the mating lateral edges of trapezoidal faces or weld (in case of plastic or metallic sheet) all the mating lateral edges of trapezoidal faces

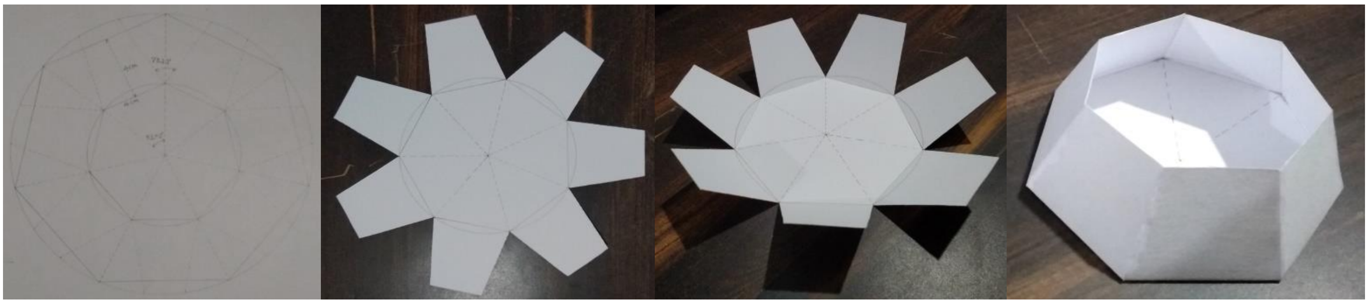
Some typical examples of making pyramidal flat containers with regular pentagonal, heptagonal & octagonal bases from paper sheets are shown in the pictures below

1. Pyramidal flat container with regular pentagonal base of side $5\ cm$, slant height $4\ cm$, angle of intersection i.e. interior angle of regular pentagon, $\alpha = 108^o$, angle of inclination of lateral trapezoidal faces with the plane of base, $\theta = 80^o$ & V-cut angle, $\boldsymbol{\delta = 57.62^o}$(Pictures below depict steps to make desired flat container)



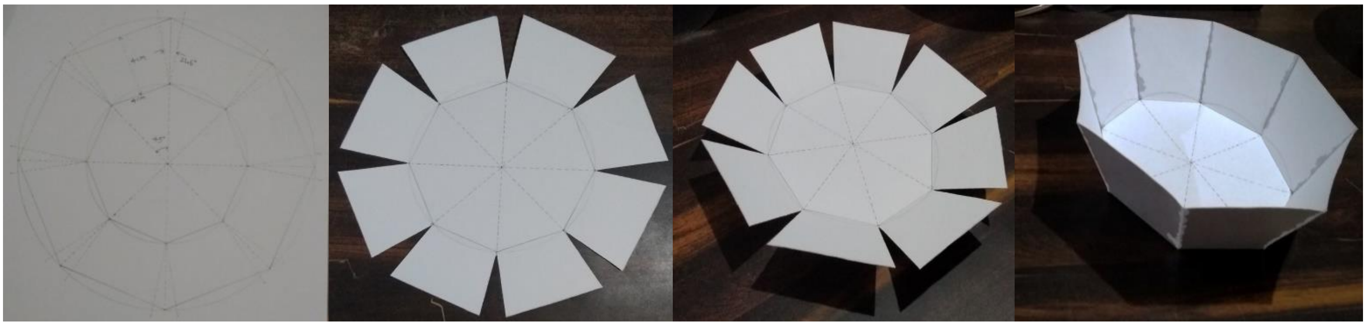
I. Drawing pentagon   II. Cutting V-parts & trimming   III. Folding faces about edges   IV. Gluing mating edges

2. Pyramidal flat container with regular heptagonal base of side $4\ cm$, slant height $4\ cm$, angle of intersection i.e. interior angle of regular heptagon, $\alpha = 128.57^o$, angle of inclination of lateral trapezoidal faces with the plane of base, $\theta = 110^o$ & V-cut angle, $\boldsymbol{\delta} = 70.13^o$(Pictures below depict steps to make desired flat container)



I. Drawing heptagon   II. Cutting V-parts & trimming   III. Folding faces about edges   IV. Gluing mating edges

3. Pyramidal flat container with regular octagonal base of side $4\ cm$, slant height $4\ cm$, angle of intersection i.e. interior angle of regular octagon, $\alpha = 135^o$, angle of inclination of lateral trapezoidal faces with the plane of base, $\theta = 60^o$ & V-cut angle, $\boldsymbol{\delta = 21.6^o}$ (Pictures below depict steps to make desired flat container)



I. Drawing octagon II. Cutting V-parts & trimming III. Folding faces about edges IV. Gluing mating edges

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

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